Acta Polytechnica **62**(1):211–221, 2022
https://doi.org/10.14311/AP.2022.62.0211

TIME-DEPENDENT MASS OSCILLATORS: CONSTANTS OF MOTION AND SEMICLASICAL STATES

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Abstract. This work reports the construction of constants of motion for a family of time-dependent mass oscillators, achieved by implementing the formalism of form-preserving point transformations. The latter allows obtaining a spectral problem for each constant of motion, one of which leads to a non-orthogonal set of eigensolutions that are, in turn, coherent states. That is, eigensolutions whose wavepacket follows a classical trajectory and saturate, in this case, the Schrödinger-Robertson uncertainty relationship. Results obtained in this form are relatively general, and some particular examples are considered to illustrate the results further. Notably, a regularized Caldirola-Kanai mass term is introduced in an attempt to amend some of the unusual features found in the conventional Caldirola-Kanai case.

Keywords: Time-dependent mass oscillators, Caldirola-Kanai oscillator, quantum invariants, coherent states, semiclassical dynamics.

1. Introduction

The search for exact solutions for time-dependent (nonstationary) quantum models is challenging task as compared to the stationary (time-independent) counterpart. In the stationary case, the dynamical law (Schrödinger equation) reduces to an eigenvalue equation associated with the energy observable, the Hamiltonian, for which several methods can be implemented to obtain exact solutions. Particularly, new exactly solvable models can be constructed from previously known ones through Darboux transformations [1] (also known as SUSY-QM). In the nonstationary case, it is still possible to recover an eigenvalue problem for the Hamiltonian if one restricts to the *adiabatic approximation* [2, 3]. However, in general, the latter is not feasible, and other workarounds should be implemented. Despite all these challenges, time-dependent phenomena find exciting applications in physical systems such as electromagnetic traps of charged particles and plasma physics [4–8].

The parametric oscillator is perhaps the most well-known exactly solvable nonstationary model in quantum mechanics. A straightforward method to solve such a problem was introduced by Lews and Riesenfeld [9] by noticing that the appropriate constant of motion (quantum invariant) admits a nonstationary eigenvalue equation with time-dependent solutions and constant eigenvalues. In this form, nonstationary models can be addressed similarly to their stationary counterparts. This paved the way to solve other time-dependent problems [10–14].

Recently, the Darboux transformation has been adapted into the quantum invariant scheme to construct new time-dependent Hamiltonians, together with the corresponding quantum invariant and the set of solutions [15–17]. Alternatively, other methods exist to build new time-dependent models, such as the modified Darboux transfomation introduced by Bagrov et al. [18], which relies on a differential operator that intertwines a known Schrödinger equation with an unknown one. This has led to new results in the nonstationary Hermitian regime [19–21]. A non-Hermitian PT-symmetric extension has been discussed in [22], and some further models were reported in [23, 24].

On the other hand, the point transformations formalism [25] has been proved useful to construct and solve time-dependent oscillators. This was achieved by implementing a geometrical deformation that transforms the stationary oscillator Schrödinger equation into one with time-dependent frequency and mass [26, 27]. This allows obtaining further information such as the constants of motion, which are preserved throughout the point transformation [25], leading to a straightforward way to get such constants of motion without imposing any ansatz. A further extension for non-Hermitian systems was introduced in [28], whereas a non-Hermitian extension of the generalized Caldirola-Kanai oscillator was discussed in [29].

In this work, the point transformation formalism is exploited to construct and study the dynamics of semiclassical states associated with time-dependent mass oscillators. This is achieved by using the aforementioned preservation of constants of motion and identifying their corresponding spectral problem. Notably, it is shown that one constant of motion leads to an orthogonal set of solutions, whereas a different one leads to nonorthogonal solutions that behave like semiclassical states. That is, Gaussian wavepackets whose maximum point follows the corresponding classical trajectory and minimize, in this case, the

Schrödinger-Robertson uncertainty principle. Two particular examples are considered to illustrate the usefulness of the approach further.

2. Materials and methods

Throughout this manuscript, the time-dependent mass $m(t)$ and frequency $\Omega^2(t)$ oscillator subjected to an external driving force $F(t)$ is considered. Such a model is characterized by the time-dependent Hamiltonian

$$\hat{H}_{\mathrm{ck}}(t) = \frac{\hat{p}^2}{2m(t)} + \frac{m(t)\Omega^2(t)}{2}\hat{x}^2 + F(t)\hat{x}, \quad (1)$$

with $\hat{x}$ and $\hat{p}_x$ the canonical position and momentum operators, respectively, with $[\hat{x}, \hat{p}_x] = i\hbar\mathbb{I}$. Henceforth, the identity operator $\mathbb{I}$ is omitted each time it multiplies a constant or a function. The corresponding Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m(t)}\frac{\partial^2\psi}{\partial x^2} + \frac{m(t)\Omega^2(t)x^2}{2}\psi + F(t)x\psi, \quad (2)$$

is recovered by using the coordinate representation $p_x \equiv -i\hbar\frac{\partial}{\partial x}$ and $\hat{x} \equiv x \in \mathbb{R}$.

The solutions of Eq. (2) have been discussed by several authors, see [27, 30–32]. Here, a brief summary of the point transformation approach discussed in [26, 27] is provided. This eases the discussion of semiclassical states and dynamics to be presented later in Section 3.

2.1. Point transformations

In general, the method of form-preserving point transformations relies on a geometrical deformation that maps an initial differential equation with variable coefficients into another one of the same form but with different coefficients. To illustrate this, be the stationary oscillator Hamiltonian

$$\hat{H}_{\mathrm{osc}} = \frac{\hat{p}_y^2}{2m_0} + \frac{m_0 w_0^2 \hat{y}^2}{2}, \quad [\hat{y}, \hat{p}_y] = i\hbar, \quad (3)$$

with $\hat{y}$ and $\hat{p}_y$ another couple of canonical position and momentum observables, respectively. The corresponding Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial\tau} = -\frac{\hbar^2}{2m_0}\frac{\partial^2\Psi}{\partial y^2} + \frac{m_0 w_0^2 y^2}{2}\Psi, \quad (4)$$

admits the well-known solutions [2]

$$\Psi_n(y,\tau) = e^{-iw_0\left(n+\frac{1}{2}\right)\tau}\Phi_n(y), \quad (5)$$

where

$$\Phi_n(y) = \sqrt{\frac{1}{2^n n!}\sqrt{\frac{m_0 w_0}{\pi\hbar}}}\, e^{-\frac{m_0 w_0}{2\hbar}y^2} H_n\left(\sqrt{\frac{m_0 w_0}{\hbar}}y\right), \quad (6)$$

with $H_n(z)$ the Hermite polynomials [33], fulfills the stationary eigenvalue problem

$$H_{\mathrm{osc}}\Phi_n(y) = E_n^{(\mathrm{osc})}\Phi_n(y), \quad E_n^{(\mathrm{osc})} = \hbar w_0\left(n + \frac{1}{2}\right), \quad (7)$$

with H_{osc} the coordinate representation of $\hat{H}_{\mathrm{osc}}$, i.e., a second-order differential operator that admits a Sturm-Liouville problem.

To implement the point transformation, one imposes a set of relationships between the coordinates, time paramaters, and solutions of both systems in consideration [25]. In general one has

$$y(x,t), \quad \tau(x,t), \quad \Psi(y(x,t),\tau(x,t)) \equiv G(x,t;\psi), \quad (8)$$

where $G(x,t;\psi)$ is a reparametrization of Ψ as an explicit function of x, t, and ψ.

In the case under consideration, some further conditions are required to preserve the linearity and the Hermiticity of $\hat{H}_{\mathrm{osc}}$ and $\hat{H}_{\mathrm{ck}}(t)$. A detailed discussion on the matter can be found in [27]. Here, the final form of the point transformation is used, leading to

$$y(x,t) = \frac{\mu(t)x + \gamma(t)}{\sigma(t)}, \quad \tau(t) = \int^t \frac{dt'}{\sigma^2(t')}, \quad (9)$$

and

$$\Psi(y(x,t),\tau(t)) \equiv G(x,t;\psi) = A(x,t)\psi(x,t), \quad (10)$$

with $m(t) = \mu^2(t)$, together with $\sigma(t)$ and $\gamma(t)$ some real-valued functions to be determined.

By substituting (9) into the Schrödinger equation (4), and after some calculations, one arrives to a new partial differential equation for $\psi(x,t)$ that takes the exact form in (2). The latter allows obtaining

$$\begin{aligned} A(x,t) &= \sqrt{\frac{\sigma}{\mu}}\exp\mathcal{A}(x,t), \\ \mathcal{A}(x,t) &:= \left(i\frac{m_0 w_0}{\hbar}\frac{\mu}{\sigma}\left(\frac{W_\mu}{2}x^2 + W_\gamma x\right) + i\eta\right), \end{aligned} \quad (11)$$

where $\mathcal{A}(x,t)$ is a local time-dependent complex-phase and [27]

$$\begin{aligned} \eta(t) &:= \frac{m_0}{2\hbar}\frac{\gamma(t)W_\gamma(t)}{\sigma(t)} - \frac{1}{2\hbar}\int^t dt'\frac{F(t')}{\mu(t')}, \\ W_\mu(t) &= \sigma(t)\dot{\mu}(t) - \dot{\sigma}(t)\mu(t), \\ W_\gamma(t) &= \sigma(t)\dot{\gamma}(t) - \dot{\sigma}(t)\gamma(t), \end{aligned} \quad (12)$$

with $\dot{f}(t) \equiv \frac{df(t)}{dt}$ a short-hand notation for the time derivative. In the latter, $\sigma(t)$ and $\gamma(t)$ fulfill the nonlinear Ermakov equation

$$\ddot{\sigma}(t) + \left(\Omega^2(t) - \frac{\ddot{\mu}(t)}{\mu(t)}\right)\sigma(t) = \frac{w_0^2}{\sigma^3(t)}, \quad (13)$$

and non-homogeneous equation

$$\ddot{\gamma}(t) + \left(\Omega^2(t) - \frac{\ddot{\mu}(t)}{\mu(t)}\right)\gamma(t) = \frac{F(t)}{m_0\mu(t)}, \quad (14)$$

The solutions of the Ermakov equation are well-known [34–36] and computed from two linearly independent solutions of the associated linear equation

$$\ddot{q}_j(t) + \left(\Omega^2(t) - \frac{\ddot{\mu}(t)}{\mu(t)}\right)q_j(t) = 0, \quad j = 1,2, \quad (15)$$

through the nonlinear combination

$$\sigma(t) = \left[aq_1^2(t) + bq_1(t)q_2(t) + cq_2^2(t)\right]^{\frac{1}{2}}, \qquad (16)$$

with $b^2 - 4ac = -4\frac{w_0^2}{\mathcal{W}_0^2}$ and $\mathcal{W}_0 = Wr(q_1(t), q_2(t)) \neq 0$ the Wronskian of two linearly independent solutions of (15), which is in general a time-independent complex constant. The previous constraint on a, b, and c guarantees that $\sigma(t)$ is different from zero [26] for $t \in \mathbb{R}$.

In this form, one obtains a set of solutions $\{\psi_n(x,t)\}_{n=0}^{\infty}$ to the Schrödinger equation (2), where

$$\psi_n(x,t) = \sqrt{\frac{\mu(t)}{\sigma(t)}}[\mathcal{A}(x,t)]^{-1} e^{-iw_0\left(n+\frac{1}{2}\right)\tau(t)} \times e^{-\frac{m_0 w_0}{2\hbar}\left(\frac{\mu(t)x+\gamma(t)}{\sigma(t)}\right)^2} H_n\left(\sqrt{\frac{m_0 w_0}{\hbar}}\frac{\mu(t)x+\gamma(t)}{\sigma(t)}\right). \qquad (17)$$

From (10)-(11) it follows that

$$(\psi_m, \psi_n) := \int_{\mathbb{R}} dx\, \psi_m^*(x,t)\psi_n(x,t) = \int_{\mathbb{R}} dy \Psi_m^*(y,\tau)\Psi_n(y,\tau) = \delta_{n,m}, \qquad (18)$$

with z^* the complex conjugate of z. That is, the inner product is preserved and thus the set $\{\psi_n(x,t)\}_{n=0}^{\infty}$ is orthonormal in $L^2(\mathbb{R}, dx)$.

The expressions presented so far are general, and specific result may be obtained once the time-dependent mass and frequency terms are specified. This is discussed in the following sections.

Before concluding, an explicit expression for $\tau(t)$ can be determined in terms of the two linearly independent solutions $q_1(t)$ and $q_2(t)$ as well. One gets

$$\tau(t) = w_0^{-1} \arctan\left[\frac{\mathcal{W}_0}{2w_0}\left(b + 2c\frac{q_2(t)}{q_1(t)}\right)\right]. \qquad (19)$$

3. Results: Constants of motion and semiclassical states

Additional information can be extracted from the stationary oscillator into the time-dependent model. Particularly, point transformations preserve *first-integrals* of the initial equation [25]. In the context of the Schrödinger equation, such first-integrals correspond to constants of motion, also known as *quantum invariant*, associated with the physical models under consideration. From the stationary oscillator, it is straightforward to realize that the Hamiltonian $\hat{H}_{\text{osc}}$ is a constant of motion that characterize the energy observable. In the time-dependent case, $\hat{H}_{\text{ck}}(t)$ is no longer a constant of motion, as $\frac{d\hat{H}_{\text{ck}}(t)}{dt} \neq 0$. This implies that an eigenvalue problem associated with $\hat{H}_{\text{ck}}$ is not possible[1].

[1] One can still link an eigenvalue problem with $\hat{H}_{\text{ck}}(t)$ under the adiabatic approximation [3]. This work focuses on exact solutions and such an approach will be disregarded.

On the other hand, an orthonormal set of solutions $\{\psi_n(x,t)\}_{n=0}^{\infty}$ has been already identified, and it is still unclear the eigenvalue problem that such a set solves. This problem was addressed by Lewis-Riesenfeld [9] while solving the dynamics of the *parametric oscillator*. They notice that even in the time-dependent regime, there may be a constant of motion $\hat{I}_0(t)$ that admits a spectral problem

$$\hat{I}_0(t)\phi(x,t) = \lambda\phi(x,t), \qquad (20)$$

where the eigenvalues λ are time-independent. The existence and uniqueness of such a quantum invariant is not necessarily ensured. Still, for the parametric oscillator, Lewis and Riesenfeld managed to find the quantum invariant and solve the related spectral problem.

Here, some quantum invariants associated with $\hat{H}_{\text{ck}}$ can be found through point transformations. First, notice that the point transformation was implemented in the Schrödinger equation to get the time-dependent counterpart. The same transformation can be applied to a constant of motion of the harmonic oscillator to get the corresponding one on the time-dependent model. Particularly, by consider the eigenvalue problem (7), and after some calculations, one gets a first quantum invariant of the form

$$\begin{aligned}\hat{I}_1(t) := {} & \frac{\sigma^2(t)}{2m_0\mu^2(t)}\hat{p}_x^2 + \frac{m_0}{2}\left(W_\mu^2(t) + w_0^2\frac{\mu^2(t)}{\sigma^2(t)}\right)\hat{x}^2 \\ & + \frac{\sigma W_\mu(t)}{2\mu(t)}(\hat{x}\hat{p}_x + \hat{p}_x\hat{x}) + \frac{\sigma W_\gamma(t)}{\mu(t)}\hat{p}_x \\ & + m_0\left(W_\gamma(t)W_\mu(t) + w_0^2\frac{\mu(t)\gamma(t)}{\sigma^2(t)}\right)\hat{x} \\ & + \left(\frac{m_0}{2}W_\gamma^2(t) + \frac{\gamma^2(t)}{\sigma^2(t)}\right). \end{aligned} \qquad (21)$$

It is straightforward to show that $\hat{I}_1(t)$ is indeed a quantum invariant,

$$\frac{i}{\hbar}[\hat{H}_{\text{ck}}, \hat{I}_1(t)] + \frac{\partial \hat{I}_1(t)}{\partial t} = 0. \qquad (22)$$

Moreover, $I_1(t)$, the coordinate representation of $\hat{I}_1(t)$, defines a Sturm-Liouville problem with time-dependent coefficients,

$$I_1(t)\psi_n(x,t) = \hbar w_0\left(n + \frac{1}{2}\right)\psi_n(x,t), \qquad (23)$$

which justifies the existence of the orthogonal set of solutions found in Section 2. Note that orthogonality has been alternatively proved in (18) using the preservation of the inner product.

Remarkably, there are still more quantum invariants to be exploited. To see this, let us consider the operators

$$\hat{a} = \sqrt{\frac{m_0 w_0}{2\hbar}}\hat{y} + i\frac{\hat{p}_y}{\sqrt{2m_0\hbar w_0}}, \quad \hat{a}^\dagger = \sqrt{\frac{m_0 w_0}{2\hbar}}\hat{y} - i\frac{\hat{p}_y}{\sqrt{2m_0\hbar w_0}}, \qquad (24)$$

which factorize the stationary oscillator Hamiltonian as $\hat{H}_{\mathrm{osc}} = \hbar w_0(\hat{a}^\dagger\hat{a} + \frac{1}{2})$ and fulfill the commutation relationship $[\hat{a}, \hat{a}^\dagger] = 1$. Although $\hat{a}$ and $\hat{a}^\dagger$ are not constants of motion of $\hat{H}_{\mathrm{osc}}$, one can introduce a new pair of operators

$$\hat{\mathfrak{a}} := e^{iw_0\tau}\hat{a}, \quad \hat{\mathfrak{a}}^\dagger := e^{-iw_0\tau}\hat{a}^\dagger, \tag{25}$$

where the straightforward calculations show that $\frac{i}{\hbar}[\hat{H}_{\mathrm{osc}}, \hat{\mathfrak{a}}] + \frac{\partial\hat{\mathfrak{a}}}{\partial\tau} = 0$, and similarly for $\hat{\mathfrak{a}}^\dagger$. That is, $\mathfrak{a}$ and $\mathfrak{a}^\dagger$ are quantum invariants of $\hat{H}_{\mathrm{osc}}$.

The latter can now be mapped into the time-dependent model, leading straightforwardly to new quantum invariants of $\hat{H}_{\mathrm{ck}}(t)$ of the form

$$\begin{aligned}\hat{I}_{\mathfrak{a}}(t) = e^{iw_0\tau(t)}\Bigg[&\frac{i}{\sqrt{2m_0\hbar w_0}}\frac{\sigma(t)}{\mu(t)}\hat{p}_x \\ &+\left(\sqrt{\frac{m_0w_0}{2\hbar}}\frac{\mu(t)}{\sigma(t)} + i\sqrt{\frac{m_0}{2\hbar w_0}}W_\mu(t)\right)\hat{x} \\ &+\left(\sqrt{\frac{m_0w_0}{2\hbar}}\frac{\gamma(t)}{\sigma(t)} + i\sqrt{\frac{m_0}{2\hbar w_0}}W_\gamma(t)\right)\Bigg],\end{aligned} \tag{26}$$

and its adjoint $\hat{I}_{\mathfrak{a}}^\dagger(t)$.

Before proceeding, it is worth to recalling that two arbitrary quantum invariants $\hat{I}(t)$ and $\widetilde{\widetilde{I}}(t)$ of a given Hamiltonian $\hat{H}(t)$ can be used to construct further invariants. This follows from the fact that the linear combination $\ell\hat{I}(t) + \widetilde{\ell}\widetilde{\widetilde{I}}(t)$ and the product $\overline{\ell}\hat{I}(t)\widetilde{\widetilde{I}}(t)$ of quantum invariants are also quantum invariants of the same Hamiltonian $\hat{H}(t)$, for ℓ, $\widetilde{\ell}$, and $\overline{\ell}$ time-independent coefficients.

In this form, $\hat{I}_{\mathfrak{a}}(t)$ and $\hat{I}_{\mathfrak{a}}^\dagger(t)$ generate $\hat{I}_1(t)$ through

$$\hat{I}_1(t) = \hbar w_0\left(\hat{I}_{\mathfrak{a}}^\dagger(t)\hat{I}_{\mathfrak{a}}(t) + \frac{1}{2}\right), \tag{27}$$

which is analogous to the factorization of the stationary oscillator. Similarly, the commutation relationship $[\hat{a}, \hat{a}^\dagger] = 1$ of the stationary oscillator is preserved. One thus get $[\hat{I}_{\mathfrak{a}}(t), \hat{I}_{\mathfrak{a}}^\dagger(t)] = 1$ together with

$$[\hat{I}_1(t), \hat{I}_{\mathfrak{a}}(t)] = -\hbar w_0\hat{I}_{\mathfrak{a}}(t), \; [\hat{I}_1(t), \hat{I}_{\mathfrak{a}}^\dagger(t)] = \hbar w_0\hat{I}_{\mathfrak{a}}^\dagger(t), \tag{28}$$

which means that $\hat{I}_{\mathfrak{a}}(t)$ and $\hat{I}_{\mathfrak{a}}^\dagger(t)$ are annihilation and creation operators, respectively, for the eigensolutions of $\hat{I}_1(t)$. The latter leads to

$$\begin{aligned}\hat{I}_{\mathfrak{a}}(t)\psi_{n+1}(x,t) &= \sqrt{\hbar w_0(n+1)}\psi_n(x,t), \\ \hat{I}_{\mathfrak{a}}^\dagger(t)\psi_n(x,t) &= \sqrt{\hbar w_0(n+1)}\psi_{n+1}(x,t),\end{aligned} \tag{29}$$

for $n = 0, \ldots$.

On the other hand, the orthonormal set $\{\psi_n(x,t)\}_{n=0}^\infty$ can be used as a basis to expand any arbitrary solution $\psi(x,t)$ of (2) through

$$\psi(x,t) = \sum_{n=0}^\infty \mathcal{C}_n\psi_n(x,t), \quad \mathcal{C}_n := (\psi_n(x,t), \psi(x,t)). \tag{30}$$

Now, from the above results, one may investigate the spectral problem related to the remaining quantum invariants $\hat{I}_{\mathfrak{a}}(t)$ and $\hat{I}_{\mathfrak{a}}^\dagger(t)$. By considering the annihilation operator $\hat{I}_{\mathfrak{a}}(t)$, one obtains the eigenvalue problem

$$\hat{I}_{\mathfrak{a}}(t)\xi_\alpha(x,t) = \alpha\xi_\alpha(x,t), \tag{31}$$

where the eigensolution $\xi_\alpha(x,t)$ can be expanded as the linear combination

$$\xi_\alpha(x,t) = \sum_{n=0}^\infty \widetilde{\mathcal{C}}_n(\alpha)\psi_n(x,t), \quad \alpha \in \mathbb{C}. \tag{32}$$

This corresponds to the construction of *coherent states* using the Barut-Girardelo approach [37]. The complex coefficients $\widetilde{\mathcal{C}}_n(\alpha)$ are determined by using the action of the ladder operators (29) and exploiting the orthonormality of the set $\{\psi_n(x,t)\}_{n=0}^\infty$. After substituting the linear combination $\xi_\alpha(x,t)$ into the corresponding eigenvalue problem in (31), one obtains the one-parameter normalized eigensolutions

$$\xi_\alpha(x,t) = \exp\left(-\frac{|\alpha|^2}{2\hbar w_0}\right)\sum_{n=0}^\infty\left(\frac{\alpha}{\sqrt{\hbar w_0}}\right)^n\frac{\psi_n(x,t)}{\sqrt{n!}}. \tag{33}$$

Henceforth, the latter are called *time-dependent coherent states* or semiclassical states interchangeably.

Similar to Glauber coherent states [38], the eigensolutions of the annihilation operator $\hat{I}_{\mathfrak{a}}$ are not orthogonal among themselves. This follows from the overlap between two solutions with different eigenvalues, let say α and β, leading to

$$|(\xi_\beta, \xi_\alpha)|^2 = \exp\left(-\frac{|\alpha|^2 + |\beta|^2 - 2\,\mathrm{Re}(\alpha^*\beta)}{\hbar w_0}\right), \tag{34}$$

which is different from zero for every $\alpha, \beta \in \mathbb{C}$, with the inner product defined in (18). Interestingly, the eigensolution $\xi_\alpha(x,t)$ can be brought into an alternative and handy expression by using the explicit form of $\psi_n(x,t)$ given in (17), together with the well-known summation rules for the Hermite polynomials. By doing so one gets

$$\begin{aligned}\xi_\alpha(x,t) \equiv &\sqrt{\frac{\mu(t)}{\sigma(t)}\sqrt{\frac{m_0w_0}{\pi\hbar}}}[\mathcal{A}(x,t)]^{-1}e^{-i\frac{w_0\tau(t)}{2}} \\ &\times\exp\left[i\sqrt{\frac{2m_0w_0}{\hbar}}\left(\frac{\mu(t)x+\gamma(t)}{\sigma(t)}\right)\mathrm{Im}\,\widetilde{\alpha}(t)\right] \\ &\times\exp\left[-\frac{m_0w_0}{2\hbar}\left(\frac{\mu(t)x+\gamma(t)}{\sigma(t)} - \sqrt{\frac{2\hbar}{m_0w_0}}\,\mathrm{Re}\,\widetilde{\alpha}(t)\right)^2\right],\end{aligned} \tag{35}$$

with $\widetilde{\alpha}(t) = \alpha e^{-iw_0\tau(t)}$. Thus, $\xi_\alpha(x,t)$ is a normalized Gaussian wavepacket with time-dependent width. The complex constants α plays the role of the initial conditions of the wavepacket at a given time t_0. See the discussion in the following section.

Despite the lack of orthogonality in the elements of the set $\{\xi_\alpha(x,t)\}_{\alpha\in\mathbb{C}}$, they can be still used as a non-orthogonal basis so that any arbitrary solution of (2) can be constructed through the appropriate linear superposition. That is, a given solution $\psi(x,t)$ of (2) expands as

$$\psi(x,t) = \int_{\mathbb{C}}\frac{d^2\alpha}{\pi\hbar}\mathfrak{C}(\alpha)\xi_\alpha(x,t), \tag{36}$$

where $\mathfrak{C}(\alpha)=(\xi_\alpha(x,t),\psi(x,t))$.

So far, the spectral problem related to the quantum invariants $\hat{I}_1(t)$ and $\hat{I}_{\mathfrak{a}}(t)$ has led to a discrete and a continuous representation, respectively, in which any solution of (2) can be expanded.

Although the eigenvalue problem related to the quantum invariant $\hat{I}_{\mathfrak{a}}^\dagger(t)$ can be established, it leads to non-finite norm solutions and is thus discarded.

3.1. Semiclassical dynamics

With the time-dependent coherent states already constructed, one can now study the evolution on time of such state and its relation with physical observables such as position and momentum $\hat{x}$ and $\hat{p}_x$, respectively. To this end, note that the quantum invariants obtained through point transformations preserve the commutation relation (28) of the corresponding operators of the stationary oscillator. That is, the set $\{\hat{I}_{\mathfrak{a}}(t),\hat{I}_{\mathfrak{a}}^\dagger(t),\hat{I}_{\mathfrak{a}}^\dagger(t)\hat{I}_{\mathfrak{a}}(t)\}$ fulfill the Weyl-Heisenberg algebra [39]. This allows the construction of a unitary displacement operator of the form [39]

$$\mathfrak{D}(\alpha;t)=e^{\alpha\hat{I}_{\mathfrak{a}}^\dagger(t)-\alpha^*\hat{I}_{\mathfrak{a}}(t)}=e^{-\frac{|\alpha|}{2}}e^{\alpha\hat{I}_{\mathfrak{a}}^\dagger}e^{-\alpha^*\hat{I}_{\mathfrak{a}}},\quad \alpha\in\mathbb{C},\tag{37}$$

so that

$$\begin{aligned}\mathfrak{D}^\dagger(\alpha,t)\hat{I}_{\mathfrak{a}}(t)\mathfrak{D}(\alpha,t)&=\hat{I}_{\mathfrak{a}}(t)+\alpha,\\ \mathfrak{D}^\dagger(\alpha;t)\hat{I}_{\mathfrak{a}}^\dagger(t)\mathfrak{D}(\alpha;t)&=\hat{I}_{\mathfrak{a}}^\dagger(t)+\alpha^*.\end{aligned}\tag{38}$$

It follows that the action of the first relationship acted on $\psi_0(x,t)$ leads to $\hat{I}_{\mathfrak{a}}(t)\mathfrak{D}(\alpha,t)\psi_0(x,t)=\alpha\mathfrak{D}(\alpha,t)\psi_0(x,t)$, from which one recovers the eigenvalue equation previously analyzed in (31) by identifying $\xi_\alpha(x,t)=\mathfrak{D}(\alpha,t)\psi_0(x,t)$. This corresponds to the coherent states construction of Perelomov [39].

So far, two different and equivalent ways to construct the solutions $\xi_\alpha(x,t)$ have been identified, a property akin to Glauber coherent states. To further explore the time-dependent coherent states, one can take the unitary transformations (38) and combine them with the relationship between the ladder operators and the physical position $\hat{x}$ and momentum $\hat{p}_x$ observables presented in (26). After some calculations one obtains

$$\langle\hat{x}\rangle_\alpha(t)=\sqrt{\frac{2\hbar}{m_0w_0}}\frac{\sigma(t)}{\mu(t)}r\cos\left(w_0\tau(t)-\theta\right)-\frac{\gamma(t)}{\mu(t)},\tag{39}$$

where $\alpha=re^{i\theta}$. By using (19) and some elementary trigonometric identities, one recovers an explicit expression in terms of $q_1(t)$ and $q_2(t)$ as

$$\begin{aligned}\langle\hat{x}\rangle_\alpha(t)=&-\frac{\gamma(t)}{\mu(t)}\\ &+\sqrt{\frac{2\hbar w_0}{m_0c}}\frac{r}{\mathcal{W}_0}\left[\left(\cos\theta+\frac{\mathcal{W}_0}{2w_0}c\sin\theta\right)\frac{q_1(t)}{\mu(t)}\right.\\ &\left.+\frac{\mathcal{W}_0}{w_0}c\sin\theta\frac{q_2(t)}{\mu(t)}\right].\end{aligned}\tag{40}$$

Similarly, the calculations for the momentum observable leads to

$$\begin{aligned}\langle\hat{p}_x\rangle_\alpha(t)=&-m_0\frac{\mu(t)}{\sigma(t)}(W_\mu(t)\langle\hat{x}\rangle_\alpha(t)+W_\gamma(t))\\ &-\sqrt{2m_0\hbar w_0}\frac{\mu(t)}{\sigma(t)}r\sin\left(w_0\tau(t)-\theta\right)).\end{aligned}\tag{41}$$

In the latter, $\langle\hat{\mathcal{O}}\rangle_\alpha(t)\equiv(\xi_\alpha(x,t),\hat{\mathcal{O}}\xi_\alpha(x,t))$ stands for the average value of the observable $\hat{\mathcal{O}}$ computed through the time-dependent coherent state $\xi_\alpha(x,t)$.

The expectation value of the momentum (41) can be further simplified so that it simply rewrites as

$$\langle\hat{p}_x\rangle_\alpha(t)=m(t)\frac{d}{dt}\langle\hat{x}\rangle_\alpha(t),\quad m(t)=m_0\mu^2(t),\tag{42}$$

which is an analogous relation to that obtained from the canonical equations of motion of the corresponding classical Hamiltonian. This is also consequence of the quadratic nature of the time-dependent Hamiltonian $\hat{H}_{\text{ck}}(t)$ and the Ehrenfest theorem.

From the expectation values obtained in (39)-(42), a relationship between the complex parameter $\alpha=re^{i\theta}$ and the expectation values at a given initial time $t=t_0$ can be established. The straightforward calculations lead to

$$\begin{aligned}\begin{pmatrix}\operatorname{Re}\widetilde{\alpha}_{t_0}\\ \operatorname{Im}\widetilde{\alpha}_{t_0}\end{pmatrix}=&\begin{pmatrix}\sqrt{\frac{m_0w_0}{2\hbar}}\frac{\gamma_{t_0}}{\sigma_{t_0}}\\ \sqrt{\frac{m_0}{2\hbar w_0}}W_{\gamma_{t_0}}\end{pmatrix}\\ &+\begin{pmatrix}\sqrt{\frac{m_0w_0}{2\hbar}}\frac{\mu_{t_0}}{\sigma_{t_0}} & 0\\ \sqrt{\frac{m_0}{2\hbar w_0}}W_{\mu_{t_0}} & \frac{1}{\sqrt{2m_0\hbar w_0}}\frac{\sigma_{t_0}}{\mu_{t_0}}\end{pmatrix}\begin{pmatrix}\langle\hat{x}\rangle_{t_0}\\ \langle\hat{p}_x\rangle_{t_0}\end{pmatrix},\end{aligned}\tag{43}$$

with $\widetilde{\alpha}_{t_0}=\alpha e^{-iw_0\tau_{t_0}}=re^{i(\theta-w_0\tau_{t_0})}$, $\tau_{t_0}=\tau(t_0)$, $\sigma_{t_0}=\sigma(t_0)$, $\gamma_{t_0}=\gamma(t_0)$, $W_{\gamma_{t_0}}=W_\gamma(t_0)$, $W_{\mu_{t_0}}=W_\mu(t_0)$, $\langle\hat{x}\rangle_{t_0}=\langle\hat{x}\rangle_\alpha(t_0)$, and $\langle\hat{p}_x\rangle_{t_0}=\langle\hat{p}_x\rangle_\alpha(t_0)$.

On the other hand, one can write the probability density associated with the time-dependent coherent state in terms of $\langle\hat{x}\rangle_\alpha(t)$ through

$$\begin{aligned}\mathcal{P}_\alpha(x,t):=|\xi_\alpha(x,t)|^2=&\sqrt{\frac{m_0w_0}{\pi\hbar}}\frac{\mu(t)}{\sigma(t)}\\ &\times\exp\left[-\frac{m_0w_0}{2\hbar}\frac{\mu^2(t)}{\sigma^2(t)}\left(x-\langle\hat{x}\rangle_\alpha(t)\right)^2\right],\end{aligned}\tag{44}$$

which is a Gaussian wavepacket whose maximum follows the classical trajectory. That is, the time-dependent coherent state is considered as a *semiclassical state.*

Before concluding this section, it is worth exploring the corresponding uncertainty relations associated with the canonical observables, which can be computed by using (26), (31), and (38). After some calculations one gets

$$\begin{aligned}(\Delta\hat{x})^2_\alpha&=\frac{\hbar}{2m_0w_0}\frac{\sigma^2(t)}{\mu^2(t)},\\ (\Delta\hat{p}_x)^2_\alpha&=\frac{m_0\hbar w_0}{2}\frac{\mu^2(t)}{\sigma^2(t)}\left(1+\frac{\sigma^2(t)W_\mu^2(t)}{w_0^2\mu^2(t)}\right),\end{aligned}\tag{45}$$

from which the uncertainty relation reduces to

$$(\Delta\hat{x})^2_\alpha (\Delta\hat{p}_x)^2_\alpha = \frac{\hbar^2}{4}\left(1+\frac{\sigma^2(t)W_\mu^2(t)}{w_0^2\mu^2(t)}\right), \quad (46)$$

where it is clear that, in general, $\xi_\alpha(x,t)$ does not minimize the Heisenberg uncertainty relationship, except for those times t' at which $W_\mu(t')=0$. The latter follows from the fact that $\sigma\neq 0$ for $t\in\mathbb{R}$. Still, there are two special cases for which Eq. (46) minimizes at all times.

- For $\mu(t)=\mu_0$ and $\Omega(t)=w_1$, one can always find a constant solution $\sigma^4(t)=w_0^2/w_1^2$ so that $W_\mu=0$. The uncertainty relationship (46) is minimized, and the time-dependent Hamiltonian becomes

$$\hat{H}_{\text{ck}}(t)=\frac{\hat{p}_x^2}{2m_0\mu_0^2}+\frac{m_0\mu_0^2 w_1^2}{2}\hat{x}^2+F(t)\hat{x}, \quad (47)$$

  which is nothing but a stationary oscillator with an external time-dependent driving force $F(t)$[2]. Thus, the uncertainty relation gets minimized in the stationary limit, as expected.
- For $\Omega^2(t)=w_0^2\mu^{-4}(t)$, there is a solution $\sigma(t)=\mu(t)$ for which $W_\mu=0$. This leads to a Hamiltonian of the form

$$\hat{H}_{\text{ck}}(t)=\frac{1}{\mu^2(t)}\left(\frac{\hat{p}_x^2}{2m_0}+\frac{m_0 w_0^2}{2}\hat{x}^2+\mu^2(t)F(t)\hat{x}\right). \quad (48)$$

Although the solutions $\xi_\alpha(x,t)$ minimize the Heisenberg uncertainty relation only on some restricted cases, one can still explore the *Schrödinger-Robertson inequality* [40, 41]. This is defined for a pair of observables $\hat{A}$ and $\hat{B}$ through

$$\left(\Delta\hat{A}\right)^2\left(\Delta\hat{B}\right)^2 \geq \frac{|\langle[\hat{A},\hat{B}]\rangle|^2}{4}+\sigma_{A,B}^2, \quad (49)$$

where $\sigma_{A,B}:=\frac{1}{2}\langle\hat{A}\hat{B}+\hat{B}\hat{A}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle$ stands for the correlation function.

For the canonical position $\hat{x}$ and momentum $\hat{p}_x$ observables one gets

$$\sigma_{x,p_x}^2=\frac{\hbar^2}{4w_0^2}\frac{\sigma^2(t)W_\mu^2(t)}{\mu^2(t)}, \quad (50)$$

when computed through $\xi_\alpha(x,t)$. Thus, the semiclassical states $\xi_\alpha(x,t)$ minimize the Schrödinger-Robertson relationship for $t\in\mathbb{R}$.

4. Discussion: Conventional and regularized Caldirola-Kanai oscillators

So far, the most general setup has been addressed for a time-dependent mass oscillator. Two particular examples are considered in this section to further illustrate the usefulness and behavior of the so-constructed solutions and coherent states. Henceforth, all calculations are carried on by working in units of $\hbar=1$ to simplify the ongoing discussion. Throughout the rest of this manuscript, the following two time-dependent masses are considered:

$$\mu_{ck}(t)=e^{-\kappa t}, \qquad \kappa\geq 0, \quad (51a)$$

$$\mu_{rck}(t)=e^{-\kappa t}+\mu_0, \qquad \kappa,\mu_0\geq 0. \quad (51b)$$

The first one corresponds to the well-known Caldirola-Kanai oscillator [42, 43], which contains a mass-term that asymptotically approaches to zero. This is a rather unrealistic scenario in the context of the Schrödinger equation. Still, one can study the dynamics on a given time range, let say $t\in[0,T]$, where T denotes the time spent by the mass to reduce its initial value in a factor e^{-1}. In other words, $T=\kappa^{-1}$ is equivalent to the lifetime of a decaying system. One thus may disregard the dynamics for $t>T$. To amend such issue, the second mass term $\mu_{rck}(t)$ has been introduced, which transits from $\mu_{rck}(0)=1$ to $\mu_{ck}(t\to\infty)=\mu_0$. Thus, there is no need to introduce any artificial truncation on the time domain. The Hamiltonian associated with this mass-term will be called *regularized Caldirola-Kanai oscillator*. Despite the apparent advantages of the regularized system, analytic expressions for $\sigma(t)$ are significantly more complicated with respect to those obtained from $\mu_{ck}(t)$. Still, exact result can be obtained. The discussion is thus divided for each case separately.

4.1. Caldirola-Kanai case

The so-called Caldirola-Kanai system is another well-known nonstationary problem, characterized by time-dependent mass decaying exponentially on time. It was independently introduced by Caldirola [42] and Kanai [43] in an attempt to describe the quantum counterpart of a damped oscillator. This model has been addressed by different means, such using a Fourier transform to map the map it into a parametric oscillator [32], and using the *quantum Arnold transformation* [30].

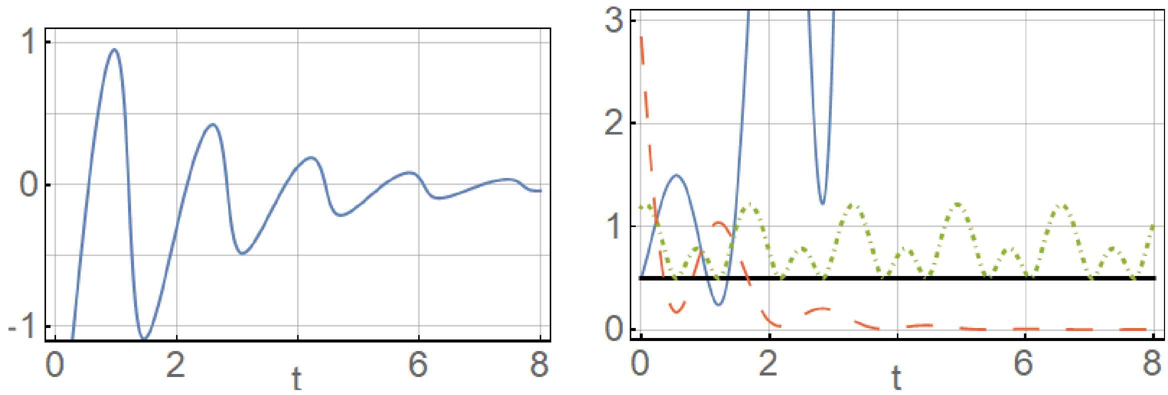
For this particular case, a constant frequency $\omega^2(t)=w_1^2$ and a driven force $F(t)=\mathcal{A}_0\cos(\nu t)$, for $\nu,\mathcal{A}_0\in\mathbb{R}$, are considered. This leads to a forced Caldirola-Kanai oscillator Hamiltonian [10, 44] of the form

$$\hat{H}_{\text{ck}}(t)=\frac{e^{2\kappa t}}{2m_0}\hat{p}_x^2+e^{-2\kappa t}\frac{m_0 w_1^2}{2}\hat{x}^2+\mathcal{A}_0\cos(\nu t)\hat{x}. \quad (52)$$

From the results obtained in previous sections, one gets the solutions to the Ermakov and nonhomogeneous equations as

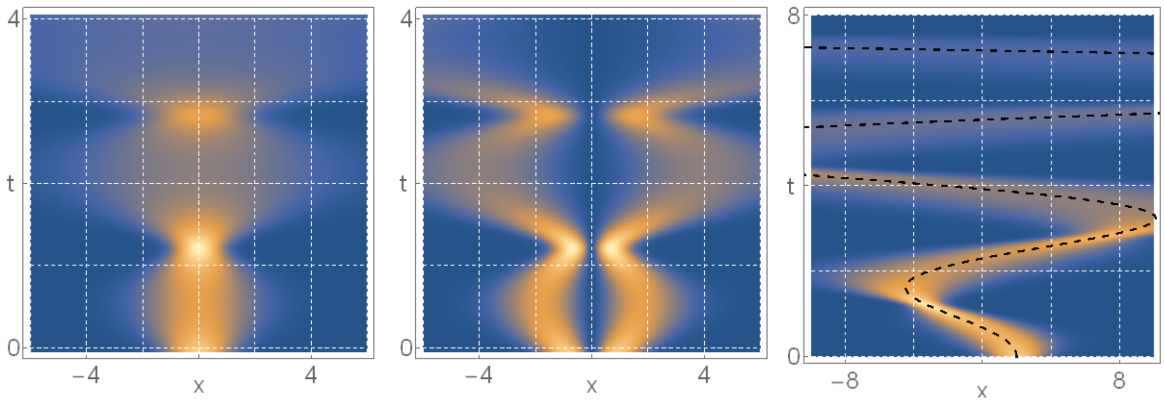
$$\begin{aligned}\sigma(t)&=\left(aq_1^2(t)+bq_1(t)q_2(t)+cq_2^2(t)\right)^{\frac{1}{2}},\\ \gamma(t)&=\gamma_1 q_1(t)+\gamma_2 q_2(t)+\gamma_p(t),\end{aligned} \quad (53)$$

[2] The Hamiltonian (47) is essentially stationary, for the term $F(t)$ can be absorbed through an appropriate reparametrization of the canonical coordinate.



(A) . $W_\mu(t)$ (B).

Figure 1. (A) $W_\mu(t) = \sigma(t)\dot{\mu}(t) - \dot{\sigma}(t)\mu(t)$ for the Caldirola-Kanai mass term $\mu_{ck}(t)$. (B) Variances $(\Delta\hat{x})^2_\alpha$ (solid-blue), $(\Delta\hat{p}_x)^2_\alpha$ (dashed-red), the Schrödinger-Robertson uncertainty minimum (dotted-green), and the Heisenberg uncertainty minimum (thick-solid-black) associated with the coherent states $\xi_\alpha(x,t)$ and the mass term $\mu_{ck}(t)$. The parameters have been fixed as $a = c = w_0 = 1$, $w_1 = 2$, and $\kappa = 0.5$.



(A) . $n = 0$ (B) . $n = 1$ (C).

Figure 2. Probability density $\mathcal{P}_n = |\psi_n(x,t)|^2$ for $n = 0$ (A), $n = 1$ (B), and $\mathcal{P}_\alpha = |\psi_n(x,t)|^2$ (C) associated with the Caldirola-Kanai mass term $\mu_{ck}(t)$. For simplicity, the external force $F(t)$ and $\gamma(t)$ have been fixed to zero. The rest of parameters have been fixed as $w_0 = a = b = 1$, $w_1 = 2$, and $\kappa = 0.5$.

respectively, with γ_1 and γ_2 arbitrary real constants, $b^2 - 4ac = -16\frac{w_0^2}{w_1^2-\kappa^2}$, and

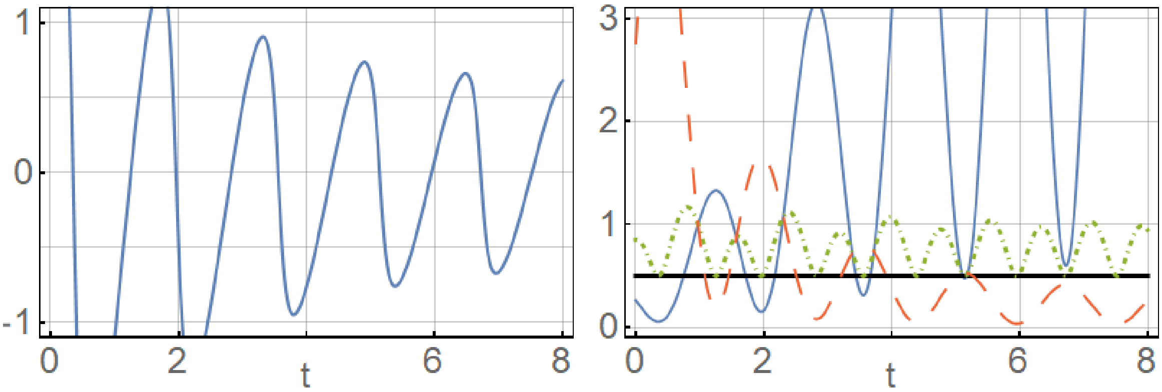
$$q_1(t) = \cos(\sqrt{w^2-\kappa^2}\,t), \quad q_2(t) = \sin(\sqrt{w^2-\kappa^2}\,t),$$
$$\gamma_p(t) = \mathcal{A}_0 e^{-kt}\frac{(w_1^2-\nu^2)\cos(\nu t) - 2\kappa\nu\sin(\nu t)}{(w_1^2+\nu^2)^2 - 4\nu^2(w_1^2-\kappa^2)}. \tag{54}$$

In the sequel, $\kappa = 0.5$ is consider so that the Caldirola-Kanai oscillator is constrained to the time interval $t \in [0, 2]$. Further discussions concerning the dynamics will be restricted to such a time interval.

It is worth recalling that the zeros of $W_\mu(t)$ correspond to the times for which the Heisenberg uncertainty relationship saturates. Although the expression for $W_\mu(t)$ is rather simple in this case, determining the zeros consist of solving a transcendental equation. Thus, to get further insight, one may analyze Figure 1a, which depicts the behavior of such a function for $\mu_{ck}(t)$ (solid-blue). From the latter, one can see that zeroes do exist indeed, and thus one should expect points in time for which the Heisenberg inequality saturates. Despite the latter, the Schrödinger-Robertson inequality saturates at all times.

In Figure 1b, one can see the behavior of the variances, from which it is clear that the variance in the position blows up at time pass by, whereas the momentum variance squeezes indefinitely, approaching asymptotically to zero. This odd behavior results from a mass term that quickly decays, approaching zero but never converging to it. For those reasons, a truncation on the time interval was previously introduced in the form of a mean lifetime, which in this case becomes $T = \kappa^{-1} = 2$. In this form, one still has a realistic behavior for $t \in (0, 2)$.

The previous results can be verified by looking at the probability density associated with the solutions $\psi_n(x,t)$ and the coherent state $\xi_\alpha(x,t)$, which is depicted in Figure 2. From those probability densities, one may see the increase on the position variance $(\Delta\hat{x})^2_\alpha$, for the wavepacket spreads rapidly on time,



(A) . $W_\mu(t)$ (B).

FIGURE 3. (A) $W_\mu(t) = \sigma(t)\dot{\mu}(t) - \dot{\sigma}(t)\mu(t)$ for the regularized Caldirola-Kanai mass term $\mu_{rck}(t)$. (B) Variances $(\Delta\hat{x})^2_\alpha$ (solid-blue), $(\Delta\hat{p}_x)^2_\alpha$ (dashed-red), the Schrödinger-Robertson uncertainty minimum (dotted-green), and the Heisenberg uncertainty minimum (thick-solid-black) associated with the coherent states $\xi_\alpha(x,t)$ and the mass term $\mu_{rck}(t)$. The parameters have been fixed as $a = c = w_0 = 1$, $w_1 = 2$, $\mu_0 = 0.3$, and $\kappa = 0.5$.

to the point that, for times $t > 4$ is almost indistinguishable. For completeness, the classical trajectory is depicted as a dashed-black curve in Figure 2c, where the initial conditions $\langle\hat{x}\rangle_{t_0} = 2$ and $\langle\hat{p}_x\rangle_{t_0} = 0$ have been used.

4.2. Regularized Caldirola-Kanai

In this section, the *regularized Caldirola-Kanai oscillator* is introduced so that it amends the difficulties found in the Caldirola-Kanai for $t >> T$. This model is characterized by a constant frequency $\Omega^2(t) = w_1^2$ and a mass term $\mu_{rck}(t) = \mu_0 e^{-\kappa t} + \mu_1$, with $w_1, \mu_0, \mu_1, \kappa > 0$. The mass term will converge at a constant value (different from zero) and the anomalies found in the conventional Caldirola-Kanai case will be fixed. The main consequence of the mass regularization is that the classical equation of motion is not as trivial as in Section 4. In turn one has

$$\ddot{q}(t) + \left(w_1^2 - \frac{\kappa^2}{1+\mu_0 e^{\kappa t}} \right) q(t) = 0. \qquad (55)$$

Two linearly independent solutions to the corresponding linear equation (15) can be found as

$$\overline{q}_1(t) = z(t)^{i\frac{w_1}{k}} {}_2F_1 \left(\begin{array}{c} A_1, A_2 \\ 1 - 2i\dfrac{w_1}{k} \end{array} \middle| \frac{-1}{z(t)} \right), \qquad (56)$$
$$\overline{q}_2(t) = \overline{q}_1^*(t),$$

where $z(t) = \mu_0 e^{\kappa t}$, $A_1 = -i\frac{w_1}{k} - i\sqrt{\frac{w_1^2}{k^2} - 1}$, and $A_2 = -i\frac{w_1}{k} + i\sqrt{\frac{w_1^2}{k^2} - 1}$. On the other hand, ${}_2F_1(a,b;c;Z)$ stands for the hypergeometric function [33], which converges in the complex unit-disk $|Z| < 1$. Given that $z(t) : \mathbb{R} \to (1,\infty)$, the solutions $\overline{q}_{1,2}(t)$ in (56) converge for $t \in \mathbb{R}$.

Since both solutions in (56) are complex-valued, with $\overline{q}_2 = \overline{q}_1^*$, one can construct a real-valued solution to the Ermakov equation by taking $q_1 = \mathrm{Re}(\overline{q}_1)$ and $q_2 = \mathrm{Im}(\overline{q}_1)$. To simplify the ongoing discussion, the external force is considered null, $F(t) = 0$. One thus obtains

$$\sigma^2(t) = a\,\mathrm{Re}[q_1(t)]^2 + b\,\mathrm{Re}[q_1(t)]\,\mathrm{Im}[q_1(t)] + c\,\mathrm{Im}[q_1(t)]^2, \qquad (57)$$

$$\gamma(t) = \gamma_1\,\mathrm{Re}[q_1(t)] + \gamma_2\,\mathrm{Im}[q_1(t)], \qquad (58)$$

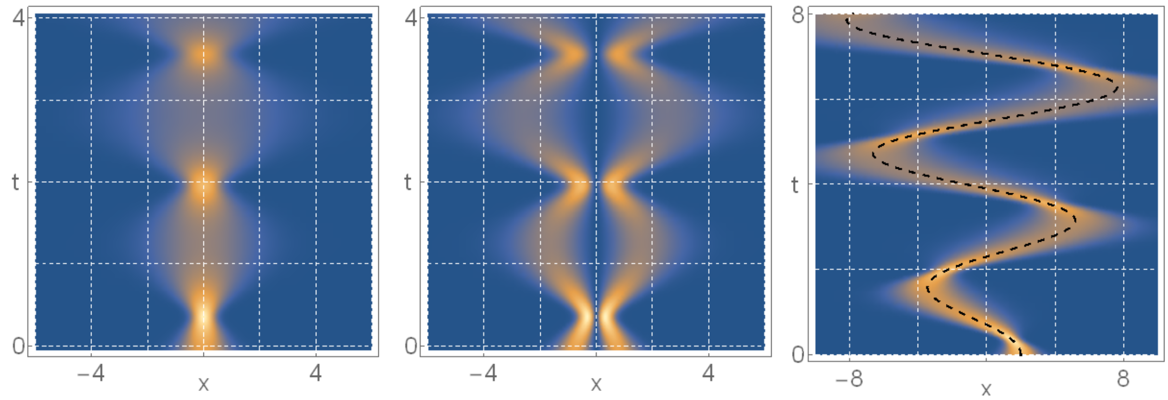
where the Wronskian of the two linearly independent solutions $\mathrm{Re}\,q_1$ and $\mathrm{Im}\,q_1$ becomes $\mathcal{W}_0 = w_1$, leading to the constraint $b^2 - 4ac = -4\frac{w_0^2}{w_1^2}$.

Similarly to the Caldirola-Kanai case, the Heisenberg uncertainty relation saturates for times t_m such that $W_\mu(t_m) = 0$. In this case, an analytic expression for such points is fairly complicated. Instead, one may look at the behavior of $W_\mu(t)$ depicted in Figure 3a, from which it is clear that such points exist. On the other hand, Figure 3b reveals that, in contradistinction to the Caldirola-Kanai case, the position variance does not grow indefinitely in time. This is rather expected as, for asymptotic times $t >> 1$, the mass term converges to a finite value different from zero. That is, the Hamiltonian becomes stationary for asymptotic values.

Before conclude, the probability density for $\psi_n(x,t)$ and $\xi_\alpha(x,t)$ are shown in Figure 4. In the latter, it can be verified that the width of the wavepackets oscillates in a bounded way for times $t > 2$. Particularly, for the coherent state case of Figure 4c, the dynamics of the wavepacket can be identified clearly, where the maximum point follows the corresponding classical trajectory (dashed-black). Therefore, there is no need to introduce a truncation time T, for the mass converges to a constant value different from zero, remaining physically reasonable at all times.

5. Conclusions

In this work, the class of form-preserving point transformations has been used to construct the constants of



(A) . $n = 0$ (B) . $n = 1$ (C).

FIGURE 4. Probability density $\mathcal{P}_n = |\psi_n(x,t)|^2$ for $n = 0$ (A), $n = 1$ (B), and $\mathcal{P}_\alpha = |\psi_n(x,t)|^2$ (C) associated with the regularized Caldirola-Kanai mass term $\mu_{rck}(t)$. The rest of parameters have been fixed as $w_0 = a = b = 1$, $w_1 = 2$, $\mu_0 = 0.3$, and $\kappa = 0.5$.

motion for the family of time-dependent mass oscillators. This was achieved by exploiting the preservation of first-integrals on the initial stationary oscillator model. Since several constants of motion are already known for the initial system, the corresponding counterparts for the time-dependent model are straightforwardly constructed by implementing the appropriate mappings. Notably, three different constants of motion were identified, one that admits an orthogonal set of eigensolutions, another that permits non-orthogonal eigensolutions, and the third one that does not admit finite-norm solutions.

Interestingly, the non-orthogonal eigensolutions are actually coherent states, for they are constructed from the annihilation operator of the time-dependent oscillator. Furthermore, by exploiting the underlying Weyl-Heisenberg algebra fulfilled by the quantum invariants, it was possible to find exact expressions for the expectation values of the position and momentum observables. The latter revealed the coherent states are represented by Gaussian wavepacket whose maximum follows the corresponding classical trajectory.

Besides the latter properties, it was also found that, in general, the Schrödinger-Robertson uncertainty relation saturates for all times, whereas the Heisenberg one gets minimized only for some times. Still, two special time-dependent Hamiltonians exist so that the Heisenberg inequality saturates at all times, one of which is the stationary limit case, as expected.

Remarkably, the newly introduced regularized Caldirola-Kanai mass term admits exact solutions that regularize the unusual behavior observed in the conventional Caldirola-Kanai case. More precisely, the variances become bounded as well as the expectation values. This allows obtaining localization of particles, which is desired in physical implementations such as traps of charged particles.

ACKNOWLEDGEMENTS

The author acknowledges the support from the project "Physicist on the move II" (KINEÓ II), Czech Republic, Grant No. CZ.02.2.69/0.0/0.0/18_053/0017163. This research has been also funded by Consejo Nacional de Ciencia y Tecnología (CONACyT), Mexico, Grant No. A1-S-24569.

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