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ON SOME ALGEBRAIC FORMULATIONS WITHIN UNIVERSAL ENVELOPING ALGEBRAS RELATED TO SUPERINTEGRABILITY

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Abstract. We report on some recent purely algebraic approaches to superintegrable systems from the perspective of subspaces of commuting polynomials in the enveloping algebras of Lie algebras that generate quadratic (and eventually higher-order) algebras. In this context, two algebraic formulations are possible; a first one strongly dependent on representation theory, as well as a second formal approach that focuses on the explicit construction within commutants of algebraic integrals for appropriate algebraic Hamiltonians defined in terms of suitable subalgebras. The potential use in this context of the notion of virtual copies of Lie algebras is briefly commented.

Keywords: Enveloping algebras, commutants, quadratic algebras, superintegrability.

1. Introduction

Both the study of (quasi-)exactly solvable systems, as well as that of super-integrable systems make an extensive use of the universal enveloping algebras of Lie algebras, either in the context of the so-called hidden algebras or as symmetry algebras of the system. Of particular interest are those systems that, beyond super-integrability properties, also belong to the class of (quasi-)exactly solvable systems [1–5]. In particular, quadratic subalgebras have been shown to be a powerful tool for classifying and comparing super-integrable systems, as shown in [6], where the scheme of superintegrable systems on a two-dimensional conformally flat space has been characterized in terms of contractions. Additional examples in higher dimensions [7] lead us to suspect that n-dimensional super-integrable systems are somehow associated to (higher rank) polynomials in a suitable enveloping algebra [8], further stimulating the search of alternative algebraic approaches based on the structural properties of enveloping algebras. Although the precise fundamental properties of enveloping algebras of generic semidirect sums of simple and solvable Lie algebras are still far from being completely understood, a purely formal ansatz applied to the case of the Schrödinger algebras $\widehat{S}(n)$ has recently been shown to provide some interesting features [9].

In this work we comment on some purely algebraic approaches formulated in the enveloping algebras of Lie algebras for the identification or construction of quadratic algebras that may lead to super-integrable systems, once a suitable realization of the enveloping algebra by first-order differential operators has been chosen. The motivation for this analysis lies primarily on the inspection of super-integrable systems from the point of view of the algebraic properties of first integrals seen as elements of an enveloping algebra, as well as an attempt to determine to which extent these integrals are characterized algebraically by the hidden algebra [10]. This moreover suggests a realization-free description of systems in terms of commutants of algebraic Hamiltonians in enveloping algebras [11], in which elements of the coadjoint representation of Lie algebras may be useful to simplify computations.

2. First algebraic reformulation

In the context of (quasi)-exactly solvable problems, the Hamiltonians are described as differential operators in p variables that admit an expression as elements in the enveloping algebra of a Lie algebra $\mathfrak{g}$, commonly known as the hidden algebra, not necessarily associated to any symmetry algebra of the system. The main requirement is the existence of a representation of $\mathfrak{g}$ that is invariant for the Hamiltonian, a constraint that allows us to determine its spectrum (either partially or completely) using algebraic methods [12]. So, for example, the universal enveloping algebra of the simple Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and its realization as first-order differential operators on the real line provide a characterization of quasi-exactly solvable one-dimensional systems [13]. A second type of systems that uses the structural properties of enveloping algebras is given by super-integrable systems, where both the Hamiltonian and the constants of the motion are interpreted in the enveloping algebra of some Lie algebra $\mathfrak{g}$. Merely integrable n-dimensional systems can be interpreted as the image, via a realization Φ by first-order differential operators, of an Abelian subalgebra $\mathcal{A}$ of $\mathcal{U}(\mathfrak{g})$, while super-integrable systems would correspond to non-Abelian extensions of $\mathcal{A}$. The problem under what conditions a system both exhibits super-integrability and (quasi-)exact solvability has been analyzed in detail, and large classes of super-integrable systems that are exactly solvable have been found (see [3, 14, 15] and references therein).

A first algebraic formulation, as developed in [10], is motivated by the use of quadratic algebras in the context of super-integrable (and exactly solvable) systems with a given hidden algebra $\mathfrak{g}$ [3]. To this extent, we consider a Hamiltonian $\mathcal{H}$ expressed in terms of a subalgebra $\mathfrak{m} \subset \mathfrak{g}$ via a realization Φ by differential operators of the Lie algebra $\mathfrak{g}$:

$$\mathcal{H} = \sum_{i,j=1}^{\dim \mathfrak{m}} \alpha_{ij} \Phi(X_i)\Phi(X_j) + \sum_{k=1}^{\dim \mathfrak{m}} \beta_k \Phi(X_k) + \gamma_0, \quad (1)$$

where $\alpha_{ij}, \beta_k, \gamma_0$ are constants and $\{X_1, \ldots, X_{\dim \mathfrak{m}}\}$ is a basis of $\mathfrak{m}$. In this context, the Hamiltonian $\mathcal{H}$ is obtained as the image of a quadratic element $\mathcal{H}_a$ in the universal enveloping algebra $\mathcal{U}(\mathfrak{m}) \subset \mathcal{U}(\mathfrak{g})$. Similarly, the (independent) constants of the motion $\varphi_1, \ldots, \varphi_s$ can also be rewritten as the image of elements in the enveloping algebra $\mathcal{U}(\mathfrak{g})$. As differential operators they satisfy the commutators

$$[\mathcal{H}, \varphi_j] = 0, \ 1 \leq j \leq s. \quad (2)$$

The commutators $[\varphi_i, \varphi_j]$ provide additional (dependent) higher-order constants of the motion. A specially interesting case is given whenever the first integrals generate a quadratic algebra.

Abstracting from the specific realization Φ, and focusing merely on the underlying algebraic formulation, the formal polynomial

$$\mathcal{H}_a = \sum_{i,j=1}^{\dim \mathfrak{m}} \alpha_{ij} X_i X_j + \sum_{k=1}^{\dim \mathfrak{m}} \beta_k X_k + \gamma_0$$

in the enveloping algebra $\mathcal{U}(\mathfrak{m})$ of $\mathfrak{m}$ allows us to recover Hamiltonian $\mathcal{H}$ of the system once the generators are realized by the differential operators. In analogous form, we can find elements $J_1, \ldots, J_s$ in $\mathcal{U}(\mathfrak{g})$ that correspond, via the realization Φ, to the first integrals $\varphi_1, \ldots, \varphi_s$ of the system. While for the initial system the relations

$$[\mathcal{H}, \varphi_k] = 0, \quad 1 \leq k \leq s,$$

are ensured, there is no necessity that the polynomials J_k commute with $\mathcal{H}_a$ in $\mathcal{U}(\mathfrak{g})$, although the relation

$$[\mathcal{H}_a, J_k] = 0 \pmod{\Phi} \quad (3)$$

is satisfied. Similarly, for the polynomial relations $[\varphi_i, \varphi_j] = \alpha_{ij}^{k\ell} \varphi_k \varphi_\ell + \beta_{ij}^k \varphi_k$ of the first integrals, the commutators in $\mathcal{U}(\mathfrak{g})$ lead to the relation

$$[J_i, J_j] = \alpha_{ij}^{k\ell} J_k J_\ell + \beta_{ij}^k J_k \pmod{\Phi} \quad (4)$$

If equations (3) and (4) are satisfied for any realization Φ, then the problem is entirely characterized algebraically by the reduction chain $\mathfrak{m} \subset \mathfrak{g}$. It should be observed that this situation is rather exceptional, as the analysis of the exactly solvable systems described in [3] from the point of view of the first algebraic formulation indicates that, in general, the first integrals of the system do not correspond, at the level of the enveloping algebra of the hidden algebra, to polynomials that commute with the algebraic Hamiltonian, showing that the commutativity properties are a consequence of the realization by differential operators.

Using the correspondence existing between the representations of $\mathfrak{g}$ and those of its enveloping algebra $\mathcal{U}(\mathfrak{g})$ (see e.g. [11]) and identifying a Lie algebra $\mathfrak{g}$ with the first-order (left-invariant) differential operators on a Lie group G admitting $\mathfrak{g}$ as its Lie algebra, it follows that the universal enveloping algebra can be seen as the set of (left-invariant) differential operators on G of arbitrary order. Therefore, if $\Phi : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^n)$ is some realization of the Lie algebra by first-order differential operators, it can be uniquely extended to a realization $\widehat{\Phi} : \mathcal{U}(\mathfrak{g}) \to \mathfrak{X}(\mathbb{R}^n)$.

In this context, this first algebraic reformulation of the system is still strongly related to the representation theory of Lie algebras. More precisely, supposed that $\mathcal{H}_a \in \mathcal{U}(\mathfrak{m})$ is an algebraic Hamiltonian defined in the enveloping of some subalgebra $\mathfrak{m} \subset \mathfrak{g}$ and that the (independent) polynomials $J_1, \ldots, J_s$ generate a quadratic algebra, that is, satisfy the conditions

$$[J_i, J_j] = \alpha_{ij}^{k\ell} J_k J_\ell + \beta_{ij}^k J_k, \quad (5)$$

we consider the (two-sided) ideal $\mathcal{I}$ in $\mathcal{U}(\mathfrak{g})$ generated by the polynomials

$$Q_i := [\mathcal{H}_a, J_i], \quad 1 \leq i \leq s.$$

The problem is now to analyze whether there exists an equivalence class of (faithful) representations $\Phi : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^n)$ such that for the corresponding extension $\widehat{\Phi} : \mathcal{U}(\mathfrak{g}) \to \mathfrak{X}(\mathbb{R}^n)$, the image of the ideal $\mathcal{I}$ is contained in the kernel $\ker \widehat{\Phi}$, ensuring that the realized polynomials $\widehat{\Phi}(Q_i)$ correspond to first integrals of the Hamiltonian in the given realization. In some sense, this is a special case of an important and still unsolved problem, namely the embedding of a Lie algebra $\mathfrak{g}$ into the enveloping algebra $\mathcal{U}(\mathfrak{k})$ of another Lie algebra $\mathfrak{k}$, for which currently only the case of embeddings $\iota : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ for $\mathfrak{g}$ semisimple has been completely solved [16], using techniques of deformation theory [17].

We illustrate the preceding procedure considering the six-dimensional non-solvable Lie algebra $\mathfrak{r} \subset \mathfrak{sl}(3, \mathbb{R})$ with basis $\{X_1, \ldots, X_6\}$ and nonvanishing commutators

$$\begin{array}{lll} [X_1, X_2] = X_1, & [X_1, X_5] = X_4, & [X_2, X_5] = X_5, \\ [X_2, X_6] = -X_6, & [X_3, X_4] = -X_4, & [X_3, X_5] = -X_5, \\ [X_3, X_6] = X_6 & [X_4, X_6] = X_1, & [X_5, X_6] = X_2 - X_3. \end{array}$$

Superintegrable systems based on this hidden algebra $\mathfrak{r}$ and the vector field realization

$$X_1 = \partial_t, \ X_2 = t\partial_t - \frac{N}{3}, \ X_3 = su\partial_u - \frac{N}{3}, \quad (6)$$
$$X_4 = \partial_u, \ X_5 = t\partial_u, \ X_6 = u\partial_t,$$

have been extensively studied in [4], where in addition their exact solvability was analyzed. We consider a special case of the generic Hamiltonians studied there. Taking the values $s = k = \omega = 1$, $a = b = -\frac{1}{2}$ and $N = 0$, we obtain the Hamiltonian h_1 and two quadratic integrals

$$\begin{aligned} h_1 &= -4t\partial_t^2 - 8u\partial_{tu}^2 - 4u\partial_u^2 + 4t\partial_t + 4u\partial_u, \\ \varphi_1 &= 4u(u-t)\partial_u^2,\ \varphi_2 = 4(t-u)\left(\partial_t^2 - \partial_t\right). \end{aligned} \tag{7}$$

Now $h_1, \varphi_1, \varphi_2$ are the image by the realization of the following polynomials in the enveloping algebra of $\mathfrak{r}$:

$$\begin{aligned} H_1 &= 4X_2(1-X_1) + 8(1-X_3)X_1 + 4(1-X_4)X_3, \\ P_1 &= -4X_3X_5 + 4X_3^2 - 4X_3, \\ Q_1 &= 4(X_2X_1 - X_6X_1 + X_6 - X_2). \end{aligned}$$

At the purely algebraic level we have

$$[H_1, P_1] \neq 0, \quad [H_1, Q_1] \neq 0,$$

showing that the polynomials P_1 and Q_1 do not belong to the commutant of H_1 in $\mathcal{U}(\mathfrak{r})$. Therefore, the origin of the quadratic integrals of system (7) is not algebraic, but a consequence of the specific realization (6).

If we maintain the algebraic Hamiltonian as given above and search for quadratic polynomials in $\mathcal{U}(\mathfrak{r})$ commuting with it, we find that only two such operators exist (see [10] for the general case), given by

$$\begin{aligned} A_1 &= X_4 - X_3 - X_6 + X_1(1 + X_3 + X_6) + (X_3 + X_6)X_4, \\ B_1 &= -4X_1 - X_2 + X_6 + X_1X_2 + X_1X_3 - X_1X_6 - X_6X_4. \end{aligned}$$

These polynomials are not independent, as they satisfy the relation $A_1 + B_1 + \frac{1}{4}H_1 = 0$. Now, if we extend the analysis to cubic polynomials in $\mathcal{U}(\mathfrak{r})$, we find the following operator C_1 that commutes with H_1:

$$\begin{aligned} C_1 = 3X_1 &- 2X_3 - X_5 - 4X_6 + 2X_1X_3 + 4X_1X_6 + X_3^2 \\ &+ X_2X_4 + X_3^2 - X_3X_5 + X_3X_6 + X_6X_4 - X_6X_5 \\ &- X_1X_3^2 - X_1X_3X_6 + X_2X_3X_4 + X_2X_6X_4. \end{aligned}$$

The operators A_1 and C_1 generate a finite-dimensional polynomial algebra in $\mathcal{U}(\mathfrak{r})$, with explicit nonvanishing commutators

$$\begin{aligned} &[A_1, C_1] = D_1, \quad [A_1, D_1] = D_1, \quad [B_1, C_1] = -D_1, \\ &[C_1, D_1] = \frac{1}{2}\{B_1, D_1\} - \frac{1}{2}\{A_1, D_1\} - 12A_1 \\ &\qquad - 12A_1 + 4B_1 + 4C_1 - 2\{A_1, B_1\}, \end{aligned}$$

where $\{\circ, \circ\}$ is the anticommutator.

Now, as the operators H_1, A_1, C_1 commute at the algebraic level, for any realization of $\mathfrak{r}$ by vector fields they give rise to a Hamiltonian system possessing a quadratic and a cubic integral, respectively.[1] For the particular realization (6), it follows that the resulting system is actually equivalent to the initial one (7), as the image of the ideal $\mathcal{J}$ generated by A_1, B_1, C_1, D_1 is properly contained in the ideal spanned by φ_1 and φ_2, thus being functionally dependent on these integrals.

[1] Provided that the transformed operators are independent.

3. Commutants in enveloping algebras and coadjoint representations

A second algebraic approach, of a more general nature, can be proposed considering chain reductions $\mathfrak{g}' \subset \mathfrak{g}$ of (reductive) Lie algebras, and analyzing the structure of the commutant of $\mathfrak{g}'$ in the enveloping algebra $\mathcal{U}(\mathfrak{g})$, in order to identify polynomial (in particular, quadratic) subalgebras [9]. In the generic analysis of commutants, elements of the theory the coadjoint representation of Lie algebras can be used, in order to simplify some of the computations in enveloping algebras. If $\mathfrak{g}$ is a Lie algebra with generators $\{X_1, \ldots, X_n\}$ and commutators $[X_i, X_j] = C_{ij}^k X_k$, the X_i's are realized in the space $C^\infty(\mathfrak{g}^*)$ by means of the first-order differential operators:

$$\widehat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{8}$$

where $\{x_1, \ldots, x_n\}$ are the coordinates of a covector in a dual basis of $\mathbb{R}\{X_1, \ldots, X_n\}$. The invariants of $\mathfrak{g}$ (in particular, the Casimir operators) correspond to the solutions of the following system of partial differential equations:

$$\widehat{X}_i F = 0, \quad 1 \leq i \leq n. \tag{9}$$

For an embedding of Lie algebras $f : \mathfrak{g}' \to \mathfrak{g}$, a basis $\{X_1, \ldots, X_r\}$ of the subalgebra can be extended to a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$. Therefore, we can consider the subsystem formed by the first r equations of (9), corresponding to the generators of the subalgebra $\mathfrak{g}'$. The solutions of this subsystem, that in particular encompass the invariants of $\mathfrak{g}'$, are usually called subgroup scalars [18].

By means of the standard symmetrization map

$$\Lambda\left(x_{i_1} \ldots x_{i_p}\right) = \frac{1}{p!} \sum_{\sigma \in S_p} X_{\sigma(i_1)} \ldots X_{\sigma(i_p)} \tag{10}$$

polynomial solutions of the subsystem correspond to elements in the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ that commute with the subalgebra $\mathfrak{g}'$. If we now define an algebraic Hamiltonian

$$\mathcal{H} = \mathcal{H}\left(X_1, \ldots, X_r\right) \in \mathcal{U}(\mathfrak{g}'), \tag{11}$$

in terms of the subalgebra generators, the commutant

$$C_{\mathcal{U}(\mathfrak{g})}(\mathcal{H}) = \{U \in \mathcal{U}(\mathfrak{g}) \mid [\mathcal{H}, U] = 0\}$$

certainly includes the solutions of (9) common to the $\mathfrak{g}'$-generators, i.e.

$$C_{\mathcal{U}(\mathfrak{g})}(\mathcal{H}) \supset \left\{\Lambda(\varphi) \mid \widehat{X}_1(\varphi) = \cdots = \widehat{X}_r(\varphi) = 0\right\},$$

where $\varphi(x_1, \ldots, x_n) \in C^\infty(\mathfrak{g}^*)$.

Depending on the structure of $\mathfrak{g}$ and the subalgebra $\mathfrak{g}'$, as well as on the choice of $\mathcal{H}$, two possible cases arise for a polynomial $P \in C_{\mathcal{U}(\mathfrak{g})}(\mathcal{H})$:

(1.) P commutes with all $X_1, \dots, X_r$.

(2.) There is an index k_0 with $[P, X_{k_0}] \neq 0$.

Polynomials P in the first case actually commute with the Hamiltonian $\mathcal{H}$, and thus belong to the two-sided ideal $\langle \mathcal{I} \rangle$ generated by the set $\mathcal{I} = \{J_1, \dots, J_s\}$ of elements corresponding to the symmetrization of independent polynomials satisfying the subsystem of (9) corresponding to $\mathfrak{g}'$. For these elements, it follows at once that $[J_k, J_\ell]$ belongs to $\mathcal{I}$. In the general case, the Hamiltonian $\mathcal{H}$ does not commute with all X_j-generators, and in order to find the commutant $C_{\mathcal{U}(\mathfrak{g})}(\mathcal{H})$, we can restrict the analysis to the determination of a basis of the factor module $C_{\mathcal{U}(\mathfrak{g})}(\mathcal{H})/\langle \mathcal{I} \rangle$. Although the problem is computationally cumbersome, certain algorithms in terms of Gröbner bases have been developed that allow its precise determination [19].

A (restricted) systematic procedure that circumvents the above-mentioned obstruction and allows us to analyze polynomial algebras with respect to a reduction chain $\mathfrak{g}' \subset \mathfrak{g}$ can be proposed starting from the polynomials in $\mathcal{U}(\mathfrak{g})$ that commute with all the generators intervening in the expression of the algebraic Hamiltonian $\mathcal{H} \in \mathcal{U}(\mathfrak{g}')$. More precisely, if the Hamiltonian $\mathcal{H}$ is given as a polynomial $P(X_{i_1}, \dots, X_{i_s})$ in terms of the generators of the subalgebra $\mathfrak{g}'$ with basis $\{X_1, \dots, X_r\}$, we consider the subsystem of (9) given by

$$\widehat{X}_{i_j} F(x_1, \dots, x_n) = 0, \quad 1 \leq j \leq s.$$

We then extract a maximal set of independent polynomial solutions $\{Q_1, \dots, Q_p\}$ of (9), which in the reductive case forms an integrity basis for the solutions. Symmetrizing these functions we obtain elements M_j in the commutant $C_{\mathcal{U}(\mathfrak{g})}(\mathcal{H})$. Starting from the set of polynomials $\mathcal{S} = \{\mathcal{H}, M_1, \dots, M_p\}$, we inspect their commutators and determine whether, either adjoining new (dependent) elements to $\mathcal{S}$ or discarding some elements of $\mathcal{S}$, a finite-dimensional quadratic algebra $\mathcal{A}$ can be found. Although there is some ambiguity in the construction, as there is no quadratic algebra "canonically" associated to the reduction chain $\mathfrak{g}' \subset \mathfrak{g}$, it provides an alternative method that does not require a specific realization by vector fields, as the integrability condition is guaranteed by the commutant.

This ansatz has been successfully applied in [9] to the enveloping algebra of the Schrödinger algebras $\widehat{S}(n)$ for arbitrary values of $n \geq 1$ and various choices of algebraic Hamiltonian, showing that the construction is formally of use for the analysis of hidden algebras that are not reductive.

4. Virtual copies in enveloping algebras

In the solution of the embedding problem into enveloping algebras for semisimple algebras, the vanishing of the first cohomology group with values in $\mathcal{U}(\mathfrak{g})$ plays an important role, as it allows to provide a general solution for the perturbation problem [16]. For nonsemisimple Lie algebras, the application of the procedure is quite complicated for both computational reasons and the currently incomplete understanding of the precise structure of the corresponding enveloping algebras. However, for certain types of semidirect sums of simple and solvable Lie algebras, some analogous statements may be proposed, providing copies of semisimple Lie algebras in the enveloping algebra of a semidirect sum, up to a polynomial factor.

Supposed that $\mathfrak{s}$ is the Levi subalgebra of a semidirect sum $\mathfrak{g} = \mathfrak{s} \overrightarrow{\oplus}_\Gamma \mathfrak{r}$, we seek for elements of degree $d \geq 2$ in the generators in $\mathcal{U}(\mathfrak{g})$ that transform according to the structure tensor of $\mathfrak{s}$, up to a (polynomial) factor. The procedure can be summarized as follows: Consider a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{s}$ with commmutators

$$[X_i, X_j] = C_{ij}^k X_k. \tag{12}$$

and extend it to a basis $\{X_1, \dots, X_n, Y_1, \dots, Y_m\}$ of of the semidirect sum $\mathfrak{g}$. We now define operators

$$X_i' = X_i \, R\left(Y_1, \dots, Y_m\right) + P_i\left(Y_1, \dots, Y_m\right) \tag{13}$$

in $\mathcal{U}(\mathfrak{g})$, where P_i and R are still undetermined polynomials. In order to simplify computations, they can be considered as homogeneous polynomials of degrees k and $k-1$ respectively, so that X_i' is homogeneous of degree k. We require that these operators commute with the generators Y_k of the radical $\mathfrak{r}$, so that the identity

$$[X_i', Y_j] = 0, \quad 1 \leq i \leq n, \ 1 \leq j \leq m$$

is satisfied for all indices. Expanding the latter leads to the expression

$$\begin{aligned}[X_i', Y_j] &= [X_i R, Y_j] + [P_i, Y_j] \\ &= X_i \left[R, Y_j\right] + \left[X_i, Y_j\right] R + \left[P_i, Y_j\right].\end{aligned}$$

Taking into account the homogeneity degree of the terms with respect to the generators of $\mathfrak{s}$ and the representation space, it follows that $X_i \left[R, Y_j\right]$ can be seen as a polynomial of degree $(k-1)$ in the variables $\{Y_1, \dots, Y_m\}$. On the other hand the terms of $\left[X_i, Y_j\right] R + \left[P_i, Y_j\right]$ have degree k, allowing us to further separate the commutator as

$$\begin{aligned}[R, Y_j] &= 0, \\ \left[X_i, Y_j\right] R + \left[P_i, Y_j\right] &= 0.\end{aligned} \tag{14}$$

From the first equation we conclude that the factor R commutes with all generators Y_i, thus defines an invariant of the solvable Lie algebra $\mathfrak{r}$. We further require that the operators X_i' transform by the action of $\mathfrak{s}$ as the generators of the latter algebra, i.e.

$$[X_i', X_j] \quad = [X_i, X_j]' := C_{ij}^k \left(X_k R + P_k\right). \tag{15}$$

As this relation must hold for all the generators of the semidirect sum $\mathfrak{g}$, further structural constraints on the polynomials R and P_i are obtained. Expanding the left-hand term of condition (15) yields

$$[X_i', X_j] = \left[X_i, X_j\right] R - X_i \left[X_j, R\right] + \left[P_i, X_j\right].$$

As the Y_j are the generators of the representation space Γ, it follows that the term $[X_i, X_j]\, R - X_i\, [X_j, R]$ is linear in the generators of $\mathfrak{s}$ and of degree $(k-1)$ in the Y_j's, while $[P_i, X_j]$ does not involve generators of $\mathfrak{s}$. Comparing now with the right-hand side of (15), the condition again separates into two parts:

$$\begin{aligned} [X_i, X_j]\; R - X_i\, [X_j, R] &= C_{ij}^k X_k R, \\ [P_i, X_j] &= C_{ij}^k P_k. \end{aligned} \tag{16}$$

Simplifying the first equations shows that

$$X_i\, [X_j, R] = 0,$$

hence implying that R also commutes with the generators of the Lie algebra. As R corresponds simultaneously to an invariant polynomial of the radical, it must correspond to an invariant of $\mathfrak{g}$ that depends only on the generators of its maximal solvable ideal.[2] The second equation shows that the polynomials P_i transform according to the adjoint representation of the semisimple Lie algebra $\mathfrak{s}$. Supposed that all the conditions are satisfied, we obtain the commutators of the operators X_i' in the enveloping algebra $\mathcal{U}(\mathfrak{g})$ as

$$\begin{aligned} \left[X_i', X_j'\right] &= [X_i R + P_i, X_j R + P_j] \\ &= [X_i R + P_i, X_j R] + [X_i R + P_i, P_j] \\ &= C_{ij}^k X_k R^2 + C_{ij}^k P_k R + [X_i', P_j]\,. \end{aligned} \tag{17}$$

As the X_i' commute with the Y_j, it follows from equation (17) that $[X_i', P_j] = 0$ and therefore that $\left[X_i', X_j'\right] = [X_i, X_j]' R$, showing that the operators reproduce the commutators of $\mathfrak{s}$, up to the invariant factor R. It should be emphasized that R is not necessarily a central element, but an invariant of $\mathfrak{g}$ that solely depends on the generators of the characteristic representation Γ.

It follows in particular from this construction that the operators $\{R, X_1', \dots, X_n'\}$ generate a finite dimensional quadratic algebra $\mathcal{A}$ in the enveloping algebra $\mathcal{U}(\mathfrak{g})$, with commutators

$$[R, X_i'] = 0,\; \left[X_i', X_j'\right] = C_{ij}^k X_k' R, \quad 1 \le i, j, k \le n.$$

Under some specific conditions, these so-called virtual copies of semisimple Lie algebras in enveloping algebras can be used to construct (formal) Hamiltonians with first integrals given by some of the operators X_i'. Let us outline one possibility, based on the branching rules of representations of semisimple Lie algebras. To this extent, we fix a semisimple subalgebra $\mathfrak{s}'$ of the Levi factor $\mathfrak{s}$ of the semidirect sum $\mathfrak{g}$. Further suppose that the adjoint representation $\mathrm{ad}(\mathfrak{s})$ decomposes, as a representation of $\mathfrak{s}'$, as follows

$$\mathrm{ad}(\mathfrak{s}) \downarrow \mathrm{ad}(\mathfrak{s}') + \Gamma_1 + \cdots + \Gamma_s, \tag{18}$$

where $\Gamma = \Gamma_1 + \cdots + \Gamma_s$ is the so-called characteristic representation [20]. Suppose that the trivial representation Γ_0 of $\mathfrak{s}'$ has multiplicity $k > 0$ in the decomposition (18). This means specifically that we can find k generators $\left\{\tilde{X}_1, \dots, \tilde{X}_k\right\}$ of $\mathfrak{s}$ that commute with the subalgebra $\mathfrak{s}'$. Now, by condition (15), for the corresponding operators $\tilde{X}_s$ $(1 \le s \le k)$ we have that

$$\left[\tilde{X}_i', Z\right] = \left[\tilde{X}_i, Z\right]' = 0, \quad Z \in \mathfrak{s}', \tag{19}$$

from which it follows that for any algebraic Hamiltonian $\mathcal{H} \in \mathcal{U}(\mathfrak{s}')$ the integrability condition

$$\left[\tilde{X}_i', \mathcal{H}\right] = [R, \mathcal{H}] = 0,\; 1 \le i \le k \tag{20}$$

is satisfied. On the other hand, by condition (17), it is straightforward to verify that

$$\left[\mathcal{H}, \left[\tilde{X}_i', \tilde{X}_j'\right]\right] = 0. \tag{21}$$

This last identity implies that the terms appearing in the commutator $\left[\tilde{X}_i', \tilde{X}_j'\right]$ also transform according to the trivial representation of the subalgebra $\mathfrak{s}'$.

We conclude that the set $\left\{R, \tilde{X}_1', \dots, \tilde{X}_k'\right\}$ generates a finite-dimensional quadratic algebra in the enveloping algebra $\mathcal{U}(\mathfrak{g})$ that are (formal) first integrals for the Hamiltonian $\mathcal{H}$. Whether or not these integrals are sufficient for guaranteeing (super-)integrability, essentially depends on the subalgebra $\mathfrak{s}'$ and the associated branching rule. In any case, the preceding construction determines the maximal number of operators X_i' that commute with the Hamiltonian $\mathcal{H}$, independently of any realization of the hidden algebra $\mathfrak{g}$ by first-order differential operators. For the case where the characteristic representation Γ does not contain the trivial representation of the subalgebra $\mathfrak{s}'$, i.e., when no generators of $\mathfrak{s}$ simultaneously commute with the elements of $\mathfrak{s}'$, the integrability condition for the operators would not be a consequence of the structure of the enveloping algebra, but the specific consequence of a realization of $\mathfrak{g}$, relating this approach with the first algebraic formulation.

We finally observe that the construction presented here, that depends essentially on the homogeneity of the operators X_i', is specially suitable for semidirect sums admitting a nonvanishing centre and the class of one-dimensional non-central extensions of double inhomogeneous Lie algebras [21, 22], while the argument is not valid whenever the Levi factor $\mathfrak{s}$ and the radical do not have nonconstant invariants in common. Due to this obstruction, it is formally conceivable to propose a generalized construction by skipping the homogeneity assumption. It should however be taken into account that using operators of different degrees in (13) may lead to incompatibilities in the commutators, as equations (14)-(16) cease to hold, and more general constraints depending on the particular degrees of each P_i would be required. If and under what specific assumptions a solution can be found for a generalized inhomogeneous set of generators (13), is still an unanswered question that is currently being studied in detail.

[2] This fact actually provides information concerning the dimension of the characteristic representation Γ in the semidirect sum.

5. Conclusions

Two possible approaches to the problem of determining quadratic algebras as subalgebras of the enveloping algebra of a Lie algebra have been commented. The first approach corresponds to an algebraic abstraction of already known systems, which are analyzed purely from the perspective of the Hamiltonian and the integrals as the image by a realization of differential operators of elements in some enveloping algebra, trying to determine to which extent such integrals are realization-dependent [10]. In a the second algebraic formulation, commutants of subalgebras $\mathfrak{g}' \subset \mathfrak{g}$ in the enveloping algebra of $\mathfrak{g}$ are considered, from which quadratic algebras formed by polynomials that commute with a given algebraic Hamiltonian defined in $\mathcal{U}(\mathfrak{g}')$ are deduced. In order to simplify the computations in the enveloping algebra, distinguished elements in the commutant can be deduced from the coadjoint representation. For the subalgebras found with this method, a realization by vector fields of an appropriate number of variables automatically provides a (super-)integrable system for the given Hamiltonian [9]. The method of virtual copies, initially introduced in the context of invariant theory, provides an additional approach that combines elements of the two algebraic formulations, and refers to a number of still open problems, such as the general solution of the embedding problem of Lie algebras into enveloping algebras [16], as well the classification problem of realizations of Lie algebras in terms of differential operators [23]. Whether these approaches are compatible or can be combined with other procedures like the quadratic deformations of Lie algebras or the formalism of Racah algebras (see e.g. [8, 24, 25] and references therein) is a problem worthy to be inspected. We hope to report on some progress in these directions in a near future.

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