

DERIVATION OF ENTROPY PRODUCTION IN A FLUID FLOW IN A GENERAL CURVILINEAR COORDINATE SYSTEM

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ABSTRACT. The paper deals with the derivation of the entropy production in the fluid flow performed in a general curvilinear coordinate system. The derivation of the entropy production is based on the thermodynamics laws as well as on the balances of mass, momentum, and energy. A brief description of the differential geometry used in general curvilinear coordinates is presented here as well to define the used notation.

The application of this approach is then shown in the evaluation of the entropy production along the suction side of the blade, where the calculation was performed using available experimental data.

KEYWORDS: Entropy production, fluid flow, general curvilinear coordinates, linear blade cascade, experimental data.

1. INTRODUCTION

A general curvilinear description of a flow is useful to describe the flow in highly curved channels, e.g. turbine blade passages. It is better for understanding the processes that occur in the flow field because the researcher can make some predictions about the flow without solving the equations at all. Meaning, that it can be deduced from the form of equations whose terms will affect the results mostly and, next, if the geometry of the problem under investigation will be varied, how this geometry variation can affect the obtained data.

The derivation of the equations of a fluid flow in a general curvilinear coordinate system was performed several times. For the first time, it was performed in [1], where curvilinear components of velocity vector were introduced. A more detailed description can be found in book [2]. The derivation of the momentum equation in general curvilinear coordinates as well as the description of differential geometry with application to physics was introduced in [3]. There are, of course, papers focused on the application of this approach to the numerical simulations, see e.g. [4], or [5]. These papers, however, are considering only an incompressible fluid flow.

The aim of this paper is to go a little bit further and derive not only the equations of the fluid flow but unite the fluid mechanics and thermodynamics to obtain the entropy production in the flowing fluid in the general coordinate system. This result was not published so far according to the author's knowledge. The entropy production in a flowing fluid was recently derived in [6], however, this derivation was performed in a vector form.

2. BASICS OF DIFFERENTIAL GEOMETRY

The definition of the tensor¹ has to be mentioned in the following paragraph. The tensor is a quantity that exists independently whether there is some observer present or not. Tensor is then given by its components multiplied by basis vectors. The sum of these multiplied components with basis vector has to be invariant under basis transformation (Equation (1)), i.e. that tensors of the first order (vector) obey the transformation rule (Equation (2)):

$$\mathbf{v} = v^i \mathbf{g}_i = \hat{v}^j \hat{\mathbf{g}}_j, \quad (1)$$

$$v^i = A_j^i \hat{v}^j, \quad (2)$$

where \mathbf{v} is the vector, v^i are the components of the vector in the new basis, \mathbf{g}_i are the transformed basis vectors, $\hat{\mathbf{g}}_j$ are the original basis vectors, \hat{v}^j are the components of the vector v in the original basis and A_j^i is the transformation matrix.

The basis vectors \mathbf{g}_i can be defined as (see any textbook about differential geometry, e.g. [3]):

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \eta^i}, \quad (3)$$

where η^i are the coordinates and \mathbf{x} is the position vector. Similarly, co-basis vectors can be defined as:

$$\mathbf{g}^j = \frac{\partial \eta^j}{\partial \mathbf{x}}. \quad (4)$$

Covariant, contravariant metric coefficients and the Kronecker symbol can be obtained as:

¹Here vectors and scalars are included in the definition of the tensors as the tensors of the first and zeroth order, respectively.

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad (5)$$

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j, \quad (6)$$

$$\mathbf{g}^i \cdot \mathbf{g}_j = g_{il} g^{lj} = \delta_j^i. \quad (7)$$

Now, derivatives of the tensors in general curvilinear coordinates have to be determined. The gradient of the vector function in the general curvilinear coordinate has a form:

$$\frac{d\mathbf{v}}{dx} = \frac{\partial u^i}{\partial \eta^j} \mathbf{g}_i \mathbf{g}^j + u^i \frac{\partial \mathbf{g}_i}{\partial \eta^j} \mathbf{g}^j. \quad (8)$$

Christoffel symbols of the second kind can be defined based on this relationship as:

$$\Gamma_{ij}^k = \frac{\partial \mathbf{g}_i}{\partial \eta^j} \cdot \mathbf{g}^k = \frac{1}{2} g^{kl} (g_{li,j} + g_{lj,i} - g_{ij,l}). \quad (9)$$

Substituting this result into Equation (8) gives:

$$\frac{d\mathbf{v}}{dx} = \frac{\partial u^i}{\partial \eta^j} \mathbf{g}_i \mathbf{g}^j + u^i \Gamma_{ij}^k \mathbf{g}_k \mathbf{g}^j. \quad (10)$$

Finally, the covariant and contravariant derivatives of the vector and second order tensor, which will be frequently used in the following text, are defined:

$$v_{;j}^i = \frac{\partial u^i}{\partial \eta^j} + u^k \Gamma_{kj}^i, \quad (11)$$

$$v_{i;j} = \frac{\partial u_i}{\partial \eta^j} - u^k \Gamma_{ij}^k, \quad (12)$$

$$T_{;k}^{ij} = T_{,k}^{ij} + T^{lj} \Gamma_{lk}^i + T^{il} \Gamma_{lk}^j, \quad (13)$$

$$T_{ij;k} = T_{i,j,k} - T_{lj} \Gamma_{ik}^l - T_{il} \Gamma_{jk}^l, \quad (14)$$

$$T_{j;k}^i = T_{j,k}^i + T_j^l \Gamma_{lk}^i - T_l^i \Gamma_{jk}^l, \quad (15)$$

where partial derivatives are noted as:

$$v_{,j}^i = \frac{\partial u^i}{\partial x^j}. \quad (16)$$

All of the operations, such as divergence, gradient, curl etc., can be obtained from these relationships, see again e.g. [3].

3. LAWS OF THERMODYNAMICS

Thermodynamics is a scientific discipline concerning the transformation of thermal energy into its other forms. It is based on three laws, that were obtained thanks to a combination of theoretical and experimental research.

3.1. THE FIRST LAW OF THERMODYNAMICS

The formulation of the first law was described in detail by Kvasnica [7]. If the system is free of chemical reactions, the first law can be written in the form:

$$de_{\text{in}} = dq + dw, \quad (17)$$

where e_{in} is the internal energy, q is the heat and w is the work. In other words, the heat given to the system can be transformed into the system's internal energy and work performed by this system. Note that the number of the particles in the system has to be the same throughout the process. It must also be noted that the heat, as well as the work, are not total differentials, and therefore they are dependent on the integral path between the starting point and the ending state.

It is useful to write down another formulation of the first law, where enthalpy is defined as:

$$dh = de_{\text{in}} + pdv + vdp, \quad (18)$$

where p is the thermodynamic pressure and v is the volume.

3.2. THE SECOND LAW OF THERMODYNAMICS

The formulation of the second law was motivated by the research of R. Clausius, who followed the work of Carnot and described the real processes in nature by the quantity named entropy. This quantity is defined as:

$$ds = \frac{dq}{T}, \quad (19)$$

where T is the thermodynamic temperature.

This definition is valid for reversible processes. Entropy is then a total differential, therefore, for the reversible process between states 1 and 2, the following equation holds:

$$s_2 - s_1 = \int_{(1)}^{(2)} \frac{dq}{T}. \quad (20)$$

In the case of the reversible cycle, the integral on the R.H.S. of this equation is equal to zero. Note that at this moment, the integral constant has to be determined. This can be done thanks to the Nernst theorem:

$$\lim_{T \rightarrow 0} \Delta s = 0. \quad (21)$$

The real processes are, however irreversible and therefore the Equations (19) and (20) are actually inequalities:

$$ds \geq \frac{dq}{T}. \quad (22)$$

Now, entropy increase can be easily obtained (e.g. Bejan [8])

$$s_{\text{gen}} = s_2 - s_1 - \int_{(1)}^{(2)} \frac{dq}{T}. \quad (23)$$

4. BALANCES LAWS

If some quantity has to be conserved in time, then its time derivative has to be equal to zero², see e.g. Maršík [9]:

$$\frac{D\psi}{dt} = \mathcal{J}(\psi) + \mathcal{P}(\psi) = 0, \quad (24)$$

where $\mathcal{J}(\psi)$ is the flux of quantity ψ through a boundary of the control volume and $\mathcal{P}(\psi)$ is its production within the control volume.

4.1. CONTINUITY EQUATION

Creation or destruction of mass cannot occur in the control volume, this means that the production term in Equation (24) has to be equal to zero, therefore:

$$\frac{D}{dt} \int_{\mathcal{V}} dm = \frac{D}{dt} \int_{\mathcal{V}} \varrho d\mathcal{V} = 0, \quad (25)$$

where ϱ is the fluid density and \mathcal{V} is the control volume. Changing the sequence of the derivatives and integration (thanks to the Leibniz rule for integration), the following equation is obtained:

$$\int_{\mathcal{V}} \frac{D\varrho}{dt} d\mathcal{V} = 0. \quad (26)$$

This relationship has to hold in any control volume $d\mathcal{V} \neq 0$, therefore:

$$\frac{D\varrho}{dt} = \partial_t \varrho + u^i \varrho_{;i} + \varrho u^i_{;i} = 0, \quad (27)$$

where u^i are contravariant components of the velocity vector.

4.2. MOMENTUM CONSERVATION LAW

The momentum conservation law describes in Newtonian mechanics that the time derivative of the momentum has to be equal to all of the forces acting on the control volume:

$$\frac{D\mathcal{M}^i}{dt} = \mathcal{F}^i + \mathcal{S}^i, \quad (28)$$

where \mathcal{F}^i are components of the volume forces and \mathcal{S}^i are components of the surface forces, respectively, that are defined as:

²Note about the notations of derivatives:

in literature, there usually is a difference in the notation of derivative. In Lagrangian description material derivative of quantity, ψ is $d\psi/dt = \partial\psi/\partial t$, in Euler description, the material derivative is given as $D\psi/Dt = \partial\psi/\partial t + u^j \partial\psi/\partial x_j$. Here, the material derivative in the sense of Euler description will be noted as $D\psi/dt$ to emphasise the fact that the derivations are performed in general curvilinear coordinates, and therefore the convective term in the material derivative is covariant derivative, that contains Christoffel symbols of the second kind.

$$\mathcal{M}^i = \int_{\mathcal{V}} \varrho u^i d\mathcal{V}, \quad (29)$$

$$\mathcal{F}^i = \int_{\mathcal{V}} \varrho a^i d\mathcal{V}, \quad (30)$$

$$\mathcal{S}^i = \oint_{\mathcal{A}} \sigma^{ij} d\mathcal{A}_j, \quad (31)$$

where a^i are the components of the acceleration vector and σ^{ij} are the components of the stress tensor.

If the control volume bonded by the surface \mathcal{A} is free of discontinuity or singular points, then integral (28) can be transformed by the Gauss divergence theorem into:

$$\oint_{\mathcal{A}} \sigma^{ij} d\mathcal{A}_j = \int_{\mathcal{V}} \sigma^{ij}_{;j} d\mathcal{V}. \quad (32)$$

Substituting Equations (29) – (32) into Equation (28), the balance of momentum is given by:

$$\frac{D}{dt} \int_{\mathcal{V}} \varrho u^i d\mathcal{V} = \int_{\mathcal{V}} \varrho a^i d\mathcal{V} + \int_{\mathcal{V}} \sigma^{ij}_{;j} d\mathcal{V}. \quad (33)$$

Another change in the sequence of derivation and integration transforms this relationship into the form:

$$\int_{\mathcal{V}} \left[\varrho a^i + \sigma^{ij}_{;j} - \frac{D(\varrho u^i)}{dt} \right] d\mathcal{V}. \quad (34)$$

This again has to be true in any control volume $d\mathcal{V} \neq 0$, therefore:

$$\frac{D(\varrho u^i)}{dt} = \varrho (\partial_t u^i + u^j u^i_{;j}) = \varrho a^i + \sigma^{ij}_{;j}, \quad (35)$$

no special assumptions about fluid flow were made to this point.

The stress tensor can be decomposed into two parts:

$$\sigma^{ij} = -g^{ij} p + \tau^{ij}, \quad (36)$$

where p is the thermodynamic pressure, g^{ij} is the inverse of the metric tensor, and τ^{ij} are the components of the viscous tensor. The negative sign in the thermodynamic pressure means that the force caused by this pressure acts in the opposite direction of the outer normal. If the fluid is at rest, a viscous tensor is zero and the thermodynamics pressure is equal to the hydrostatic pressure. At last, the momentum balance can be written in the form of:

$$\varrho (\partial_t u^i + u^j u^i_{;j}) = \varrho a^i - g^{ij} p_{;j} + \tau^{ij}_{;j}. \quad (37)$$

4.3. ENERGY CONSERVATION LAW

Energy cannot be created nor destroyed by any known mechanism, the law of balance of energy tells us, that the energy is transforming between its different forms. The change of total energy of the flowing fluid is given by:

$$\frac{D}{dt} \int_{\mathcal{V}} \varrho e_{\text{tot}} d\mathcal{V} = \frac{D}{dt} \int_{\mathcal{V}} \varrho (e_{\text{in}} + e_{\text{kin}}) d\mathcal{V} = \quad (38)$$

$$= - \oint_a g^{jl} q_l dA_j + \oint_a g_{il} u^l \sigma^{ij} dA_j + \int_{\mathcal{V}} \varrho g_{ij} u^i a^j d\mathcal{V},$$

where total energy e_{tot} is a sum of internal energy e_{in} and kinetic energy e_{kin} . Next, q_l is the covector of the heat flux and g_{il} is the metric tensor. On the L.H.S. of the Equation (38), the sequence of the derivation and integration will be switched and on the R.H.S. of the Equation (38), the Gauss divergence theorem will be used to convert the surface integrals into volume integrals, resulting into:

$$\int_{\mathcal{V}} \frac{D}{dt} [\varrho (e_{\text{in}} + e_{\text{kin}})] d\mathcal{V} = \quad (39)$$

$$= \int_{\mathcal{V}} \left[- (g^{il} q_l)_{;i} + (g_{jl} u^l \sigma^{ij})_{;i} + \varrho g_{ij} u^i a^j \right] d\mathcal{V}.$$

Again, this has to hold in any control volume, therefore:

$$\frac{D}{dt} [\varrho (e_{\text{in}} + e_{\text{kin}})] = - (g^{il} q_l)_{;i} + (g_{jl} u^l \sigma^{ij})_{;i} + \varrho g_{ij} u^i a^j. \quad (40)$$

4.4. ENTROPY BALANCE

Entropy balance can be written in the form of the balance Equation (24) as:

$$\frac{Ds}{dt} = \mathcal{J}(s) + \mathcal{P}(s) = 0, \quad (41)$$

from where entropy production can be obtained as:

$$\mathcal{P}(s) = \frac{Ds}{dt} - \mathcal{J}(s) \geq 0. \quad (42)$$

Entropy contained in the control volume can be calculated as:

$$S = \int_{\mathcal{V}} \varrho s d\mathcal{V}, \quad (43)$$

where s is the specific entropy. Using Clausius-Duhem inequality and by performing the time derivative of Equation (43), the integral value of the entropy production in the control volume can be obtained as:

$$\begin{aligned} \mathcal{P} &= \int_{\mathcal{V}} p d\mathcal{V} = \int_{\mathcal{V}} \varrho \frac{Ds}{dt} d\mathcal{V} + \oint_{\mathcal{A}} \frac{g^{ij} q_j}{T} dA_i = \\ &= \int_{\mathcal{V}} \left[\varrho \frac{Ds}{dt} + \frac{g^{ij} q_{j;i}}{T} \right] d\mathcal{V}, \end{aligned} \quad (44)$$

where the flux through the boundary is given by covector of the flux of heat q_j . Equation (44) has to be true in any control volume, therefore:

$$p = \varrho \frac{Ds}{dt} + \frac{g^{ij} q_{j;i}}{T}, \quad (45)$$

where the continuity equation was taken into account.

5. ENTROPY PRODUCTION IN FLOWING FLUID

Relationships from Section 3 are used for the determination of the entropy production in the flowing fluid. By applying the time derivative on Equations (17)–(19), the following relationships are obtained:

$$\frac{De_{\text{in}}}{dt} = \frac{Dq}{dt} - \frac{p}{\varrho^2} \frac{D\varrho}{dt}, \quad (46)$$

$$\frac{Dh}{dt} = \frac{De_{\text{in}}}{dt} + \frac{1}{\varrho} \frac{Dp}{dt} - \frac{p}{\varrho^2} \frac{D\varrho}{dt}, \quad (47)$$

$$\frac{Ds}{dt} = \frac{1}{T} \frac{Dq}{dt} \quad (48)$$

where, for the work, the following relationship holds:

$$dw = p d \left(\frac{1}{\varrho} \right). \quad (49)$$

Time change of entropy can be obtained after some elementary manipulations as:

$$\varrho T \frac{Ds}{dt} = \varrho \frac{Dh}{dt} - \frac{Dp}{dt}. \quad (50)$$

The kinetic energy of the flowing fluid can be obtained by taking the dot product between the momentum equation and velocity in the form:

$$\varrho g_{il} u^l \frac{Du^i}{dt} = \varrho \frac{De_{\text{kin}}}{dt} = g_{il} u^l \sigma_{;j}^{ij} + \varrho g_{il} u^l a^i. \quad (51)$$

Extracting Equation (51) from Equation (40), the internal energy is obtained:

$$\begin{aligned} \varrho \frac{De_{\text{in}}}{dt} &= - (g^{ij} q_j)_{;i} + (g_{jl} u^l \sigma^{ij})_{;i} - g_{jl} u^l \sigma_{;i}^{ij} = \\ &= g_{jl} u_{;i}^l \sigma^{ij} - (g^{ij} q_j)_{;i}. \end{aligned} \quad (52)$$

Substituting Equation (52) into Equation (47) and subsequently substituting this into Equation (50), the following relation for the time change of entropy is obtained:

$$\varrho T \frac{Ds}{dt} = g_{jl} u_{;i}^l \sigma^{ij} - (g^{ij} q_j)_{;i}. \quad (53)$$

The continuity equation was used as well during the manipulations. Comparing the relationships Equation (45) and Equation (53), the local entropy production can be obtained as:

$$p = g_{jl} u_{;i}^l \sigma^{ij}. \quad (54)$$

This means, that irreversible processes in the fluid flow are connected with the stress tensor.

6. NEWTONIAN FLUID

No special assumption was made to this point about the stress tensor σ^{ij} except that if the fluid is at rest, the stress tensor is equal to the thermodynamic pressure and if this fluid is in the external force field that is acting on that fluid, the thermodynamic pressure has to be equal to the hydrostatic pressure. This transforms the problem of establishing the stress tensor to the problem of determination of the viscous tensor τ^{ij} . Let's suppose, that this viscous tensor is a linear combination of the strain rate tensor $e^{lm} = g^{kl}e_k^m$, where the mixed type of tensor is given by his definition through the covariant derivatives of the velocity field, then in general, the relationship between these two tensors can be written as:

$$\tau^{ij} = ag^{ij}g_{kl}e^{kl} + b \left(g_k^i g_l^j + g_l^i g_k^j \right) e^{kl} + c \left(g_k^i g_l^j + g_l^i g_k^j \right) e_m^k e^{ml}, \quad (55)$$

where the non-linear third term on the R.H.S. of this equation will be neglected in the following text, therefore:

$$\tau^{ij} = ag^{ij}g_{kl}e^{kl} + b \left(g_k^i g_l^j + g_l^i g_k^j \right) e^{kl}. \quad (56)$$

This relationship, however, tells nothing about the properties of the tensor τ^{ij} , meaning mainly about its symmetry. The coefficients are then functions of the invariants of the tensor of deformation, see e.g. [2]. If the fluid is homogeneous and isotropic, and if the viscous tensor is a continuous function of the stress tensor, then:

$$\tau^{ij} = ag^{ij} + \lambda g^{ij}e_k^k + 2\mu e^{ij}. \quad (57)$$

This tensor has to be symmetric due to the law of conservation of moment of momentum. Now strain rate can be substituted into the Equation (57):

$$e_k^k = u_{;k}^k, \quad (58)$$

$$e^{ij} = \frac{1}{2} \left(g^{jk}u_{;k}^i + g^{il}u_{;l}^j \right). \quad (59)$$

The stress tensor can be written in the form:

$$\sigma^{ij} = -g^{ij}p + \lambda g^{ij}u_{;k}^k + \mu g^{jk}u_{;k}^i + \mu g^{il}u_{;l}^j, \quad (60)$$

where λ is the second viscosity and μ is the dynamic viscosity. Substituting Equation (60) into (37), Navier-Stokes equation is obtained:

$$\varrho \left(\partial_t u^i + u^j u_{;j}^i \right) = -g^{ij}p_{;j} + \left(\lambda g^{ij}u_{;k}^k + \mu g^{jk}u_{;k}^i + \mu g^{il}u_{;l}^j \right)_{;j} + \varrho a^i. \quad (61)$$

Coefficients of viscosity are, in general, functions of temperature and pressure, however, in most of the cases, their variations in the flowing fluid are minimal (see e.g. [10]), therefore, they can be considered constant. Finally, the Navier-Stokes equation is obtained in a general curvilinear coordinate system in the form:

$$\varrho \left(\partial_t u^i + u^j u_{;j}^i \right) = -g^{ij}p_{;j} + (\lambda + \mu) g^{ij}u_{;k;j}^k + \mu g^{jk}u_{;k;j}^i + \varrho a^i. \quad (62)$$

6.1. ENTROPY PRODUCTION IN NEWTONIAN FLUID

Entropy production in Newtonian fluid can be easily obtained by substituting the stress tensor into the Equation (54). Prior this step, however, the vector of the heat flux will be obtained from the constitutive relation:

$$q^i = kg^{ij}T_{;j}. \quad (63)$$

Substituting Equations (63) and (60) into (54) gives:

$$\varrho T \frac{Ds}{dt} = -kg^{ij}T_{;ji} + g_i l u_{;j}^l \left(-g^{ij}p_{;j} + \lambda g^{ij}u_{;k}^k + \mu \left[g^{jl}u_{;l}^i + g^{im}u_{;m}^j \right] \right). \quad (64)$$

Using Relationship (7) and after some manipulations, time change of entropy can be determined as:

$$\varrho T \frac{Ds}{dt} = -kg^{ij}T_{;ji} - pu_{;i}^i + \lambda u_{;i}^i u_{;j}^j + 2\mu u_{;j}^i u_{;i}^j. \quad (65)$$

By a comparison of Equations (65) and (45), local entropy production can be obtained as:

$$p = -pu_{;i}^i + \lambda \left(u_{;i}^i \right)^2 + 2\mu u_{;j}^i u_{;i}^j. \quad (66)$$

It can be seen that the entropy production in the flowing fluid will be dependent only on the stress tensor. Moreover, if the fluid will be incompressible, the entropy production will depend only on the last term on the R.H.S. of the Equation (66), as the other terms will be zero due to the continuity equation, where the divergence of the velocity vector will be zero.

7. APPLICATION TO LINEAR BLADE CASCADE

Application of the curvilinear coordinates for calculation of the entropy production is demonstrated on a linear blade cascade, which was experimentally investigated by Perdichizzi & Dossena [11]. In this case, the two-dimensional flow is considered to demonstrate the usability of curvilinear coordinates. The blade under investigation is depicted in Figure 1, where the suction side is approximated by the fourth-order polynomial function. The positions of the pressure taps are highlighted by red points.

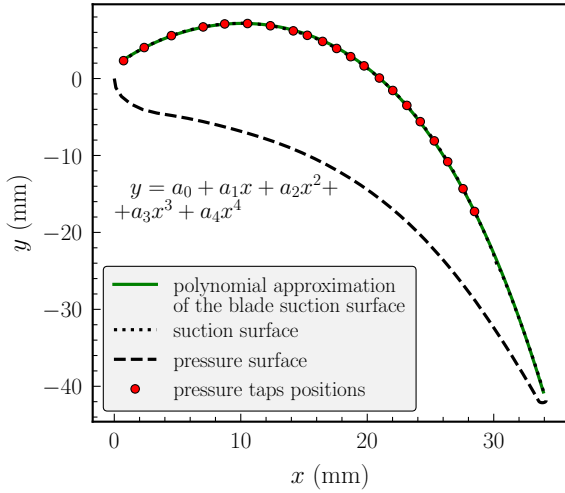


FIGURE 1. Blade under investigation.

The coefficients in the polynomial function have values $a_0 = 1.64$, $a_1 = 1.16$, $a_2 = 7.14 \times 10^{-2}$, $a_3 = 1.56 \times 10^{-3}$ and $a_4 = 4.55 \times 10^{-5}$. The curvilinear coordinate system of the blade can be introduced if the blade geometry is known. This system is shown in Figure 2.

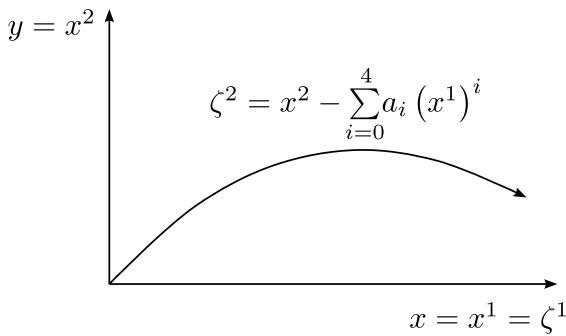


FIGURE 2. Blade curvilinear coordinates.

The relations between the Cartesian and curvilinear coordinates are given by:

$$x^1 = \zeta^1, \quad (67)$$

$$x^2 = \zeta^2 + \sum_{i=1}^4 a_i (\zeta^1)^i, \quad (68)$$

$$\zeta^1 = x^1, \quad (69)$$

$$\zeta^2 = x^2 - \sum_{i=1}^4 a_i (x^1)^i. \quad (70)$$

By introducing substitution:

$$\sigma = \sum_{i=1}^4 a_i (x^1)^i, \quad (71)$$

the metric tensor as well as its inverse can be calculated as:

$$g_{ij} = \begin{pmatrix} 1 & -\sigma \\ -\sigma & 1 + \sigma^2 \end{pmatrix}, \quad (72)$$

$$g^{ij} = \begin{pmatrix} 1 + \sigma^2 & \sigma \\ \sigma & 1 \end{pmatrix}. \quad (73)$$

Christoffel symbols of the second kind can be then calculated using Equation (8). At first, the derivatives of the metric components, with respect to the coordinate x^i , are calculated as:

$$\partial_{x^1} g_{11} = 2\sigma\beta, \quad (74)$$

$$\partial_{x^1} g_{12} = \partial_{x^1} g_{21} = \beta, \quad (75)$$

$$\partial_{x^1} g_{22} = 0, \quad (76)$$

$$\partial_{x^2} g_{11} = 0, \quad (77)$$

$$\partial_{x^2} g_{12} = \partial_{x^2} g_{21} = 0, \quad (78)$$

$$\partial_{x^2} g_{22} = 0, \quad (79)$$

where:

$$\beta = \sum_{i=2}^4 i(i-1)a_i (x^1)^{(i-2)}. \quad (80)$$

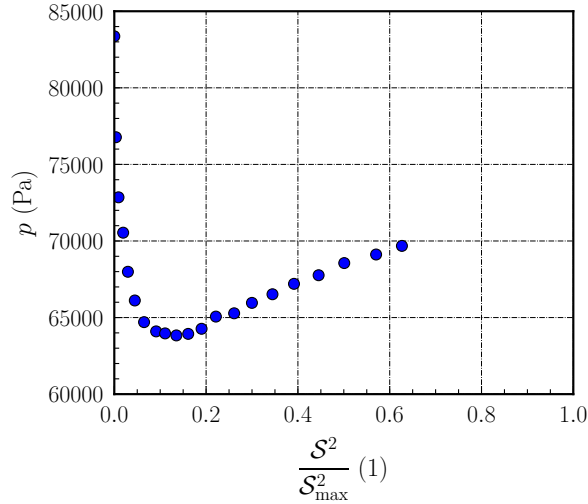
Then, all of the Christoffel symbols are equal to zero except $\Gamma_{11}^2 = \beta$.

Having all of these pieces of information, the calculation of the entropy production can be directly performed. This calculation was performed in the region near the blade surface (out of the boundary layer), where the gradients of the parameters along the coordinate ζ^2 were considered only (in other directions, their values can be neglected). The boundary conditions were not specified in the paper [11], i.e. both the stagnation temperature and the pressure, however, based on the wind tunnel type, their values can be estimated as $T_0 = 297.15$ K and $p_0 = 98000$ Pa.

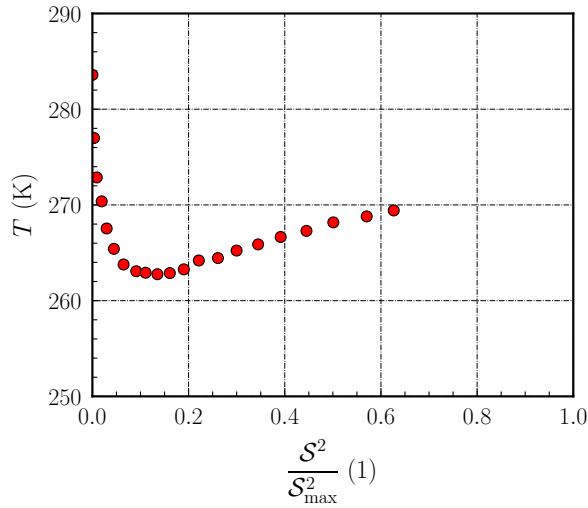
The pressure distribution on the blade surface (denoted by coordinate \mathcal{S}) can be calculated from the Mach number distribution. The pressure, temperature, and velocity distributions are shown in Figure 3.

Distribution of the entropy production $\varrho Ds/dt$ along the blade surface is shown in Figure 4. Note, that the bulk viscosity λ was obtained by the Stokes hypothesis, that $\lambda + 2/3\mu = 0$. It is obvious that most of the entropy production occurred in the leading edge region, where the flow acceleration was the highest, and therefore the velocity gradients were the largest.

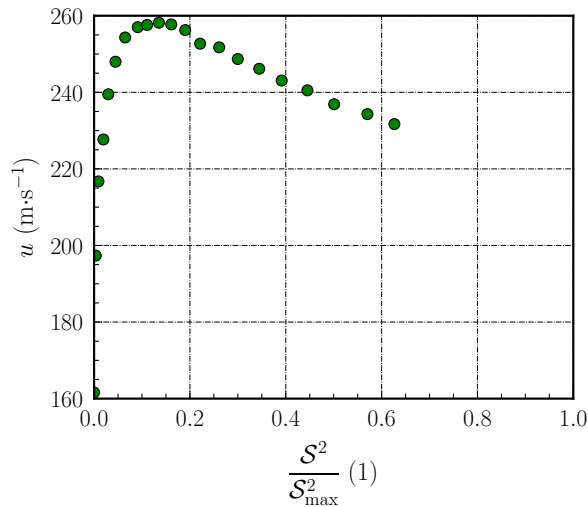
Overall entropy production along the coordinate ζ^2 was calculated as an integral along this curve and was $\mathcal{P} = 0.0327$ J·K⁻¹·m⁻²·s⁻¹.



(A). Pressure distribution



(B). Temperature distribution



(C). Velocity distribution

FIGURE 3. Distribution of the flow parameters along the blade surface.

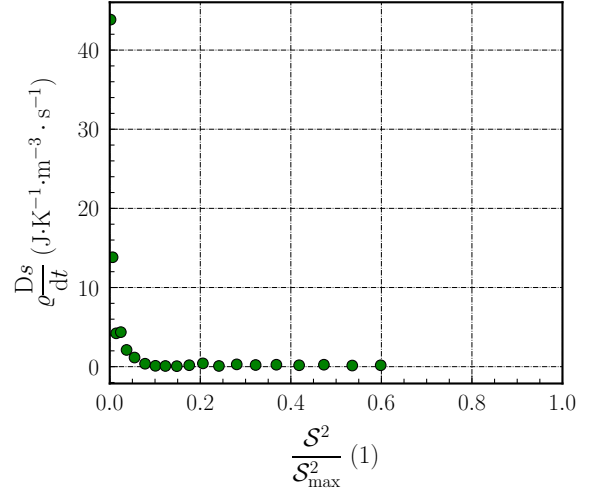


FIGURE 4. Entropy production.

8. CONCLUSION

A derivation of the entropy production in a flowing fluid in a general curvilinear coordinate system was performed in this paper. Basic relationships from the differential geometry used during the derivation were presented as well to clarify used notations. Curvilinear coordinates were chosen to describe this phenomenon in a more general case than is usually used in the literature. The results obtained here were used to describe the flow within the linear blade cascades in a simplified manner and the distribution of the entropy production along one coordinate near the blade suction surface was obtained. The integral value of the entropy production was then evaluated from this distribution.

The aim of this paper was to prove that this approach is capable to evaluate some experimental data along the suction side of the blade. This approach can be used in close proximity to the blade. The polynomial function will be changing with increasing distance from the blade surface, as the opposite surface of the blade will be given by a polynomial with different coefficients. Future work will, therefore, be dealing with the generalisation of this approach to a more complicated situation, where the polynomial coefficients a^i will not be constants.

LIST OF SYMBOLS

- a^i components of acceleration vector [m s^{-2}]
- e strain rate tensor [s^{-1}]
- \mathcal{F} volume force [N]
- $\mathbf{g}^i, \mathbf{g}_i$ basis and co-basis vectors [1]
- g^{ij}, g_{ij} components of metric tensor [1]
- h specific enthalpy [J kg^{-1}]
- $\mathcal{J}(s)$ entropy flux [$\text{J kg}^{-1} \text{K}^{-1} \text{s}^{-1}$]
- m mass [kg]
- \mathcal{M} momentum [kg m s^{-1}]
- p thermodynamic pressure [Pa]
- $\rho(s)$ local entropy production [$\text{J m}^{-2} \text{s}^{-1} \text{K}^{-1}$]

$\mathcal{P}(s)$ integral entropy production [$\text{J m}^{-2} \text{s}^{-1} \text{K}^{-1}$]
 q^i components of heat flux vector [$\text{Jm}^{-2} \text{s}^{-1}$]
 s specific entropy [$\text{J kg}^{-1} \text{K}^{-1}$]
 S blade surface coordinate [m]
 t time [s]
 T thermodynamic temperature [K]
 u^i components of velocity vector [m s^{-1}]
 \mathcal{V} control volume [m^3]
 $x, y, z = x^i$ Cartesian coordinates [m]
 Γ Christoffel symbols [1]
 ζ^i curvilinear blade coordinates [m]
 μ Dynamic viscosity [Pa s]
 λ Bulk viscosity [Pa s]
 ρ density [kg m^{-3}]

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