

A NOTE ON ENTANGLEMENT CLASSIFICATION FOR TRIPARTITE MIXED STATES

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ABSTRACT. We study the classification of entanglement in tripartite systems by using Bell-type inequalities and principal basis. By using Bell functions and the generalized three dimensional Pauli operators, we present a set of Bell inequalities which classifies the entanglement of triqutrit fully separable and bi-separable mixed states. By using the correlation tensors in the principal basis representation of density matrices, we obtain separability criteria for fully separable and bi-separable $2 \otimes 2 \otimes 3$ quantum mixed states. Detailed example is given to illustrate our criteria in classifying the tripartite entanglement.

KEYWORDS: Bell inequalities, separability, principal basis.

1. INTRODUCTION

One of the most remarkable features that distinguishes quantum mechanics from classical mechanics is the quantum entanglement. Entanglement was first recognized by EPR [1], with significant progress made by Bell [2] toward the resolution of the EPR problem. Since Bell's work, derivation of new Bell-like inequalities has been one of the important and challenging subjects. CHSH generalized the original Bell inequalities to a more general case for two observers [3]. In [4] the authors proposed an estimation of quantum entanglement by measuring the maximum violation of the Bell inequality without information of the reduced density matrices. In [5] series of Bell inequalities for multipartite states have been presented with sufficient and necessary conditions to detect certain entanglement. There have been many important generalizations and interesting applications of Bell inequalities [6–8]. By calculating the measures of entanglement and the quantum violation of the Bell-type inequality, a relationship between the entanglement measure and the amount of quantum violation was derived in [9]. However, for high-dimensional multiple quantum systems the results for such relationships between the entanglement and the nonlocal violation are still far from being satisfied. In [10], an upper bound on fully entangled fraction for arbitrary dimensional states has been derived by using the principal basis representation of density matrices. Based on the norms of correlation vectors, the authors in [11] presented an approach to detect entanglement in arbitrary dimensional quantum systems. Separability criteria for both bipartite and multipartite quantum states was also derived in terms of the correlation matrices [12].

In this paper by using the Bell function and the generalized three dimensional Pauli operators, we derive a quantum upper bound for $3 \otimes 3 \otimes 3$ quantum

systems. We present a classification of entanglement for triqutrit mixed states by a set of Bell inequalities. These inequalities can distinguish fully separable and bi-separable states. Moreover, we propose criteria to detect classification of entanglement for $2 \otimes 2 \otimes 3$ mixed states with correlation tensor matrices in the principal basis representation of density matrices.

2. ENTANGLEMENT IDENTIFICATION WITH BELL INEQUALITIES

We first consider relations between entanglement and non-locality for $3 \otimes 3 \otimes 3$ quantum systems. Consider three observers who may choose independently between two dichotomic observables denoted by A_i and B_i for the i -th observer, $i = 1, 2, 3$. Let \widehat{V}_i denote the measurement operator associated with the variable $V_i \in \{A_i, B_i\}$ of i -th observer. We choose a complete set of orthonormal basis vectors $|k\rangle$ to describe an orthogonal measurement of a given variable V_i . The measurement outcomes are indicated by a set of eigenvalues $1, \lambda, \lambda^2$, where $\lambda = \exp(\frac{i2\pi}{3})$ is a primitive third root of unity. Therefore the measurement operator can be represented by $\widehat{V}_i = \sum_{k=0}^2 \lambda^k |k\rangle\langle k|$. Inspired by the Bell function (the expected value of Bell operator) constructed in [13], we introduce the following Bell operator,

$$\mathcal{B} = \sum_{j=1}^2 \frac{1}{4} (\widehat{A}_1^j \otimes \widehat{A}_2^j \otimes \widehat{A}_3^j + \lambda^j \widehat{A}_1^j \otimes \widehat{B}_2^j \otimes \widehat{B}_3^j + \lambda^j \widehat{B}_1^j \otimes \widehat{A}_2^j \otimes \widehat{B}_3^j + \lambda^j \widehat{B}_1^j \otimes \widehat{B}_2^j \otimes \widehat{A}_3^j), \quad (1)$$

where \widehat{A}_i^j (\widehat{B}_i^j) denotes the j -th power of \widehat{A}_i (\widehat{B}_i).

Next we construct three Bell operators in terms of Eq. (1). Consider three dimensional Pauli opera-

tors [14] \widehat{X} and \widehat{Z} which satisfy

$$\widehat{X}|k\rangle = |k+1\rangle, \widehat{Z}|k\rangle = \lambda^k|k\rangle, \widehat{X}^3 = I, \widehat{Z}^3 = I,$$

where I denotes the identity operator. Therefore, if we replace \widehat{A}_i and \widehat{B}_i with the following unitary operators, $\widehat{A}_1 = \widehat{Z}$, $\widehat{A}_2 = \lambda^2\widehat{X}\widehat{Z}$, $\widehat{A}_3 = \widehat{X}\widehat{Z}^2$, $\widehat{B}_1 = \widehat{Z}$, $\widehat{B}_2 = \widehat{X}\widehat{Z}^2$ and $\widehat{B}_3 = \lambda^2\widehat{X}\widehat{Z}$, we obtain

$$\begin{aligned} \mathcal{B}_1 = & \sum_{j=1}^2 \frac{1}{4} [\widehat{Z}^j \otimes (\lambda^2\widehat{X}\widehat{Z})^j \otimes (\widehat{X}\widehat{Z}^2)^j + \lambda^j\widehat{Z}^j \otimes (\widehat{X}\widehat{Z}^2)^j \\ & \otimes (\lambda^2\widehat{X}\widehat{Z})^j + \lambda^j\widehat{Z}^j \otimes (\lambda^2\widehat{X}\widehat{Z})^j \otimes (\lambda^2\widehat{X}\widehat{Z})^j \\ & + \lambda^j\widehat{Z}^j \otimes (\widehat{X}\widehat{Z}^2)^j \otimes (\widehat{X}\widehat{Z}^2)^j]. \end{aligned} \tag{2}$$

If we choose unitary operators as follows, $\widehat{A}_1 = \lambda^2\widehat{X}\widehat{Z}$, $\widehat{A}_2 = \widehat{X}\widehat{Z}^2$, $\widehat{A}_3 = \widehat{Z}$, $\widehat{B}_1 = \widehat{X}\widehat{Z}^2$, $\widehat{B}_2 = \lambda^2\widehat{X}\widehat{Z}$ and $\widehat{B}_3 = \widehat{Z}$, we have

$$\begin{aligned} \mathcal{B}_2 = & \sum_{j=1}^2 \frac{1}{4} [(\omega^2\widehat{X}\widehat{Z})^j \otimes (\widehat{X}\widehat{Z}^2)^j \otimes \widehat{Z}^j + \lambda^j(\lambda^2\widehat{X}\widehat{Z})^j \\ & \otimes (\lambda^2\widehat{X}\widehat{Z})^j \otimes \widehat{Z}^j + \lambda^j(\widehat{X}\widehat{Z}^2)^j \otimes (\widehat{X}\widehat{Z}^2)^j \\ & \otimes (\widehat{Z})^j + \lambda^j(\widehat{X}\widehat{Z}^2)^j \otimes (\lambda^2\widehat{X}\widehat{Z})^j \otimes (\widehat{Z})^j]. \end{aligned} \tag{3}$$

Taking $\widehat{A}_1 = \lambda^2\widehat{X}\widehat{Z}$, $\widehat{A}_2 = \widehat{Z}$, $\widehat{A}_3 = \widehat{X}\widehat{Z}^2$, $\widehat{B}_1 = \widehat{X}\widehat{Z}^2$, $\widehat{B}_2 = \widehat{Z}$ and $\widehat{B}_3 = \lambda^2\widehat{X}\widehat{Z}$, we have

$$\begin{aligned} \mathcal{B}_3 = & \sum_{j=1}^2 \frac{1}{4} [(\lambda^2\widehat{X}\widehat{Z})^j \otimes \widehat{Z}^j \otimes (\widehat{X}\widehat{Z}^2)^j + \lambda^j(\lambda^2\widehat{X}\widehat{Z})^j \\ & \otimes (\widehat{Z})^j \otimes (\lambda^2\widehat{X}\widehat{Z})^j + \lambda^j(\widehat{X}\widehat{Z}^2)^j \otimes \widehat{Z}^j \\ & \otimes (\lambda^2\widehat{X}\widehat{Z})^j + \lambda^j(\widehat{X}\widehat{Z}^2)^j \otimes \widehat{Z}^j \otimes (\widehat{X}\widehat{Z}^2)^j]. \end{aligned} \tag{4}$$

Concerning the bounds on the mean values $|\langle \mathcal{B}_i \rangle|$ of the operators \mathcal{B}_i , $i = 1, 2, 3$, we have the following conclusions.

Theorem 1. For $3 \otimes 3 \otimes 3$ mixed states, we have the inequality, $|\langle \mathcal{B}_i \rangle| \leq \frac{5}{4}$, $i = 1, 2, 3$.

Proof Due to the linear property of the average values, it is sufficient to consider pure states. Any triqutrit pure state can be written as,

$$\begin{aligned} |\psi\rangle = & c_1|000\rangle + c_2|011\rangle + c_3|012\rangle + c_4|021\rangle + c_5|022\rangle \\ & + c_6|101\rangle + c_7|102\rangle + c_8|110\rangle + c_9|111\rangle \\ & + c_{10}|120\rangle + c_{11}|122\rangle + c_{12}|201\rangle + c_{13}|202\rangle \\ & + c_{14}|210\rangle + c_{15}|212\rangle + c_{16}|220\rangle + c_{17}|221\rangle \\ & + c_{18}|222\rangle, \end{aligned} \tag{5}$$

where $c_5, c_{11}, c_{13}, c_{15}, c_{16}, c_{17}$ and c_{18} are real and non-negative, $|c_1| \geq |c_i|$ for $i = 1, 2, \dots, 18$, $|c_9| \geq |c_{18}|$

and $\sum_{i=1}^{18} |c_i|^2 = 1$. Therefore,

$$\begin{aligned} |\langle \mathcal{B}_1 \rangle| = & \left| \frac{1}{4} (-c_1c_2 + 5c_1c_5 - c_2c_5 - c_6c_{10} + 2c_7c_8 \right. \\ & + 2c_9c_{11} + 2c_{12}c_{15} + 5c_{12}c_{16} - c_{13}c_{14} - c_{13}c_{17} \\ & \left. - 4c_{14}c_{17} + 5c_{15}c_{16}) \right| \\ \leq & \frac{1}{8} \times 10 \times \sum_{i=1}^{18} c_i^2 \\ = & \frac{5}{4}. \end{aligned} \tag{6}$$

Similarly one can prove that $|\langle \mathcal{B}_i \rangle| \leq \frac{5}{4}$ for $i = 2, 3$. \square

Theorem 2. If a triqutrit mixed state ρ is fully separable, then $|\langle \mathcal{B}_i \rangle| = 0$, $i = 1, 2, 3$.

The proof is straightforward. Due to the linear property of the average values, it is sufficient to consider pure states again. A fully separable pure state can be written as under suitable bases, $|\psi\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$. Therefore $|\langle \mathcal{B}_i \rangle| = |\text{tr}(\rho \mathcal{B}_i)| = 0$.

Theorem 3. For bi-separable states $\rho_{i|jk}$ under bipartition i and jk , $i \neq j \neq k \in \{1, 2, 3\}$, we have

$$\begin{aligned} |\langle \mathcal{B}_1 \rangle| \leq \frac{3}{4}, \quad |\langle \mathcal{B}_2 \rangle| = 0, \quad |\langle \mathcal{B}_3 \rangle| = 0, \\ |\langle \mathcal{B}_1 \rangle| = 0, \quad |\langle \mathcal{B}_2 \rangle| \leq \frac{3}{4}, \quad |\langle \mathcal{B}_3 \rangle| = 0, \\ |\langle \mathcal{B}_1 \rangle| = 0, \quad |\langle \mathcal{B}_2 \rangle| = 0, \quad |\langle \mathcal{B}_3 \rangle| \leq \frac{3}{4}, \end{aligned}$$

for $\rho_{1|23}$, $\rho_{3|12}$ and $\rho_{2|13}$, respectively.

Proof It is sufficient to consider pure states only. Every bi-separable pure state $\rho_{1|23}$ can be written as via a suitable choice of bases [15],

$$|\psi\rangle = |0\rangle \otimes (c_0|00\rangle + c_1|11\rangle + c_2|22\rangle),$$

where $|c_0| \geq |c_1| \geq |c_2|$ and $\sum_{i=0}^2 |c_i|^2 = 1$. Therefore, we have by direct calculation,

$$\begin{aligned} |\langle \mathcal{B}_1 \rangle| = & \left| \frac{1}{4} (5c_2c_0 - c_0c_1 - c_1c_2) \right| \\ \leq & \frac{1}{8} (5 \times (c_2^2 + c_0^2) + (c_0^2 + c_1^2) \times (c_1^2 + c_2^2)) \\ \leq & \frac{3}{4}. \end{aligned}$$

It is straightforward to prove similarly, $|\langle \mathcal{B}_2 \rangle| = 0$ and $|\langle \mathcal{B}_3 \rangle| = 0$. For bi-separable states $\rho_{3|12}$ and $\rho_{2|13}$, the results can be proved in a similar way. \square

The above relations given in Theorem 1-3 give rise to characterization of quantum entanglement based on the Bell-type violations. If we consider $|\langle \mathcal{B}_i \rangle|$, $i = 1, 2, 3$, to be three coordinates, then all the triqutrit states are confined in a cube with size $\frac{5}{4} \times \frac{5}{4} \times \frac{5}{4}$. The bi-separable states are confined in a cube with size $\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}$, see Figure 1.

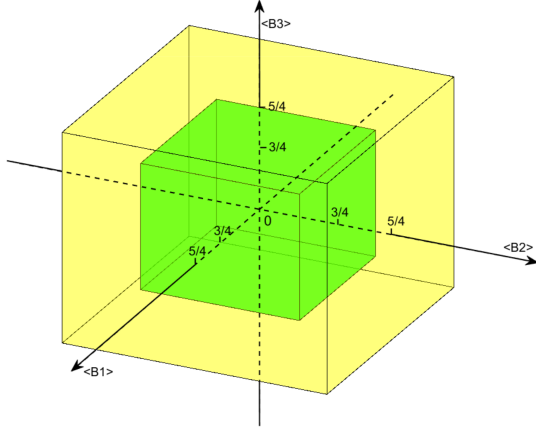


FIGURE 1. All states lie in the yellow cube, while in the green cube are bi-separable states.

3. ENTANGLEMENT CLASSIFICATION UNDER PRINCIPAL BASIS

Consider the principal basis on d -dimensional Hilbert space H with computational basis $|i\rangle$, $i = 1, 2, \dots, d$. Let E_{ij} be the $d \times d$ unit matrix with the only nonzero entry 1 at the position (i, j) . Let ω be a fixed d -th primitive root of unity, the principal basis is given by

$$A_{ij} = \sum_{m \in Z_d} \omega^{im} E_{m, m+j}, \quad (7)$$

where $\omega^d = 1$, $i, j \in Z_d$ and Z_d is Z modulo d . The set $\{A_{ij}\}$ spans the principal Cartan subalgebra of $gl(d)$. Under the stand inner product $(x|y) = \text{tr}(xy)$ of matrices x and y , the dual basis of the principal basis $\{A_{ij}\}$ is $\{(\omega^{ij}/d)A_{-i, -j}\}$, which follows also from the algebraic property of the principal matrices, $A_{ij}A_{kl} = \omega^{jk}A_{i+k, j+l}$. Namely, $A_{i, j}^\dagger = \omega^{ij}A_{-i, -j}$, and thus $\text{tr}(A_{ij}A_{kl}^\dagger) = \delta_{ik}\delta_{jl}d$ [10].

Next we consider the entanglement of $2 \otimes 2 \otimes 3$ systems. Let $\{A_{ij}\}$ and $\{B_{ij}\}$ be the principal bases of 2-dimensional and 3-dimensional Hilbert space, respectively. For any quantum state $\rho \in H_1^2 \otimes H_2^2 \otimes H_3^3$, ρ has the principal basis representation:

$$\begin{aligned} \rho = & \frac{1}{12}(I_2 \otimes I_2 \otimes I_3 + \sum_{\substack{(i,j) \\ \neq (0,0)}} u_{ij}A_{ij} \otimes I_2 \otimes I_3 \\ & + \sum_{\substack{(k,l) \\ \neq (0,0)}} v_{kl}I_2 \otimes A_{kl} \otimes I_3 + \sum_{(s,t) \neq (0,0)} w_{st}I_2 \otimes I_2 \otimes B_{st} \\ & + \sum_{\substack{(i,j),(k,l) \\ \neq (0,0)}} x_{ij,kl}A_{ij} \otimes A_{kl} \otimes I_3 + \sum_{\substack{(i,j),(s,t) \\ \neq (0,0)}} y_{ij,st}A_{ij} \\ & \otimes I_2 \otimes B_{st} + \sum_{\substack{(k,l),(s,t) \\ \neq (0,0)}} z_{kl,st}I_2 \otimes A_{kl} \otimes B_{st} \\ & + \sum_{\substack{(i,j),(k,l),(s,t) \\ \neq (0,0)}} r_{ij,kl,st}A_{ij} \otimes A_{kl} \otimes B_{st}), \end{aligned} \quad (8)$$

where I_2 (I_3) denotes the two (three) dimensional identity matrix, $u_{ij} = \text{tr}(\rho A_{ij}^\dagger \otimes I_2 \otimes I_3)$, $v_{kl} =$

$\text{tr}(\rho I_2 \otimes A_{kl}^\dagger \otimes I_3)$, $w_{st} = \text{tr}(\rho I_2 \otimes I_2 \otimes B_{st}^\dagger)$, $x_{ij,kl} = \text{tr}(\rho A_{ij}^\dagger \otimes A_{kl}^\dagger \otimes I_3)$, $y_{ij,st} = \text{tr}(\rho A_{ij}^\dagger \otimes I_2 \otimes B_{st}^\dagger)$, $z_{kl,st} = \text{tr}(\rho I_2 \otimes A_{kl}^\dagger \otimes B_{st}^\dagger)$ and $r_{ij,kl,st} = \text{tr}(\rho A_{ij}^\dagger \otimes A_{kl}^\dagger \otimes B_{st}^\dagger)$. Denote $T_1^{1|23}$, $T_2^{1|23}$, $T_1^{2|13}$, $T_2^{2|13}$, $T_1^{3|12}$ and $T_2^{3|12}$ the matrices with entries given by $r_{01,kl,st}$, $r_{11,kl,st}$, $r_{ij,01,st}$, $r_{ij,11,st}$, $r_{ij,kl,10}$ and $r_{ij,kl,20}$ ($i, j, k, l \in Z_2$, $s, t \in Z_3$), respectively. Let $\|A\|_{tr} = \sum \sigma_i = \text{tr}\sqrt{AA^\dagger}$ be the trace norm of a matrix $A \in R^{m \times n}$, where σ_i are the singular values of the matrix A .

First we note that $\|T_1^{1|23} - T_2^{1|23}\|_{tr}$ is invariant under local unitary transformations. Denote UAU^\dagger by A^U . Suppose $\rho' = \rho^{(U \otimes U_2 \otimes U_3)}$ with $U_2 \in U(2)$ and $U_3 \in U(3)$, $A_{ij}^{U_2} = \sum_{\substack{(i',j') \\ \neq (0,0)}} m_{ij,i'j'} A_{i'j'}$ and $B_{ij}^{U_3} = \sum_{\substack{(i',j') \\ \neq (0,0)}} n_{ij,i'j'} B_{i'j'}$ for some coefficients $m_{ij,i'j'}$ and $n_{ij,i'j'}$. The orthogonality of $\{A_{ij}^{U_2}\}$ and $\{B_{ij}^{U_3}\}$ requires that

$$\begin{aligned} \text{tr}(A_{ij}^{U_2} A_{kl}^{U_2}) &= \text{tr}(U_2 A_{ij} A_{kl}^\dagger U_2^\dagger) \\ &= \text{tr}(A_{ij} A_{kl}^\dagger) = 2\delta_{ik}\delta_{jl}; \\ \text{tr}(B_{ij}^{U_3} B_{kl}^{U_3}) &= \text{tr}(U_3 B_{ij} B_{kl}^\dagger U_3^\dagger) \\ &= \text{tr}(B_{ij} B_{kl}^\dagger) = 3\delta_{ik}\delta_{jl}. \end{aligned}$$

Hence, we have $M = (m_{ij,i'j'}) \in SU(3)$ and $N = (n_{ij,i'j'}) \in SU(8)$ since any two orthogonal bases are transformed by an unitary matrix. One sees that

$$\begin{aligned} & \sum_{\substack{(i,j),(k,l) \\ (s,t) \neq (0,0)}} r_{ij,kl,st} A_{ij} \otimes A_{kl}^{U_2} \otimes B_{st}^{U_3} \\ &= \sum_{\substack{(i,j),(k,l) \\ (s,t) \neq (0,0)}} \sum_{\substack{(k',l'),(s',t') \\ \neq (0,0)}} r_{ij,kl,st} m_{kl,k'l'} n_{st,s't'} A_{ij} \otimes A_{k'l'} \\ & \quad \otimes B_{s't'} \\ &= \sum_{\substack{(i,j),(k,l) \\ (s,t) \neq (0,0)}} \left(\sum_{\substack{(k',l'),(s',t') \\ \neq (0,0)}} m_{k'l',kl} r_{ij,k'l',s't'} n_{s't',st} \right) A_{ij} \otimes A_{kl} \\ & \quad \otimes B_{st}. \end{aligned}$$

We have $T_1^{1|23}(\rho') = M^t T_1^{1|23}(\rho) N$ and $T_2^{1|23}(\rho') = M^t T_2^{1|23}(\rho) N$. Therefore,

$$\|T_1^{1|23}(\rho') - T_2^{1|23}(\rho')\|_{tr} = \|T_1^{1|23}(\rho) - T_2^{1|23}(\rho)\|_{tr}, \quad (9)$$

due to that the singular values of a matrix are the same as those of $M^t T N$ when M and N are unitary matrices.

Theorem 4. If a mixed state ρ is fully separable, then $\|T_1^{1|23} - T_2^{1|23}\|_{tr} \leq \sqrt{3}$.

Proof If $\rho = |\varphi\rangle\langle\varphi|$ is fully separable, we have $|\varphi_{1|2|3}\rangle = |\varphi_1\rangle \otimes |\varphi_{23}\rangle \in H_1^2 \otimes H_{23}^6$, where $|\varphi_{23}\rangle = |\varphi_2\rangle \otimes |\varphi_3\rangle \in H_2^2 \otimes H_3^3$. Then by Schmidt decomposition, $|\varphi_{1|2|3}\rangle = t_0|0\alpha\rangle + t_1|1\beta\rangle$, where $t_0^2 + t_1^2 = 1$. Taking into account the local unitary equivalence in $H_2^2 \otimes H_3^3$ and using (9), we only need to consider that $\{|\alpha\rangle, |\beta\rangle\} = \{|00\rangle, |01\rangle\}$. Then

$|\varphi_{1|2|3}\rangle = t_0|000\rangle + t_1|101\rangle$. $T_1^{1|23}$ and $T_2^{1|23}$ are given by

$$T_1^{1|23} = \begin{bmatrix} 0 & t_0t_1 & 0 \\ 0 & t_0t_1 & 0 \\ 0 & 0 & 0 \\ 0 & t_0t_1 & 0 \\ 0 & t_0t_1\omega^2 & 0 \\ 0 & 0 & 0 \\ 0 & t_0t_1 & 0 \\ 0 & t_0t_1\omega & 0 \end{bmatrix}^t, \quad (10)$$

$$T_2^{1|23} = \begin{bmatrix} 0 & t_0t_1 & 0 \\ 0 & -t_0t_1 & 0 \\ 0 & 0 & 0 \\ 0 & t_0t_1 & 0 \\ 0 & -t_0t_1\omega^2 & 0 \\ 0 & 0 & 0 \\ 0 & t_0t_1 & 0 \\ 0 & -t_0t_1\omega & 0 \end{bmatrix}^t. \quad (11)$$

with $\omega^3 = 1$. Therefore, we have

$$\begin{aligned} & \|T_1^{1|23} - T_2^{1|23}\|_{tr} \\ &= tr \sqrt{(T_1^{1|23} - T_2^{1|23})(T_1^{1|23} - T_2^{1|23})^\dagger} \\ &= \sqrt{12t_0^2t_1^2} \leq \sqrt{3}. \end{aligned}$$

For a fully separable mixed state $\rho = \sum p_i|\varphi_i\rangle\langle\varphi_i|$, we get

$$\begin{aligned} & \|T_1^{1|23}(\rho) - T_2^{1|23}(\rho)\|_{tr} \\ &= \|T_1^{1|23}(\sum p_i|\varphi_i\rangle\langle\varphi_i|) - T_2^{1|23}(\sum p_i|\varphi_i\rangle\langle\varphi_i|)\|_{tr} \\ &\leq \sum p_i \|T_1^{1|23}(|\varphi_i\rangle\langle\varphi_i|) - T_2^{1|23}(|\varphi_i\rangle\langle\varphi_i|)\|_{tr} \\ &\leq \sqrt{3}, \end{aligned}$$

which proves the theorem. \square

Theorem 5. For any mixed state $\rho = \sum p_i|\varphi_i\rangle\langle\varphi_i| \in H_1^2 \otimes H_2^2 \otimes H_3^3$, $\sum p_i = 1$, $0 < p_i \leq 1$, we have:

- (1) If ρ is 1|23 separable, then $\|T_1^{1|23} - T_2^{1|23}\|_{tr} \leq \sqrt{6}$;
- (2) If ρ is 2|13 separable, then $\|T_1^{2|13} - T_2^{2|13}\|_{tr} \leq \sqrt{6}$;
- (3) If ρ is 3|12 separable, then $\|T_1^{3|12} - T_2^{3|12}\|_{tr} \leq \sqrt{3}$.

Proof (1) If $\rho = |\varphi\rangle\langle\varphi|$ is 1|23 separable, we have $|\varphi_{1|23}\rangle = |\varphi_1\rangle \otimes |\varphi_{23}\rangle \in H_1^2 \otimes H_{23}^6$, where $H_{23}^6 = H_2^2 \otimes H_3^3$. Then by Schmidt decomposition, one has $|\varphi_{1|23}\rangle = t_0|0\alpha\rangle + t_1|1\beta\rangle$, where $t_0^2 + t_1^2 = 1$. Taking into account the local unitary equivalence in $H_2^2 \otimes H_3^3$ and using (9), we only need to consider two cases (i) $\{|\alpha\rangle, |\beta\rangle\} = \{|00\rangle, |01\rangle\}$ and (ii) $\{|00\rangle, |11\rangle\}$.

For the first case we have $\|T_1^{1|23} - T_2^{1|23}\|_{tr} \leq \sqrt{3}$ by Theorem 4. For the second case, we have $|\varphi_{1|23}\rangle =$

$t_0|000\rangle + t_1|111\rangle$, where $T_1^{1|23}$ and $T_2^{1|23}$ are given by

$$T_1^{1|23} = \begin{bmatrix} t_0t_1 & 0 & t_0t_1 \\ t_0t_1 & 0 & -t_0t_1 \\ 0 & 0 & 0 \\ t_0t_1 & 0 & t_0t_1 \\ t_0t_1\omega^2 & 0 & -t_0t_1\omega^2 \\ 0 & 0 & 0 \\ t_0t_1 & 0 & t_0t_1 \\ t_0t_1\omega & 0 & -t_0t_1\omega \end{bmatrix}^t, \quad (12)$$

$$T_2^{1|23} = \begin{bmatrix} t_0t_1 & 0 & t_0t_1 \\ -t_0t_1 & 0 & t_0t_1 \\ 0 & 0 & 0 \\ t_0t_1 & 0 & t_0t_1 \\ -t_0t_1\omega^2 & 0 & t_0t_1\omega^2 \\ 0 & 0 & 0 \\ t_0t_1 & 0 & t_0t_1 \\ -t_0t_1\omega & 0 & t_0t_1\omega \end{bmatrix}^t. \quad (13)$$

Then we have

$$\begin{aligned} & \|T_1^{1|23} - T_2^{1|23}\|_{tr} \\ &= tr \sqrt{(T_1^{1|23} - T_2^{1|23})(T_1^{1|23} - T_2^{1|23})^\dagger} \\ &= \sqrt{24t_0^2t_1^2} \leq \sqrt{6}. \end{aligned}$$

Now consider mixed state $\rho = \sum p_i|\varphi_i\rangle\langle\varphi_i|$. We obtain

$$\begin{aligned} & \|T_1^{1|23}(\rho) - T_2^{1|23}(\rho)\|_{tr} \\ &= \|T_1^{1|23}(\sum p_i|\varphi_i\rangle\langle\varphi_i|) - T_2^{1|23}(\sum p_i|\varphi_i\rangle\langle\varphi_i|)\|_{tr} \\ &\leq \sum p_i \|T_1^{1|23}(|\varphi_i\rangle\langle\varphi_i|) - T_2^{1|23}(|\varphi_i\rangle\langle\varphi_i|)\|_{tr}, \end{aligned}$$

namely, $\|T_1^{1|23}(\rho) - T_2^{1|23}(\rho)\|_{tr} \leq \sqrt{6}$.

(2) If $\rho = |\varphi\rangle\langle\varphi|$ is 2|13 separable, we have $|\varphi_{2|13}\rangle = |\varphi_2\rangle \otimes |\varphi_{13}\rangle \in H_2^2 \otimes H_{13}^6$, where $H_{13}^6 = H_1^2 \otimes H_3^3$. Then by Schmidt decomposition, one has $|\varphi_{2|13}\rangle = t_0|0\alpha\rangle + t_1|1\beta\rangle$, where $t_0^2 + t_1^2 = 1$. Taking into account the local unitary equivalence in $H_1^2 \otimes H_3^3$, we obtain a similar equation of (9). Thus we only need to consider again the two cases (i) $\{|\alpha\rangle, |\beta\rangle\} = \{|00\rangle, |01\rangle\}$ and (ii) $\{|00\rangle, |11\rangle\}$.

In the first case, $|\varphi_{2|13}\rangle = t_0|000\rangle + t_1|101\rangle$, and $T_1^{2|13}$ and $T_2^{2|13}$ are zero matrices. In the second case, $|\varphi_{2|13}\rangle = t_0|000\rangle + t_1|111\rangle$, with $T_1^{2|13}$ and $T_2^{2|13}$ given by

$$T_1^{2|13} = \begin{bmatrix} t_0t_1 & 0 & t_0t_1 \\ t_0t_1 & 0 & -t_0t_1 \\ 0 & 0 & 0 \\ t_0t_1 & 0 & t_0t_1 \\ t_0t_1\omega^2 & 0 & -t_0t_1\omega^2 \\ 0 & 0 & 0 \\ t_0t_1 & 0 & t_0t_1 \\ t_0t_1\omega & 0 & -t_0t_1\omega \end{bmatrix}^t, \quad (14)$$

$$T_2^{2|13} = \begin{bmatrix} t_0 t_1 & 0 & t_0 t_1 \\ -t_0 t_1 & 0 & t_0 t_1 \\ 0 & 0 & 0 \\ t_0 t_1 & 0 & t_0 t_1 \\ -t_0 t_1 \omega^2 & 0 & t_0 t_1 \omega^2 \\ 0 & 0 & 0 \\ t_0 t_1 & 0 & t_0 t_1 \\ -t_0 t_1 \omega & 0 & t_0 t_1 \omega \end{bmatrix}^t. \quad (15)$$

Then we have

$$\begin{aligned} & \|T_1^{2|13} - T_2^{2|13}\|_{tr} \\ &= tr \sqrt{(T_1^{2|13} - T_2^{2|13})(T_1^{2|13} - T_2^{2|13})^\dagger} \\ &= \sqrt{24t_0^2 t_1^2} \leq \sqrt{6}. \end{aligned}$$

For the mixed state $\rho = \sum p_i |\varphi_i\rangle\langle\varphi_i|$, we have

$$\begin{aligned} & \|T_1^{2|13}(\rho) - T_2^{2|13}(\rho)\|_{tr} \\ &= \|T_1^{2|13}(\sum p_i |\varphi_i\rangle\langle\varphi_i|) - T_2^{2|13}(\sum p_i |\varphi_i\rangle\langle\varphi_i|)\|_{tr} \\ &\leq \sum p_i \|T_1^{2|13}(|\varphi_i\rangle\langle\varphi_i|) - T_2^{2|13}(|\varphi_i\rangle\langle\varphi_i|)\|_{tr} \\ &\leq \sqrt{6}. \end{aligned}$$

(3) If $\rho = |\varphi\rangle\langle\varphi|$ is 3|12 separable, we have $|\varphi_{3|12}\rangle = |\varphi_3\rangle \otimes |\varphi_{12}\rangle \in H_3^3 \otimes H_{12}^4$, where $H_{12}^4 = H_1^2 \otimes H_2^2$. Then by Schmidt decomposition, we have $|\varphi_{3|12}\rangle = t_0|0\alpha_0\rangle + t_1|1\alpha_1\rangle + t_2|2\alpha_2\rangle$, where $t_0^2 + t_1^2 + t_2^2 = 1$. Taking into account the local unitary equivalence in $H_1^2 \otimes H_2^2$, we obtain similar equation of (9). We only need to consider the case $|\varphi_{3|12}\rangle = t_0|000\rangle + t_1|101\rangle + t_2|210\rangle$. We have

$$\begin{aligned} T_1^{3|12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_0^2 - \omega t_1^2 + \omega^2 t_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ T_2^{3|12} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_0^2 - \omega^2 t_1^2 + \omega t_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \quad (16)$$

Using $1 + \omega + \omega^2 = 0$, we have

$$\begin{aligned} & \|T_1^{3|12} - T_2^{3|12}\|_{tr} \\ &= tr \sqrt{(T_1^{3|12} - T_2^{3|12})(T_1^{3|12} - T_2^{3|12})^\dagger} \\ &= \sqrt{3(t_0^2 + t_1^2)} \leq \sqrt{3}. \end{aligned}$$

For the mixed state $\rho = \sum p_i |\varphi_i\rangle\langle\varphi_i|$, we get

$$\begin{aligned} & \|T_1^{3|12}(\rho) - T_2^{3|12}(\rho)\|_{tr} \\ &= \|T_1^{3|12}(\sum p_i |\varphi_i\rangle\langle\varphi_i|) - T_2^{3|12}(\sum p_i |\varphi_i\rangle\langle\varphi_i|)\|_{tr} \\ &\leq \sum p_i \|T_1^{3|12}(|\varphi_i\rangle\langle\varphi_i|) - T_2^{3|12}(|\varphi_i\rangle\langle\varphi_i|)\|_{tr} \\ &\leq \sqrt{3}. \end{aligned}$$

As an example, let us consider the $2 \otimes 2 \otimes 3$ state,

$$\rho = x|GHZ'\rangle\langle GHZ'| + (1-x)I_{12}, \quad 0 \leq x \leq 1,$$

where $|GHZ'\rangle = \frac{1}{2}(|000\rangle + |101\rangle + |011\rangle + |112\rangle)$. By Theorem 4, we have that when $\|T_1^{1|23} - T_2^{1|23}\| = (2\sqrt{\frac{3}{2}} + 1)x > \sqrt{3}$, i.e., $0.5021 < x \leq 1$, ρ is not fully separable. By Theorem 5, when $\|T_1^{1|23} - T_2^{1|23}\| = \|T_1^{2|13} - T_2^{2|13}\| = (2\sqrt{\frac{3}{2}} + 1)x > \sqrt{6}$, i.e., $0.7101 < x \leq 1$, ρ is not separable under bipartition 1|23 or 2|13. When $\|T_1^{3|12} - T_2^{3|12}\| = \frac{7\sqrt{3}}{4}x > \sqrt{3}$, i.e., $0.5714 < x \leq 1$, ρ is not separable under bipartition 3|12.

4. CONCLUSIONS

We have presented quantum upper bounds for triqutrit mixed states by using the generalized Bell functions and the generalized three dimensional Pauli operators, from which the triqutrit entanglement has been identified. Our inequalities distinguish fully separable states and three types of bi-separable states for triqutrit states. Moreover, any triqutrits states are confined in a cube with size $\frac{5}{4} \times \frac{5}{4} \times \frac{5}{4}$ and the bi-separable states are in a cube with the size $\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4}$. We have also studied the classification of quantum entanglement for $2 \otimes 2 \otimes 3$ systems by using the correlation tensors in the principal basis representation of density matrices. By considering the upper bounds on some the trace norms, we have obtained the criteria which detect fully separable and bi-separable $2 \otimes 2 \otimes 3$ quantum mixed states. Detailed example has been given to show the classification of tripartite entanglement by using our criteria.

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