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LINEARISATION OF A SECOND-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION

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Abstract. We analyse nonlinear second-order differential equations in terms of algebraic properties by reducing a nonlinear partial differential equation to a nonlinear second-order ordinary differential equation via the point symmetry $f(v)\partial_v$. The eight Lie point symmetries obtained for the second-order ordinary differential equation is of maximal number and a representation of the $sl(3, R)$ algebra. We extend this analysis to a more general nonlinear second-order differential equation and we obtain similar interesting algebraic properties.

Keywords: Lie symmetries, integrability, linearisation.

1. Introduction

Nonlinear differential equations are ubiquitous in mathematically orientated scientific fields, such as physics, engineering, epidemiology etc. Therefore, the analysis and closed-form solutions of differential equations are important to understand natural phenomena. In the search for solutions of differential equations, one discovers the beauty of the algebraic properties that the equations possess. Even though closed-form solutions are the primary objective, one cannot ignore the interesting properties of the equations [1–6]. In recent years, one such area in relativistic astrophysics involves the embedding of a four-dimensional differentiable manifold into a higher dimensional Euclidean space which gives rise to the so-called Karmarkar condition for Class I spacetimes [7]. The Karmarkar condition leads to a quadrature, which reduces the problem of determinig the gravitational behaviour of a gravitating system to a single generating function. This is then used to close the system of field equations in order to get a full description of the thermodynamical and gravitational evolution of the model. In a recent approach, Nikolaev and Maharaj [8] investigated the embedding properties of the Vaidya metric [9]. The Vaidya solution is the unique solution of the Einstein field equations describing the exterior spacetime filled with null radition of a spherical mass distribution undergoing dissipative gravitational collapse. In their work, Nikolaev and Maharaj showed that the Vaidya solution is not Class I embeddable but the generalised Vaidya metric describing an anisotropic and inhomogeneous atmosphere comprising of a mixture of strings and null radiation gives rise to interesting embedding properties. Here, we consider the nonlinear partial differential equation arising from the generalised Vaidya metric be of Class I. The governing equation is

$$2r^2mm'' - r^2m'^2 - 2rmm' + 3m^2 = 0, \tag{1}$$

where the prime denotes differentiation of the dependent variable, $m(v, r)$, with respect to the independent variable, r. Equation (1) is not v-dependent explicitly and possesses the point symmetry $f(v)\partial_v$ where $f(v)$ is an arbitrary function of v only. Using this symmetry, we obtain the invariants $r = x$ and $m = y(x)$, which reduces (1) to a nonlinear nonautonomous second-order ordinary differential equation

$$2x^2yy'' - x^2y'^2 - 2xyy' + 3y^2 = 0, \tag{2}$$

where y is a function of x only. We use the Lie symmetry approach to obtain the solution of (2). Using the solution of (2), we obtain the solution of (1).

2. Preliminaries

Let (x, y) denote the variables of a two-dimensional space. Suppose that x is the independent variable and y is the dependent variable. An infinitesimal transformation in this space has the form

$$\bar{x} = x + \epsilon\xi(x, y) \tag{3}$$
$$\bar{y} = y + \epsilon\eta(x, y) \tag{4}$$

which can be regarded as generated by the differential operator

$$\Gamma = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{5}$$

Since we are concerned with point symmetries in this paper, ξ and η depend upon x and y only. Under the infinitesimal transformation (3) and (4), the nth derivative transform is given by

$$\zeta_n = \eta^{(n)} - \sum_{j=1}^{n}\binom{n}{j}y^{(n+1-j)}\xi^{(j)} \tag{6}$$

and

$$\Gamma_n = \zeta_n\frac{\partial}{\partial y^{(n)}}, \tag{7}$$

where the notation $\eta^{(n)}, \xi^{(j)}$ and $y^{(n)}$ denote the nth, jth and nth derivative of the dependent variable with respect to x. In the case of a function, $f(x, y, y', ..., y^{(n)})$, the infinitesimal transformation is generated by $\Gamma + \Gamma_1 + \Gamma_2 + ... + \Gamma_n$ which we write as $\Gamma^{[n]}$, where [10]

$$\Gamma^{[n]} = \Gamma + \sum_{i=1}^{n} \left[\eta^{(i)} - \sum_{j=1}^{i} \binom{i}{j} y^{(i+1-j)} \xi^{(j)} \right] \frac{\partial}{\partial y^{(i)}}, \quad (8)$$

is called the nth extension of Γ.

In the case of an equation

$$E(x, y, y', ..., y^{(n)}) = 0 \quad (9)$$

the equation is a constraint and the condition [11, 12] Γ a symmetry of the equation

$$\Gamma^{[n]} E|_{E=0} = 0, \quad (10)$$

i.e. the action of th nth extension of Γ on the function E is zero when the Equation (9) is taken into account. We note that $E = 0$ may be a scalar equation or a system of equations[1].

3. Symmetry analysis

The Lie point symmetries[2] of (2) are

$$\Gamma_1 = x\frac{\partial}{\partial x}$$
$$\Gamma_2 = y\frac{\partial}{\partial y}$$
$$\Gamma_3 = x^{3/2}\sqrt{y}\frac{\partial}{\partial y}$$
$$\Gamma_4 = \sqrt{xy}\frac{\partial}{\partial y}$$
$$\Gamma_5 = 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$$
$$\Gamma_6 = x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y}$$
$$\Gamma_7 = \sqrt{\frac{y}{x}}\frac{\partial}{\partial x} + \left(\frac{y}{x}\right)^{3/2}\partial_y$$
$$\Gamma_8 = \sqrt{xy}\frac{\partial}{\partial x} + \frac{3y^{3/2}}{\sqrt{x}}\frac{\partial}{\partial y}$$

which is a maximal number for a second-order ordinary differential equation and must be a representation of the $sl(3, R)$ algebra in the Mubarakzyanov Classification Scheme [21–24]. Equation (2) is linearisable to

$$\frac{d^2Y}{dX^2} = 0, \quad (11)$$

by means of a point transformation. The solution of (11) is

$$Y = AX + B, \quad (12)$$

while the solution of (2) is not exactly obvious. However, one can transform (2) to (11). We seek the transformation from (2) to (11) which casts $\Gamma_4 = \sqrt{xy}\partial_y$ into canonical form. Γ_4 assumes canonical form provided

$$\xi(x, y)\frac{\partial X}{\partial x} + \eta(x, y)\frac{\partial X}{\partial y} = 0 \quad (13)$$
$$\xi(x, y)\frac{\partial Y}{\partial x} + \eta(x, y)\frac{\partial Y}{\partial y} = 1, \quad (14)$$

where $\xi = 0$ and $\eta = \sqrt{xy}$ because (2) possesses a symmetry of the general form $\Gamma = \xi\partial_x + \eta\partial_y$.

When we apply the method of characteristics for first-order partial differential equations to (13) and (14), we obtain

$$\frac{dx}{0} = \frac{dy}{\sqrt{xy}} = \frac{dX}{0} \quad (15)$$
$$\frac{dx}{0} = \frac{dy}{\sqrt{xy}} = \frac{dY}{1} \quad (16)$$

for which the solutions are

$$X = x, \qquad Y^2 = \frac{4y}{x}. \quad (17)$$

Under the transformation (17), Equation (2) takes the form in (11). Hence we may apply (17) to (12) to obtain the solution to (2), which is

$$y(x) = \frac{1}{4}x(Ax + B)^2, \quad (18)$$

where A and B are two constants of integration.

By using the invariants $r = x$ and $m = y(x)$, the solution of (1) follows from (18) and is

$$m(v, r) = \frac{1}{4}r(A(v)r + B(v))^2, \quad (19)$$

where $A(v)$ and $B(v)$ are functions of integration.

4. The general case

We consider a general case by setting $y(x) = u^n$, where u is a function of x in Equation (2), we obtain a more general second-order equation

$$2nx^2uu'' + n(n-2)x^2u'^2 - 2nxuu' + 3u^2 = 0. \quad (20)$$

[1] An interested reader is referred to [13–16].

[2] The Mathematica add-on package SYM [17–20] was used to obtain the symmetries.

The Lie point symmetries of (20) are

$$\Lambda_1 = x\frac{\partial}{\partial x}$$
$$\Lambda_2 = \frac{\partial}{\partial x} + \frac{u}{nx}\frac{\partial}{\partial u}$$
$$\Lambda_3 = \frac{1}{n}x^{3/2}u^{1-n/2}\frac{\partial}{\partial u}$$
$$\Lambda_4 = \frac{1}{n}\sqrt{x}u^{1-n/2}\frac{\partial}{\partial u}$$
$$\Lambda_5 = 2x\frac{\partial}{\partial x} + \frac{3}{n}u\frac{\partial}{\partial u}$$
$$\Lambda_6 = x^2\frac{\partial}{\partial x} + \frac{3}{n}xu\frac{\partial}{\partial u}$$
$$\Lambda_7 = \sqrt{\frac{u^n}{x}}\frac{\partial}{\partial x} + \frac{u^{1+n/2}}{nx^{3/2}}\frac{\partial}{\partial u}$$
$$\Lambda_8 = \sqrt{xu^n}\frac{\partial}{\partial x} + \frac{3}{n}\sqrt{\frac{u^{n+2}}{x}}\frac{\partial}{\partial u}.$$

As (20) is a second-order ordinary differential equation and possesses eight Lie point symmetries, it is related to the generic second-order equation [25]

$$\frac{d^2Y}{dX^2} = 0. \tag{21}$$

When we apply the method of characteristics for first-order partial differential equations to (13) and (14), and using symmetry Λ_4, we obtain

$$\frac{dx}{0} = \frac{du}{\frac{1}{n}\sqrt{x}u^{1-n/2}} = \frac{dX}{0} \tag{22}$$
$$\frac{dx}{0} = \frac{du}{\frac{1}{n}\sqrt{x}u^{1-n/2}} = \frac{dY}{1} \tag{23}$$

for which the solutions are

$$X = x, \qquad Y^2 = \frac{4u^n}{x}. \tag{24}$$

From the solution of (21), by means of the transformation (24), we obtain the solution of (20) as

$$u(x) = \left(\frac{x}{4}\right)^{\frac{1}{n}} (C_1x + C_2)^{\frac{2}{n}}, \tag{25}$$

where C_1 and C_2 are constants of integration.

5. Conclusion

Most studies of the algebraic properties of ordinary differential equations are focused on the first, second and third order equations, which is most natural since these are the equations which arise in the modelling of natural phenomena. In this paper, we performed the symmetry analysis of Equation (2) and showed that the equation possesses the $sl(3, R)$ algebra. In turn, we reported the solution of (2) and thus obtained the solution of (1). A natural generalisation of (2) followed. By setting $m(v, r) = z^n$, where z is a function of v and r in Equation (1), we obtain a more general partial differential differential

$$2nr^2zz'' + n(n-2)r^2z'^2 - 2nrzz' + 3z^2 = 0, \tag{26}$$

where the prime denotes differentiation of the dependent variable, $z(v, r)$, with respect to the independent variable, r. We note that, as in Equation (1), (26) is not explicitly dependent on v, and therefore possesses the point symmetry $g(v)\partial_v$, where $g(v)$ is an arbitrary function of v only. We use this symmetry to obtain the invariants $r = x$ and $z = u(x)$ which reduce (26) to the second-order nonlinear Equation (20) with the solution given by (25). Using (25) and the invariants mentioned above, we obtain the solution for Equation (26) to be

$$z(v, r) = \left(\frac{r}{4}\right)^{\frac{1}{n}} (C_1(v)r + C_2(v))^{\frac{2}{n}}, \tag{27}$$

where $C_1(v)$ and $C_2(v)$ are functions of integration.

This paper demonstrates that the Equations (1), hence (26), which, at first glance, looks complicated, has some very interesting properties from the viewpoint of Symmetry analysis. Using the symmetry approach we were able to show that these equations are integrable and have closed-form solutions.

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