Mohr-Coulomb Failure Condition and the Direct Shear Test Revisited

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An alternative critical plane orientation is proposed in the Mohr-Coulomb failure criterion for soils with an extreme property. Parameter identification from the direct shear test is extended to incude the lateral normal stress.

Keywords: soil strength, shear failure, direct shear test.

1 Introduction

The Coulomb failure condition is defined by the equation

$$f = \tau + \sigma \tan\varphi - c = 0 \tag{1}$$

where τ and σ are the shear and normal traction components respectively on the critical plane in the material, c is the apparent cohesion and φ is the angle of shearing resistance (internal friction). The usual sign convention is used for the normal stress σ, compression is negative. In the classical Mohr-Coulomb formulation, the critical plane normal is inclined by the angle $\alpha = \pi/4 - \varphi/2$ from the σ_1 direction to the σ_3 direction. Ordered principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are assumed. This orientation of the plane follows from the postulated condition that the Mohr circle in the $\sigma_1 - \sigma_3$ plane touches the envelope (1) as shown in Fig. 1. Stresses σ_{cx}, σ_{cz} and τ_c are implied in the coordinate frame associated with the critical plane. The Mohr-Coulomb condition is natural but the assumed orientation of the critical plane in fact lacks a rigorous substantiation. Other orientations could be assumed. A rational modification of the Mohr-Coulomb condition can be obtained when the critical plane orientation is not a priori restrained. Instead, it can be determined so that f attains its maximum on the critical plane. The resulting criterion should be more severe than the classical one.

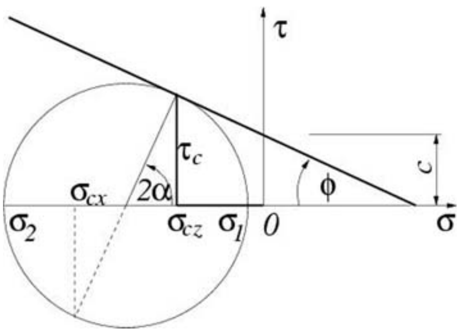

Fig. 1: Inclination α of the critical plane in the classical Mohr-Coulomb yield condition

2 Mohr-Coulomb criterion based on an extreme property

Direct notation is used in the development, and a general triaxial stress is assumed for full generality. Stress tensor $\boldsymbol{\sigma}$ is assumed to have principal stresses σ_i with direction vectors $\boldsymbol{n}_i$. The unknown critical plane normal is denoted $\boldsymbol{n}$. The normal and tangential traction components on the plane are

$$\sigma = \boldsymbol{n}\cdot\boldsymbol{n} \quad \tau = \sqrt{\boldsymbol{n}\cdot\boldsymbol{\sigma\sigma}\,\boldsymbol{n} - \sigma^2}. \tag{2}$$

The extreme of f is sought when $\boldsymbol{n}$ is subject to variation with subsidiary condition $\boldsymbol{n}\cdot\boldsymbol{n} = 1$. Lagrange multiplier λ is introduced and the extended criterion $f' = f + \lambda(\boldsymbol{n}\cdot\boldsymbol{n} - 1)$ is differentiated with respect to $\boldsymbol{n}$ to yield

$$\frac{1}{2\tau}(-4(\boldsymbol{n}\cdot\boldsymbol{\sigma n})\,\boldsymbol{\sigma n} + 2\boldsymbol{\sigma\sigma n}) - 2\boldsymbol{\sigma n}\tan\varphi + 2\lambda\boldsymbol{n} = \boldsymbol{0}. \tag{3}$$

The equation is contractively multiplied by $\boldsymbol{n}$ and the resulting scalar equation is used to eliminate λ from Eq. 3. Assuming that $\tau \neq 0$, equation

$$\begin{aligned}&\{\boldsymbol{\sigma\sigma} - 2(\boldsymbol{n}\cdot\boldsymbol{\sigma n} + \tau\tan\varphi)\,\boldsymbol{\sigma} - \\ &-[\boldsymbol{n}\cdot\boldsymbol{\sigma\sigma n} - 2((\boldsymbol{n}\cdot\boldsymbol{\sigma n})^2 + \boldsymbol{n}\cdot\boldsymbol{\sigma n}\,\tau\tan\varphi)]\}\,\boldsymbol{n} = \boldsymbol{0}\end{aligned} \tag{4}$$

is obtained for the unknown $\boldsymbol{n}$. The equation can be rewritten in a comprehensive form when tensor $\boldsymbol{\rho}$ is introduced

$$\boldsymbol{\rho} = \boldsymbol{\sigma\sigma} - 2(\boldsymbol{n}\cdot\boldsymbol{\sigma n} + \tau\tan\varphi)\,\boldsymbol{\sigma}, \tag{5}$$

$$(\boldsymbol{\rho} - \boldsymbol{n}\cdot\boldsymbol{\rho n})\,\boldsymbol{n} = \boldsymbol{0}. \tag{6}$$

The eigenvectors of $\boldsymbol{\rho}$ deliver extremes of f. Eigenvectors of $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are the same, however, so these extremes are minima ($\tau = 0$) of f. In order to find the other extremes, all variables are decomposed in terms of the eigenvalues σ_i and principal vectors $\boldsymbol{n}_i$ of $\boldsymbol{\sigma}$:

$$\begin{aligned}\boldsymbol{\sigma} &= \sum_i \sigma_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \\ \boldsymbol{\sigma\sigma} &= \sum_i \sigma_i^2 \boldsymbol{n}_i \otimes \boldsymbol{n}_i \\ \boldsymbol{n} &= \sum_i n_i\,\boldsymbol{n}_i\end{aligned} \tag{7}$$

$$\begin{aligned}s &= \boldsymbol{n}\cdot\boldsymbol{\sigma n} = \sum_i \sigma_i n_i^2 \\ \boldsymbol{n}\cdot\boldsymbol{\sigma\sigma n} &= \sum_i \sigma_i^2 n_i^2 \\ \tau &= \sqrt{\sum_i \sigma_i^2 n_i^2 - \left(\sum_i \sigma_i n_i^2\right)^2}\end{aligned} \tag{8}$$

$$\begin{aligned}\boldsymbol{\rho} &= \sum_i r_i \boldsymbol{n}_i \otimes \boldsymbol{n}_i \\ r_i &= \sigma_i(\sigma_i - 2(s + \tan\varphi\tau)).\end{aligned} \tag{9}$$

Equation (6) becomes

$$(r_i - \sum_j r_j n_j^2)\, n_i = 0 \quad i = 1, 3. \tag{10}$$

Six relevant solutions can be best presented in terms of the cyclic permutations of indices i, j, k:

$$n_i, n_j = \sqrt{\frac{1}{2}\left(1 \pm \frac{\tan\varphi}{\sqrt{4+\tan^2\varphi}}\right)}, \quad n_k = 0. \tag{11}$$

It is apparent that the critical plane normal lies always in the plane of two principal stresses directions, in the same plane as the classical Mohr-Coulomb normal. Back substitutions yield then

$$\tau = \frac{\sigma_i - \sigma_j}{\sqrt{4+\tan^2\varphi}}, \quad \sigma = \frac{1}{2}(\sigma_i + \sigma_j + \tau\tan\varphi) \tag{12}$$

and the modified Coulomb condition on the critical plane of maximum f:

$$f = \frac{1}{2}\left[(\sigma_i - \sigma_j)\frac{2+\tan\varphi}{\sqrt{4+\tan^2\varphi}} + (\sigma_i + \sigma_j)\tan\varphi\right] - c = 0 \tag{13}$$

It is interesting to compare the equation with the original Mohr-Coulomb condition

$$\frac{1}{2}\left[(\sigma_i - \sigma_j)\sqrt{1+\tan^2\varphi} + (\sigma_i + \sigma_j)\tan\varphi\right] - c = 0, \tag{14}$$

and with the Coulomb condition (1) applied on the plane of the maximum shear stress

$$\frac{1}{2}[(\sigma_i - \sigma_j) + (\sigma_i + \sigma_j)\tan\varphi] - c = 0. \tag{15}$$

The latter condition represents the third option for the critical plane orienation. For plane stress conditions $\sigma_2 = 0$ the graphic representation of all three yield locuses is in Fig. 2. The modified yield locus is the most severe, as expected.

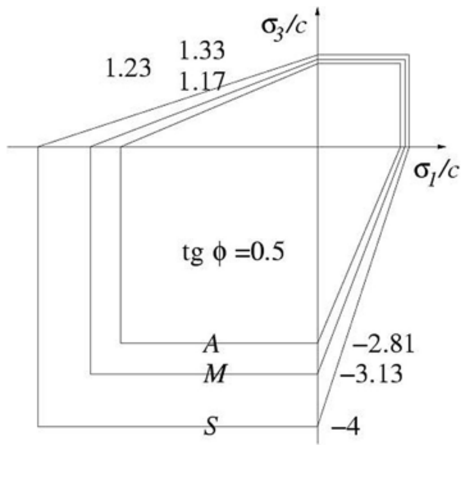

Fig. 2: The modified Mohr-Coulomb (A), the original Mohr-Coulomb (M) and the maximum shear plane condition (S) for $\tan\varphi = 0.5$, $\varphi = 26.6°$

Intersections of the modified (A) and classical (M) yield locuses with the rendulic plane $\sigma_1 = \sigma_2$ are shown in Fig. 3. Functions f_A and f_M are important for parameter calibration

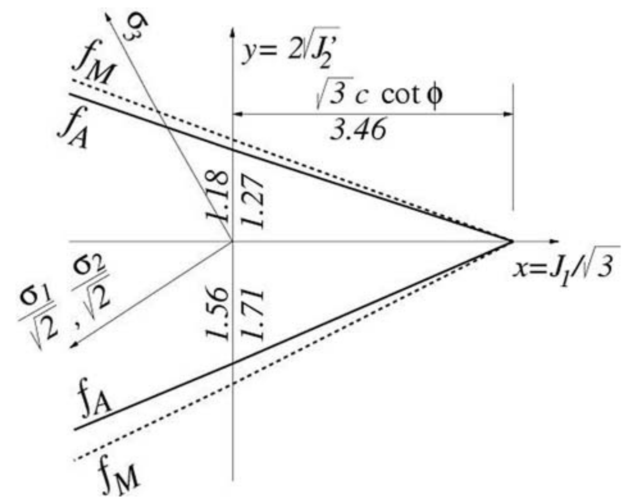

Fig. 3: The modified Coulomb (A) and the original Mohr-Coulomb (M) for $\tan\varphi = 0.5$, ($\varphi = 26.6°$), $c = 1$ in the rendulic plane $\sigma_1 = \sigma_2$. Axis σ_1 and projection of axes σ_1 and σ_2 are also shown.

of the model by the triaxial test. The modified Coulomb (A) condition intersection with the rendulic plane is:

$$f_A(x) = \frac{2\sqrt{2/3\,(4+\tan^2\varphi)}(-3c + x\sqrt{3}\tan\varphi)}{\pm(6+3\tan\varphi) + \tan\varphi\sqrt{4+\tan^2\varphi}} \tag{16}$$

whereas for the original Mohr-Coulomb

$$f_M(x) = \frac{2\sqrt{2/3}(-3c + x\sqrt{3}\tan\varphi)}{\pm\tan\varphi + 3\sqrt{1+\tan^2\varphi}}. \tag{17}$$

Positive signs pertain to the lower branches of the yield locus intersections with the rendulic plane.

The three options for the critical plane orientation distinguish three slightly different material models of the Mohr--Coulomb type. The practical value of these modifications can be assessed in connection with the solutions of actual problems. The problem tackled below is the parameter identification in the direct shear test.

3 Evaluation of the direct shear test

Most applications of constitutive equations include a) the parameter calibration and b) solution of the actual task analytically or numerically. Let us assume first that the triaxial test is used in the first step. Tests provide points in the rendulic plane and parameters c and $\tan\varphi$ are selected to best fit the points. Other procedures are available for identifying of the parameters using, for instance, the modified and alternate Mohr-Coulomb diagrams as recommended in [1] and [4]. Different parameter values are obtained for the three versions of the yield locus. The calibrated locus remains nearly the same for all versions, however. Application of the three versions in any actual problem solution does not thus make any difference in the results, in spite of the difference in the parameter values.

Differences might occur when direct shear apparatus is used in the first step, see Fig. 4. The failure plane orientation is imposed by the test arrangement in this case. Strictly speaking, there is no homogeneous stress in the specimen and from this point of view the test is not suitable for direct parameter calibration. Nevertheless, in a layer adjacent to the failure plane approximately homogeneous stress conditions can be assumed. It is assumed forthwith that normal stress σ_z and the corresponding limit shear stress τ are determined in the direct shear test in the failure plane, see Fig. 4. Corresponding techniques are specified, e.g., in [2] or [3]. The point is that the failure plane in this test does not coincide with the critical plane in the Mohr-Coulomb failure condition for any of the three versions considered here. Consequently, the line obtained by fitting the $\sigma_z{:}\tau$ points from the test is not the Mohr-Coulomb failure condition.

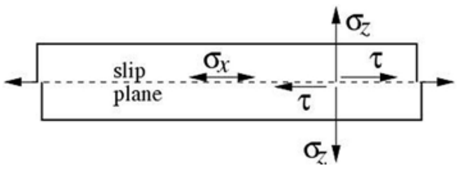

Fig. 4: Stress components at yield in the direct shear test

The confining normal stress σ_x in the slip direction is unknown. The Coulomb condition (1) does not depend on the latter stress. The Mohr-Coulomb condition with both its modifications (14)–(16) however, depends on both principal stresses in the problem plane and therefore depends on σ_z and σ_x. Consequently, the direct shear test cannot directly determine c and φ since σ_x is not known.

The arrangement of the shear test admits the approximate assumption of proportionality between the confining and active stresses $\sigma_x = \mu\sigma_z$ with constant parameter μ. The direct shear test can now be simulated with the three versions of the failure criterion.

Assuming stress components $\sigma_z, \sigma_x = \mu\sigma_z$, and τ in the layer adjacent to the slip plane at failure, standard expressions for the principal stresses are substituted in the respective failure criterion and explicit formulas for τ are derived. These formulas represent the correct failure limits. The respective limits read, for the modified Mohr-Coulomb:

$$\tau = \frac{1}{2+\tan\varphi} \times$$
$$\times \left\{ \sigma_z^2 \left[(1-\mu)^2 (4 + 4\tan\varphi) + (1+\mu)^2 \tan^4\varphi \right] + \right. \tag{18}$$
$$\left. + \left[-\sigma_z (1+\mu)\tan\varphi + c \right] 4c\,(1+\tan^2\varphi) \right\}^{\frac{1}{2}},$$

for the classical Mohr-Coulomb:

$$\tau = \frac{1}{2}\left\{ 4\sigma_z^2\mu + \right.$$
$$\left. + \frac{1}{1+\tan^2\varphi}\left[-\sigma_z^2(1+\mu)^2 - 4c\tan\varphi\,\sigma_z(1+\mu) \right] \right\}^{\frac{1}{2}}, \tag{19}$$

and for the Coulomb on the maximum shear plane:

$$\tau = \frac{1}{2}\left\{ \sigma_z^2 \left[(1-\mu)^2 + (1+\mu)^2 \tan^2\varphi \right] + 4c^2 \right\}^{\frac{1}{2}}. \tag{20}$$

Each criterion can be perceived as a batch of curves $\tau(\sigma_z)$ with parameter μ. Standard evaluation of this fictitious test would deliver a straight line from the point $\sigma_z = 2$ on the horizontal axis to the point $\tau = 1$ on the vertical axis – the conventional Mohr-Coulomb envelope. It is apparently wrong to use in the Mohr-Coulomb material models the parameter values obtained in the direct shear test by the standard evaluation. Instead, the three parameters c, φ and μ should be determined to best fit the measured data.

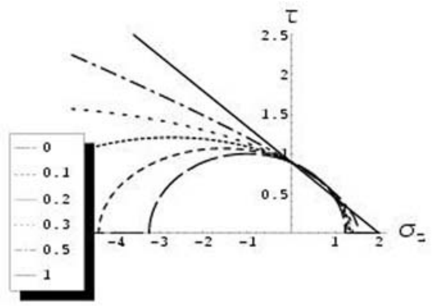

Fig. 5: Mohr-Coulomb for $\tan\varphi = 0.5$, ($\varphi = 26.6°$), $c = 1$ and several values of parameter μ

Low values $\mu < 0.3$ obviously are not realistic. The other extreme, $\mu = 1$, is closest to the conventional Mohr-Coulomb envelope that would be obtained by the standard test evaluation with the same material. However, not even this extreme curve coincides with the conventional envelope except for the maximum shear orientation of the critical plane in Fig. 7. It is worth noting that the introduction of parameter μ allows for curved locuses, which are often observed in practice [1], and that parameter μ, a side product of the parameter fitting, can be used to determine the elastic properties of the soil.

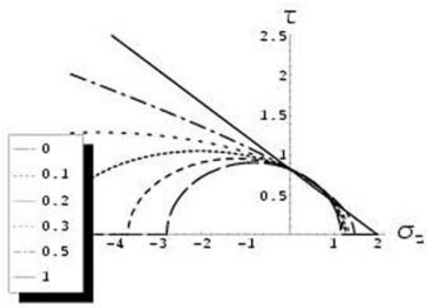

Fig. 6: Modified Mohr-Coulomb for $\tan\varphi = 0.5$, ($\varphi = 26.6°$), $c = 1$ and several values of parameter μ

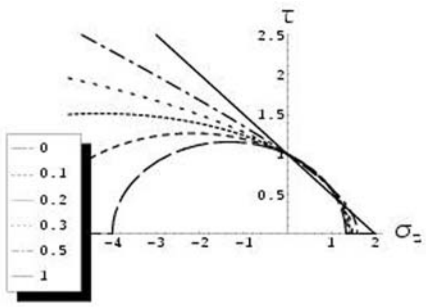

Fig. 7: Coulomb for the maximum shear plane $\tan\varphi = 0.5$, ($\varphi = 26.6°$), $c = 1$ and several values of parameter μ

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