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NOTE ON THE PROBLEM OF MOTION OF VISCOUS FLUID AROUND A ROTATING AND TRANSLATING RIGID BODY

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Abstract. We consider the linearized and nonlinear systems describing the motion of incompressible flow around a rotating and translating rigid body $\mathcal{D}$ in the exterior domain $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{D}}$, where $\mathcal{D} \subset \mathbb{R}^3$ is open and bounded, with Lipschitz boundary. We derive the L^∞-estimates for the pressure and investigate the leading term for the velocity and its gradient. Moreover, we show that the velocity essentially behaves near the infinity as a constant times the first column of the fundamental solution of the Oseen system. Finally, we consider the Oseen problem in a bounded domain $\Omega_R := B_R \cap \Omega$ under certain artificial boundary conditions on the truncating boundary ∂B_R, and then we compare this solution with the solution in the exterior domain Ω to get the truncation error estimate.

Keywords: Incompressible fluid, rigid body, exterior domain, estimates of pressure, leading terms, artificial boundary conditions.

1. Introduction

The boundary problem of Navier–Stokes equations describing flows past a rigid body translating with a constant velocity (with or without rotation) is one of the challenging problems in fluid mechanics. In recent decades, much effort has been made to analyze the properties of solutions of both stationary and non-stationary solutions, both linear and nonlinear mathematical models, both in the whole space and in exterior domains. The difficulty which arises in this type of problem is the variability of the spatial domain in time. To solve it there are two possibilities: (i) to study a problem in the time dependent domain, see Conca, Starovoitov and Tucsnak [1], Desjardins and Esteban [2], Gunzburger, Lee and Seregin [3], Hoffman and Starovoitov [4], etc. (ii) to use a transformation in order to transform the spatial domain varying in time in to a fixed domain. For this approach the global or local transformation can be applied. *The global linear transformation* implies that the whole space is rigidly rotated and shifted back to its original position at each time $t > 0$ (cf. [5]). The equations of motion of the fluid-rigid body system is in a frame attached to the rigid body, with its origin in the center of mass of the latter and coinciding with an inertial frame at time $t = 0$. (Works related to this type of transformation see [6–12]). *The local transformation* implies that the change of variables only acts in a bounded neighbourhood of the body, the solenoidal condition of the fluid velocity are preserved and the regularity of the solution are not changed. See e.g. works of Tucsnak, Cumsille and Takahashi (cf. [13–15]).

1.1. Formulation of the problem

Let us formulate our problem in the fixed domain, which is a result of applying the global linear transformation, for more details, see [5]. The systems of equations are as follows

$$\begin{aligned} -\Delta u(z) + \tau\partial_1 u(z) - (\omega \times z)\cdot\nabla u(z) + \omega \times u(z) & \\ +\tau(u(z)\cdot\nabla)u(z) + \nabla\pi(z) &= F(z) \\ \operatorname{div} u(z) = 0 \text{ for } z \in \Omega & \end{aligned} \tag{1.1}$$

$$\begin{aligned} -\Delta u(z) + \tau\partial_1 u(z) - (\omega \times z)\cdot\nabla u(z) + \omega \times u(z) + \nabla\pi(z) &= F(z) \\ \operatorname{div} u(z) = 0 \text{ for } z \in \Omega & \end{aligned} \tag{1.2}$$

where $\mathcal{D} \subset \mathbb{R}^3$ is open and bounded, with Lipschitz boundary. The systems (1.1) and (1.2) together with some boundary conditions on $\partial\Omega = \partial\mathcal{D}$ constitute the mathematical models (linear and non-linear, respectively) describing the stationary flow of a viscous incompressible fluid around a rigid body which moves at a constant velocity and rotates at a constant angular velocity. In this study we consider that the rotation is parallel to the velocity at infinity. (For more details concerning the derivation of the model, see [5, 7]. The description and the analysis in the case where the rotation is not parallel to the velocity at infinity can be found in the following works, see [16, 17]).

The aim is to obtain the L^∞ estimates for the pressure in the linear and nonlinear cases, since such estimates are missing in the literature. Only the estimates of the velocity field and the gradient of the velocity field in L^∞ are available. This implies that complete information about the decay of the solution (u, π) of the systems (1.1), (1.2) for $|x| \to \infty$. (For other works see [18], [19].)

Second, we are interested in the "Leray solutions" of (1.1), supplemented by a decay condition at infinity,

$$u(x) \to 0 \text{ for } |x| \to \infty, \tag{1.3}$$

and the suitable boundary conditions on $\partial\Omega$. Weak solutions are characterized by the conditions $u \in L^6(\Omega)^3 \cap W^{1,1}_{loc}(\Omega)^3$, $\nabla u \in L^2(\Omega)^9$ and $\pi \in L^2_{loc}(\Omega)$.

From [18] and [20], it follows that the velocity part of the Leray solution (u, π) in (1.1) and (1.3) decays for $|x| \to \infty$ as the estimates express below

$$|u(x)| \le C\left(|x|\, s(x)\right)^{-1}, \quad |\nabla u(x)| \le C\left(|x|\, s(x)\right)^{-3/2} \tag{1.4}$$

for $x \in \mathbb{R}^3$ with $|x|$ sufficiently large, where $s(x) := 1 + |x| - x_1$ $(x \in \mathbb{R}^3)$ and $C > 0$ a constant independent of x. The factor $s(x)$ may be considered as a mathematical manifestation of the wake extending downstream behind a body moving in a viscous fluid. In the work by M. Kyed, (see [21]) it was shown that

$$u_j(x) = \gamma\, E_{j1}(x) + R_j(x), \quad \partial_l u_j(x) = \gamma\, \partial_l E_{j1}(x) + S_{jl}(x) \ \ (x \in \overline{\mathfrak{D}}^c,\ 1 \le j, l \le 3), \tag{1.5}$$

where $E : \mathbb{R}^3 \backslash \{0\} \mapsto \mathbb{R}^4 \times \mathbb{R}^3$ denotes a fundamental solution to the Oseen system

$$-\Delta v + \tau\, \partial_1 v + \nabla \varrho = f, \ \ \operatorname{div} v = 0 \quad \text{in } \mathbb{R}^3. \tag{1.6}$$

The term $E_{j1}(x)$ can be expressed explicitly in terms of elementary functions. The coefficient γ is also given explicitly, its definition involving the Cauchy stress tensor. The remaining terms R and S are characterized by the relations $R \in L^q(\Omega)^3$ for $q \in (4/3, \infty)$, $S \in L^q(\Omega)^3$ for $q \in (1, \infty)$. From [22, Section VII.3] it is known that $E_{j1}|B^c_r \notin L^q(B^c_r)$ for $r > 0$, $q \in [1, 2]$, and $\partial_l E_{j1}|B^c_r \notin L^q(B^c_r)$ for $r > 0$, $q \in [1, 4/3]$, $j, l \in \{1, 2, 3\}$. The function R decays faster than E_{j1}, and S_{jl} decays faster than $\partial_l E_{j1}$, in the sense of L^q-integrability. Thus, the equations in (1.5) can be viewed in fact as asymptotic expansions of u and ∇u, respectively. Let us mention that the result in [21] are valid under the assumption that u verifies the boundary conditions

$$u(x) = e_1 + (\omega \times x) \quad \text{for } x \in \partial\Omega, \tag{1.7}$$

which is not our case.

Reference [21] does not deal with L^∞-decay of R and S, nor does it indicate whether $S = \nabla R$.

Below, in Theorem 4.1 we derive an L^∞-decay of u and ∇u respectively, which is *independent on the boundary conditions*. However, in comparison with [21] and indicated in (1.5), our leading term is less explicit than the term $\gamma\, E_{j1}(x)$ in (1.5) and instead of the fundamental solution $E_{j1}(x)$ of the stationary Oseen system, we use the time integral of the fundamental solution of the evolutionary Oseen system.

In [23] it was proved that $\mathcal{Z}_{j1}(x, 0) = E_{j1}(x)$ for $x \in \mathbb{R}^3 \backslash \{0\}$, $1 \le j \le 3$, and $\lim_{|x|\to\infty} |\partial^\alpha_x \mathcal{Z}_{jk}(x, 0)| = O\big((|x|\, s(x))^{-3/2-|\alpha|/2} \big)$ for $1 \le j \le 3$, $k \in \{2, 3\}$ ([23, Corollary 4.5, Theorem 5.1]). Thus, setting

$$\mathfrak{G}_j(x) := \sum_{k=2}^{3} \beta_k\, \mathcal{Z}_{jk}(x, 0) + \mathfrak{F}_j(x) \quad (x \in \overline{B_{S_1}}^c,\ 1 \le j \le 3), \tag{1.8}$$

we may obtain from (4.3) that

$$u_j(x) = \beta_1\, E_{j1}(x) + \Big(\int_{\partial\Omega} u \cdot n\, do_x \Big)\, x_j\, (4\,\pi\, |x|^3)^{-1} + \mathfrak{G}_j(x) \quad (x \in \overline{B_{S_1}}^c,\ 1 \le j \le 3) \tag{1.9}$$

and

$$\lim_{|x|\to\infty} |\partial^\alpha \mathfrak{G}(x)| = O\big((|x|\, s(x))^{-3/2-|\alpha|/2}\, \ln(2 + |x|) \big) \quad \text{for } \alpha \in \mathbb{N}^3_0 \text{ with } |\alpha| \le 1 \tag{1.10}$$

(Theorem 4.2, Corollary 4.3).

Comparing the coefficient γ from (1.5) in the work [21] with the coefficient β_1 from (1.9) in [24], see Theorem 4.1 below, and taking into account the boundary condition (1.7) in [21], it follows that γ and β_1 are equal.

Third, we are solving the linear system (1.2) in a truncation $\Omega_R := B_R \cap \Omega$ of the exterior domain $\mathbb{R}^3 \setminus \overline{\mathcal{D}}$ under certain artificial boundary conditions on the truncating boundary ∂B_R. Then we compare this solution with the solution of (1.2) in the exterior domain, i.e. to find the error estimates of the method of an artificial boundary condition. For this aim we use L^∞-estimates of the velocity and of the pressure.

2. Definitions and notation

Let us define

$s(y) := 1 + |y| - y_1$ for $y \in \mathbb{R}^3$, $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{D}}$, $\Omega_R := B_R \cap \Omega$, $B_R^c := \mathbb{R}^3 \setminus \overline{B_R}$,

where $B_R := \{x \in \mathbb{R}^3; |x| < R\}$, for $R > 0$ such that $B_R \supset \overline{\mathcal{D}}$.
So, Ω_R is the truncation of the exterior domain $\Omega = \mathbb{R}^3 \setminus \overline{\mathcal{D}}$ by the ball B_R. The boundary Ω_R consists of parts $\partial\Omega$ and ∂B_R, the later we call the truncating boundary.

Fix $\tau \in (0, \infty)$, $e_1 := (1, 0, 0)$, $\omega = |\omega| e_1$, $|\omega| \neq 0$, and $\Omega := |\omega| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$.

So, $\Omega \cdot z = \omega \times z$ for $z \in \mathbb{R}^3$. For $U \subset \mathbb{R}^3$ open, $u \in W^{2,1}_{\mathrm{loc}}(U)^3$, $z \in U$, put

$$(Lu)(z) := -\Delta u(z) + \tau \partial_1 u(z) - (\omega \times z) \cdot \nabla u(z) + \omega \times u(z),$$
$$(L^* u)(z) := -\Delta u(z) - \tau \partial_1 u(z) + (\omega \times z) \cdot \nabla u(z) - \omega \times u(z).$$

Put $N(x) := (4\pi |x|)^{-1}$ for $x \in \mathbb{R}^3 \backslash \{0\}$ ("Newton potential", fundamental solution of the Poisson equation in $\mathbb{R}^3$), $\mathfrak{O}(x) := (4\pi |x|)^{-1} e^{-\tau (|x| - x_1)/2}$ for $x \in \mathbb{R}^3 \backslash \{0\}$ (fundamental solution of the scalar Oseen equation $-\Delta v + \tau \partial_1 v = g$ in $\mathbb{R}^3$),

$$\text{Put } K(z,t) := (4\pi t)^{-3/2} e^{-|z|^2/(4t)} \quad (z \in \mathbb{R}^3, t \in (0,\infty)),$$
$$\Lambda(z,t) := \left(K(z,t)\delta_{jk} + \partial z_j \partial z_k \left(\int_{\mathbb{R}^3} (4\pi |z-y|)^{-1} K(y,t) dy \right) \right)_{1 \le j,k \le 3} (z \in \mathbb{R}^3, \ t > 0),$$
$$\Gamma(x,y,t) := \Lambda(x - \tau t e_1 - e^{-t\Omega} y, t) \cdot e^{-t\Omega},$$
$$\widetilde{\Gamma}(x,y,t) := \Lambda(x + \tau t e_1 - e^{t\Omega} y, t) \cdot e^{t\Omega} \quad (x, y \in \mathbb{R}^3, t > 0),$$
$$\mathcal{Z}(x,y) := \int_0^\infty \Gamma(x,y,t) dt, \ \widetilde{\mathcal{Z}}(x,y) := \int_0^\infty \widetilde{\Gamma}(x,y,t) dt, \quad (x, y \in \mathbb{R}^3, \ x \neq y).$$

$\psi(r) := \int_0^r (1 - e^{-t}) \, t^{-1} \, dt$ $(r \in \mathbb{R})$, $\Phi(x) := (4\pi\tau)^{-1} \psi\big(\tau (|x| - x_1)/2\big)$ $(x \in \mathbb{R}^3)$,

$E_{jk}(x) := (\delta_{jk} \Delta - \partial_j \partial_k) \Phi(x)$, $E_{4k}(x) := x_k (4\pi |x|^3)^{-1}$ $(x \in \mathbb{R}^3 \backslash \{0\}, \ 1 \le j,k \le 3)$ (fundamental solution of the Oseen system (1.6), with $(E_{jk})_{1 \le j,k \le 3}$ the velocity part and $(E_{4k})_{1 \le k \le 3}$ the pressure part).

For $q \in (1,2)$, $f \in L^q(\mathbb{R}^3)^3$, put

$$\mathcal{R}(f)(x) := \int_{\mathbb{R}^3} \mathcal{Z}(x,y) f(y) dy \quad (x \in \mathbb{R}^3);$$

see [25, Lemma 3.1].

We will use the space $D_0^{1,2}(\Omega)^3 := \{v \in L^6(\Omega)^3 \cap H^1_{loc}(\Omega)^3 \, : \, \nabla v \in L^2(\Omega)^9, \ v|\partial\Omega = 0\}$ equipped with the norm $\|\nabla u\|_2$, where $v|\partial\Omega$ means the trace of v on $\partial\Omega$. For $p \in (1,\infty)$, define M_p as the space of all pairs of functions (u, π) such that $u \in W^{2,p}_{loc}(\Omega)^3$, $\pi \in W^{1,p}_{loc}(\Omega)$,

$$u|\Omega_R \in W^{1,p}(\Omega_R)^3, \quad \pi|\Omega_R \in L^p(\mathcal{D}_R), \quad u|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3,$$
$$\operatorname{div} u|\Omega_R \in W^{1,p}(\Omega_R), \quad L(u) + \nabla\pi|\Omega_R \in L^p(\Omega_R)^3$$

for some $R \in (0, \infty)$ with $\overline{\Omega}^c \subset B_R$.

We write C for generic constants. In order to romove possible ambiguities, we sometimes use the notation $C(\gamma_1, ..., \gamma_n)$ in order to indicate that the constant in question depends particularly on $\gamma_1, ..., \gamma_n \in (0,\infty)$, for some $n \in \mathbb{N}$. But the relevant constant may depend on other parameters as well.

3. Decay estimates

In the first part of this section, we recall some known results from [25] and [26] about the decay of the velocity part of the solution of the system (1.2). In order to get the full decay characterization of the solution, we derive the decay of the pressure part of the solution of (1.2). In the second part of this section, we extend the result for the pressure to the non-linear case of (1.1).

3.1. Decay estimates in the linear case

Our starting point is a decay result from [26] for the velocity part u of a solution to (1.2).

Theorem 3.1. ([26, Theorem 3.12]) *Suppose that Ω^c is C^2-bounded. Let $p \in (1,\infty)$, $(u,\pi) \in M_p$. Put $F = L(u) + \nabla\pi$. Suppose there are numbers $S_1, S, \gamma \in (0,\infty)$, $A \in [2,\infty)$, $B \in \mathbb{R}$ such that $S_1 < S$,*

$$\Omega^c \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_1}, \quad u|B_S^c \in L^6(B_S^c)^3, \quad \nabla u|B_S^c \in L^2(B_S^c)^9,$$

$$A + \min\{1,B\} \geq 3, \ |F(z)| \leq \gamma |z|^{-A} s(z)^{-B} \ \textit{for } z \in B_{S_1}^c.$$

Then

$$|u(y)| \leq C\,(|y|s(y))^{-1}\, l_{A,B}(y), \tag{3.1}$$

$$|\nabla u(y)| \leq C\,(|y|s(y))^{-3/2}\, s(y)^{\max(0,7/2-A-B)}\, l_{A,B}(y) \tag{3.2}$$

for $y \in B_S^c$, where function $l_{A,B}$ is given by

$$\begin{cases} 1 & \textit{if} \quad A + \min\{1,B\} > 3 \\ \max(1, ln(y)) & \textit{if} \quad A + \min\{1,B\} = 3. \end{cases}$$

Corollary 3.2. *Let $p \in (1,\infty)$, γ, S_1, $S \in (0,\infty)$ with $\Omega^c \subset B_{S_1}$, $S_1 < S$, $A \in [2,\infty)$, $B \in \mathbb{R}$ with $A + \min\{1,B\} \geq 3$. Let $F : \Omega \mapsto \mathbb{R}^3$ be measurable with $F|\Omega_{S_1} \in L^p(\Omega_{S_1})^3$ and $|F(z)| \leq \gamma |z|^{-A} s(z)^{-B}$ for $z \in B_{S_1}^c$.*

Let $u \in W^{1,p}_{loc}(\Omega)^3$ with $u|B_S^c \in L^6(B_S^c)^3$, $\nabla u|B_S^c \in L^2(B_S^c)^9$, $\operatorname{supp}(\operatorname{div} u) \subset B_{S_1}$,

$$\int_{\mathcal{D}^c} \big[\, \nabla u \cdot \nabla\varphi + \big(\, \tau\, \partial_1 u - (\omega \times z) \cdot \nabla u + (\omega \times u) - F \,\big) \cdot \varphi \,\big]\, dz \tag{3.3}$$
$$= 0 \quad \textit{for} \ \ \varphi \in C_0^\infty(\Omega)^3 \ \ \textit{with} \ \operatorname{div}\varphi = 0.$$

Then inequalities (3.1) and (3.2) hold for $y \in B_S^c$.

Moreover $F \in L^q(\Omega)^3$ for $q \in (1,p]$. If $p \geq 6/5$, the function F may be considered as a bounded linear functional on $\mathcal{D}_0^{1,2}(\Omega)^3$, in the usual sense.

Let $\pi \in L^p_{loc}(\Omega)$ with

$$\int_{\mathcal{D}^c} \big[\, \nabla u \cdot \nabla\varphi + \big(\, \tau\, \partial_1 u - (\omega \times z) \cdot \nabla u + (\omega \times u) - F \,\big) \cdot \varphi \tag{3.4}$$
$$-\pi \operatorname{div}\varphi \,\big]\, dz = 0 \quad \textit{for} \ \ \varphi \in C_0^\infty(\Omega)^3.$$

Fix some number $S_0 \in (0,S_1)$ with $\overline{\mathcal{D}} \cup \operatorname{supp}(\operatorname{div} u) \subset B_{S_0}$. Then the relations $u|\overline{B_{S_0}}^c \in W^{2,p}_{loc}(\overline{B_{S_0}}^c)^3$, $\pi \in W^{1,p}_{loc}(\overline{B_{S_0}}^c)$ and $L(u|\overline{B_{S_0}}^c) + \nabla\pi = F|\overline{B_{S_0}}^c$ hold.

The main result of this section, dealing with the L^∞-estimates of the pressure, is stated in

Theorem 3.3. *Let p, γ, S_1, S, A, B, F, u be given as in Corollary 3.2, but with the stronger assumptions $A = 5/2$, $B \in (1/2, \infty)$ on A and B. Let $\pi \in L^p_{loc}(\Omega)$ such that (3.4) holds Then there is $c_0 \in \mathbb{R}$ such that*

$$|\pi(x) + c_0| \leq C\,|x|^{-2} \quad \textit{for} \ \ x \in B_S^c. \tag{3.5}$$

Corollary 3.4. *Let p, γ, S_1, S, A, B, F, u be given as in Corollary 3.2, but with the stronger assumptions $A \geq 5/2$, $A + \min\{1,B\} > 3$ on A and B. Let $\pi \in L^p_{loc}(\Omega)$ such that (3.4) holds. Then there is $c_0 \in \mathbb{R}$ such that inequality (3.5) is valid.*

Proof: Put $B' := A - 5/2 + \min\{1,B\}$. Since $A + \min\{1,B\} > 3$, we have $B' \in (1/2, \infty)$. Moreover, since $A \geq 5/2$, we find for $z \in B_{S_1}^c$ that

$$|F(z)| \leq \gamma\, C(S_1, A)\, |z|^{-5/2}\, s(z)^{-A+5/2-B} \leq \gamma\, C(S_1, A)\, |z|^{-5/2}\, s(z)^{-B'}.$$

Thus the assumptions of Theorem 3.3 are satisfied with B replaced by B' and with a modified parameter γ. This implies the conclusion of Theorem 3.3. □

3.2. Decay estimates in the non-linear case

Let us assume now the non-linear case, i.e. the system (1.1). First, recall the result about the decay properties of the velocity in this non-linear case:

Theorem 3.5. *[20, Theorem 1.1] Let γ, $S_1 \in (0,\infty)$, $p_0 \in (1,\infty)$, $A \in (2,\infty)$, $B \in [0, 3/2]$ with $\Omega^c \subset B_{S_1}$, $A + \min\{B,1\} > 3$, $A + B \geq 7/2$. Take $F : \mathbb{R}^3 \mapsto \mathbb{R}^3$ measurable with $F|B_{S_1} \in L^{p_0}(B_{S_1})^3$,*

$$|F(y)| \leq \gamma \cdot |y|^{-A} \cdot s(y)^{-B} \text{ for } y \in B^c_{S_1}.$$

Let $u \in L^6(\Omega)^3 \cap W^{1,1}_{loc}(\Omega)^3, \pi \in L^2_{loc}(\Omega)$ with $\nabla u \in L^2(\Omega)^9, \operatorname{div} u = 0$ and

$$\int_{\overline{\mathcal{D}}^c} [\nabla u \cdot \nabla\varphi + \tau\partial_1 u - (\omega \times z) \cdot \nabla u + \omega \times u \\ + \tau(u \cdot \nabla)u - F) \cdot \varphi - \pi \operatorname{div}\varphi] \, \mathrm{d}x = 0$$

for $\varphi \in C^\infty_0(\Omega)^3$. Let $S \in (S_1,\infty)$. Then

$$|\partial^\alpha u(x)| \leq C \left(|x|s(x)\right)^{-1-|\alpha|/2} \quad \text{for } x \in B^c_S,\ \alpha \in \mathbb{N}^3_0 \text{ with } |\alpha| \leq 1. \tag{3.6}$$

Now, using Theorems 3.3 and 3.5, we are in the position to prove the result on the decay of the pressure in the non-linear case:

Theorem 3.6. *Consider the situation in Theorem 3.5. Suppose in addition that $A \geq 5/2$. Then there is $c_0 \in \mathbb{R}$ such that inequality (3.5) holds.*

4. Leading term

In this section we study the asymptotic behavior of the velocity profile of the system (1.2). Let us recall known results from [26] and [24].

Theorem 4.1. *Let $\mathcal{D} \subset \mathbb{R}^3$ be open, $p \in (1,\infty)$, $f \in L^p(\mathbb{R}^3)^3$ with supp(f)compact. Let $S_1 \in (0,\infty)$ with $\overline{\mathcal{D}} \cup supp(f) \subset B_{S_1}$, $\Omega = \overline{\mathcal{D}}^c$.*

Let $u \in L^6(\Omega)^3 \cap W^{1,1}_{loc}(\Omega)^3$, $\pi \in L^2_{loc}(\Omega)$ with $\nabla u \in L^2(\Omega)^9$, $div\, u = 0$ and

$$\int_\Omega \Big[\nabla u \cdot \nabla\varphi \ + \ \big(\tau\,\partial_1 u + \tau\,(u\cdot\nabla)u - (\omega\times z)\cdot\nabla u + \omega\times u\big)\cdot\varphi - \pi \; div\; \varphi\Big]\, dz \tag{4.1}$$
$$= \int_\Omega f \cdot \varphi \, dz \quad \text{for } \varphi \in C^\infty_0(\Omega)^3.$$

(This means the pair (u,π) is a Leray solution to (1.2), (1.3).) Suppose in addition that

$$\Omega^c \text{ is } C^2\text{-bounded}, \quad u|\partial\Omega \in W^{2-1/p,\,p}(\partial\Omega)^3, \quad \pi|B_{S_1}\backslash\overline{\mathcal{D}} \in L^p(B_{S_1}\backslash\overline{\mathcal{D}}). \tag{4.2}$$

Let n denote the outward unit normal to Ω, and define

$$\beta_k := \int_\Omega f_k(y)\, dy \\ + \int_{\partial\Omega} \sum_{l=1}^3 \big(-\partial_l u_k(y) + \delta_{kl}\,\pi(y) + (\tau\, e_1 - \omega\times y)_l\, u_k(y) - \tau\,(u_l\, u_k)(y)\,\big)\, n_l(y)\, do_y$$

for $1 \leq k \leq 3$,

$$\mathfrak{F}_j(x) := \int_\Omega \Big[\sum_{k=1}^3 \big(\mathcal{Z}_{jk}(x,y) - \mathcal{Z}_{jk}(x,0)\,\big)\, f_k(y) - \tau\cdot \sum_{k,l=1}^3 \mathcal{Z}_{jk}(x,y)\,(u_l\,\partial_l u_k)(y)\Big]\, dy \\ + \int_{\partial\Omega} \sum_{k=1}^3 \Big[\big(\mathcal{Z}_{jk}(x,y) - \mathcal{Z}_{jk}(x,0)\,\big) \sum_{l=1}^3 \big(-\partial_l u_k(y) + \delta_{kl}\,\pi(y) + (\tau\, e_1 - \omega\times y)_l\, u_k(y)\,\big)\, n_l(y) \\ + \big(E_{4j}(x-y) - E_{4j}(x)\,\big)\, u_k(y)\, n_k(y) \\ + \sum_{l=1}^3 \big(\partial y_l \mathcal{Z}_{jk}(x,y)\,(u_k\, n_l)(y) + \tau \mathcal{Z}_{jk}(x,0)\,(u_l\, u_k\, n_l)(y)\,\big)\Big]\, do_y$$

for $x \in \overline{B_{S_1}}^c$, $1 \le j \le 3$. *The preceding integrals are absolutely convergent. Moreover* $\mathfrak{F} \in C^1(\overline{B_{S_1}}^c)^3$ *and equation*

$$u_j(x) = \sum_{k=1}^{3} \beta_k \, \mathcal{Z}_{jk}(x,0) + \Big(\int_{\partial\Omega} u \cdot n \, do_x \Big) \, x_j \, (4\,\pi\,|x|^3)^{-1} + \mathfrak{F}_j(x). \tag{4.3}$$

holds. In addition, for any $S \in (S_1, \infty)$, *there is a constant* $C > 0$ *which depends on* τ, ω, S_1, S, f, u *and* π, *and which is such that*

$$|\partial^\alpha \mathfrak{F}(x)| \le C \, \big(\, |x| \, s(x) \, \big)^{-3/2 - |\alpha|/2} \, \ln(2 + |x|) \quad \text{for } \ x \in \overline{B_S}^c, \ \alpha \in \mathbb{N}_0^3 \ \text{ with } |\alpha| \le 1. \tag{4.4}$$

Theorem 4.2. *Let* $\mathfrak{D}$, p, f, S_1, u, π *satisfy the assumptions of Theorem 4.1, including (4.2). Let* β_1, β_2, β_3 *and* $\mathfrak{F}$ *be defined as in Theorem 4.1. Define the function* $\mathfrak{G}$ *as*

$$\mathfrak{G}_j(x) := \sum_{k=2}^{3} \beta_k \, \mathcal{Z}_{jk}(x,0) + \mathfrak{F}_j(x) \quad (x \in \overline{B_{S_1}}^c, \ 1 \le j \le 3). \tag{4.5}$$

Then $\mathfrak{G} \in C^1(\overline{B_{S_1}}^c)^3$, *equation*

$$u_j(x) = \beta_1 \, E_{j1}(x) + \Big(\int_{\partial\Omega} u \cdot n \, do_x \Big) \, x_j \, (4\,\pi\,|x|^3)^{-1} + \mathfrak{G}_j(x) \quad (x \in \overline{B_{S_1}}^c, \ 1 \le j \le 3) \tag{4.6}$$

holds, and for any $S \in (S_1, \infty)$, *there is a constant* $C > 0$ *which depends on* τ, ω, S_1, S, f, u *and* π, *and which is such that*

$$|\partial^\alpha \mathfrak{G}(x)| \le C \, \big(\, |x| \, s(x) \, \big)^{-3/2 - |\alpha|/2} \, \ln(2 + |x|) \quad \text{for } \ x \in \overline{B_S}^c, \ \alpha \in \mathbb{N}_0^3 \text{ with } |\alpha| \le 1.$$

Corollary 4.3. *Take* $\mathcal{D}$, p, f, S_1, u, π *as in Theorem 4.1, but without requiring (4.2). (This means that* (u, π) *is only assumed to be a Leray solution of (1.2), (1.3).) Put* $\tilde{p} := \min\{3/2, p\}$. *Then* $u \in W^{2,\tilde{p}}_{loc}(\Omega)^3$ *and* $\pi \in W^{1,\tilde{p}}_{loc}(\Omega)$.

Fix some number $S_0 \in (0, S_1)$ *with* $\overline{\mathcal{D}} \cup supp(f) \subset B_{S_0}$, *and define* β_1, β_2, β_3 *and* $\mathfrak{F}$ *as in Theorem 4.1, but with* $\mathcal{D}$ *replaced by* B_{S_0}, *and* $n(x)$ *by* $S_0^{-1}\,x$, *for* $x \in \partial B_{S_0}$. *Moreover, define* $\mathfrak{G}$ *as in (4.5). Then all the conclusions of Theorem 4.2 are valid.*

5. Formulation of the problem with artificial boundary conditions

Recall that we defined $\Omega_R = B_R \cap \Omega$. We introduce the subspace W_R of $H^1(\Omega_R)$ denoting

$$W_R := \{ v \in H^1(\Omega_R)^3 : v|\partial\Omega = 0 \},$$

where $v|\partial\Omega$ means the trace of v on $\partial\Omega$.

Lemma 5.1. ([27, Lemma 4.1]) *The estimate*

$$\|u\|_2 \le C \, (R \, \|\nabla u\|_2 + R^{1/2} \, \|u|\partial B_R\|_2)$$

holds for $R \in (0, \infty)$ *with* $\Omega^c \subset B_R$ *and for* $u \in W_R$.

We introduce an inner product $(\cdot, \cdot)^{(R)}$ in W_R by defining

$$(v, w)^{(R)} = \int_{\Omega_R} \nabla v \cdot \nabla w \, dx + \int_{\partial B_R} (\tau/2) v \cdot w \, do_x \ \text{for} \ \ v, w \in W_R.$$

The space W_R equipped with this inner product is a Hilbert space. The norm generated by this scalar product $(\cdot, \cdot)^{(R)}$ is denoted by $|\cdot|^{(R)}$, that is

$$|v|^{(R)} := \Big(\|\nabla v\|_2^2 + (\tau/2) \, \|v|\partial B_R\|_2^2 \Big)^{1/2} \ \ \text{for } v \in W_R.$$

We define the bilinear forms

$$\begin{aligned}
&\mathcal{A}_R : H^1(\Omega_R)^3 \times H^1(\Omega_R)^3 \to \mathbb{R},\\
&\mathcal{B}_R : H^1(\Omega_R)^3 \times L^2(\Omega_R) \to \mathbb{R},\\
\mathcal{A}_R(u,w) :=& \int_{\Omega_R} [\nabla u \cdot \nabla w + \tau \partial_1 u \cdot w] dx + \frac{\tau}{2} \int_{\partial B_R} (u(x) \cdot w(x)) \Big(1 - \frac{x_1}{R}\Big)\, do_x,\\
&\int_{\Omega_R} \big[-\big((\omega \times x) \cdot \nabla \big) u(x) + \big(\omega \times u(x) \big) \big] \cdot w(x)\, dx\\
\mathcal{B}_R(w,\sigma) :=& -\int_{\Omega_R} (\operatorname{div} w)\, \sigma dx, + \big(\omega \times u(x) \big) \big] \cdot w(x)\, dx
\end{aligned}$$

for $u, w \in H^1(\Omega_R)^3$, $\sigma \in L^2(\Omega_R)$, $R \in (0,\infty)$ with $\Omega^c \subset B_R$.

Lemma 5.2. *Let $R \in (0,\infty)$ with $\Omega^c \subset B_R$. Then*

$$|\mathcal{A}_R(u,w)| \leq C(R)\, |u|^{(R)}\, |w|^{(R)}$$

for $u, w \in H^1(\Omega_R)^3$.

The key observation in this section is stated in the following lemma, which is the basis of the theory presented in this section.

Lemma 5.3. *Let $R \in (0,\infty)$ with $\Omega^c \subset B_R$, and let $w \in W_R$. Then the equation $(|w|^{(R)})^2 = \mathcal{A}_R(w,w)$ holds.*

Proof: Using the definition $\mathcal{A}_R(\cdot,\cdot)$, we get

$$\begin{aligned}
\mathcal{A}_R(w,w) &= \int_{\mathcal{D}_R} \left[|\nabla w|^2 + \tau \partial_1 \left(\frac{|w|^2}{2} \right) - (\omega \times x) \cdot \nabla \left(\frac{|w|^2}{2} \right) \right] dx\\
&\quad + \frac{\tau}{2} \int_{\partial B_R} |w(x)|^2 \Big(1 - \frac{x_1}{R}\Big)\, do_x\\
&= \int_{\mathcal{D}_R} |\nabla w|^2\, dx + \int_{\partial B_R} \left(\frac{\tau}{2} |w(x)|^2 \frac{x_1}{R} - \frac{1}{2} (\omega \times x) \cdot \frac{x}{R} |w(x)|^2 \right) do_x\\
&\quad + \frac{\tau}{2} \int_{\partial B_R} |w(x)|^2 \Big(1 - \frac{x_1}{R}\Big)\, do_x\\
&= \int_{\mathcal{D}_R} |\nabla w|^2\, dx + \frac{\tau}{2} \int_{\partial B_R} |w(x)|^2 = (|w|^{(R)})^2.
\end{aligned}$$

We applied that $(\omega \times x) \cdot x = 0$ for $x, \omega \in \mathbb{R}^3$. □

As in [28], we obtain that the bilinear form β_R is stable:

Theorem 5.4. ([28, Corollary 4.3]) *Let $R > 0$ with $\Omega^c \subset B_R$. Then*

$$\inf_{\rho \in L^2(\Omega_R), \rho \neq 0} \sup_{v \in W_R, v \neq 0} \frac{\mathcal{B}_R(v,\rho)}{|v|^{(R)} \|\rho\|_2} \geq C(R).$$

We note that functions from $W^{1,1}_{loc}(\Omega)$ with L^2-integrable gradient are L^2-integrable on truncated exterior domains:

Lemma 5.5. [29, Lemma II.6.1] *Let $w \in W^{1,1}_{loc}(\Omega)$ with $\nabla w \in L^2(\Omega)^3$, and let $R \in (0,\infty)$ with $\Omega^c \subset B_R$. Then $w|\Omega_R \in L^2(\Omega_R)$. In particular the trace of w on $\partial\Omega$ is well defined.*

The preceding lemma is implicitly used in the ensuing theorem, where we introduce an extension operator $\mathfrak{E} : H^{1/2}(\partial\Omega)^3 \mapsto W^{1,1}_{loc}(\Omega)^3$ such that $\operatorname{div} \mathfrak{E}(b) = 0$.

Theorem 5.6. [29, Exercise III.3.8] *There is an operator $\mathfrak{E}$ from $H^{1/2}(\partial\Omega)^3$ into $W^{1,1}_{loc}(\Omega)^3$ satisfying the relations $\nabla \mathfrak{E}(b) \in L^2(\Omega)^9$, $\mathfrak{E}(b)|\partial\Omega = b$ and $\operatorname{div} \mathfrak{E}(b) = 0$ for $b \in H^{1/2}(\partial\Omega)^3$.*

In view of Lemma 5.2 and 5.3 and Theorem 5.6 and 5.4, the theory of mixed variational problems yields

Theorem 5.7. *Let $S > 0$ with $\Omega^c \subset B_S$, $R \in [2S, \infty)$, $F \in L^{6/5}(\Omega_R)^3$, $b \in H^{1/2}(\partial\Omega)^3$. Then there is a uniquely determined pair of functions $(\widetilde{V}, P) = \big(\widetilde{V}(R,F,b),\, P(R,F,b)\big) \in W_R \times L^2(\Omega_R)$ such that*

$$\mathcal{A}_R(\widetilde{V}, g) + \mathcal{B}_R(g, P) = \int_{\mathcal{D}_R} F \cdot g\, dx - \mathcal{A}_R\big(\,\mathfrak{E}(b)|\Omega_R,\, g\,\big) \;\; \text{for } g \in W_R, \tag{5.1}$$

$$\mathcal{B}_R(\widetilde{V}, \sigma) = 0 \;\; \text{for } \sigma \in L^2(\Omega_R), \tag{5.2}$$

where the operator $\mathfrak{E}$ was introduced in Theorem 5.6.

Let us interpret variational problem (5.1), (5.2) as a boundary value problem. Define the expression used in the boundary condition on the artificial boundary ∂B_R :

$$\mathcal{L}_R(u,\pi)(x) := \left(\sum_{j=1}^{3} \partial_j u_k(x) \frac{x_j}{R} - \pi(x)\frac{x_k}{R} + \frac{\tau}{2}\left(1 - \frac{x_1}{R}\right) u_k(x) \right)_{1 \le k \le 3}$$

for $x \in \partial B_R,\ R \in (0,\infty)$ with $\overline{\mathcal{D}} \subset B_R,\ u \in W^{2,6/5}(\Omega_R)^3,\ \pi \in W^{1,6/5}(\Omega_R)$.

Lemma 5.8. *Assume that Ω^c is $\mathcal{C}^2$-bounded. Let $S \in (0,\infty)$ with $\Omega^c \subset B_S$, $R \in [2S,\infty)$, $F \in L^{6/5}(\Omega_R)^3$ and $b \in W^{7/6,6/5}(\partial\Omega)^3$. Put $V := \widetilde{V}(R,F,b) + \mathfrak{E}(b)|\Omega_R$, with $V(R,F,b)$ from Theorem 5.7 and $\mathfrak{E}(b)$ from Theorem 5.6. Suppose that $V \in W^{2,6/5}(\Omega_R)^3$ and $P = P(R,F,b) \in W^{1,6/5}(\Omega_R)$, with $P(R,F,b)$ also introduced in Theorem 5.7. Then*

$$\begin{gathered} -\Delta V(z) + \tau \partial_1 V(z) - (\omega \times z)\cdot \nabla V(z) + \omega \times V(z) + \nabla P(z) = F(z), \\ \operatorname{div} V(z) = 0 \end{gathered} \tag{5.3}$$

for $z \in \Omega_R$, and $V|\partial\Omega = b$, $\mathcal{L}_R(V,P) = 0$.

Theorem 5.9. *Suppose that Ω^c is C^2-bounded. Let $\gamma,\, S_1 \in (0,\infty)$ with $\Omega^c \subset B_{S_1}$, $A \in [5/2,\, \infty)$, $B \in \mathbb{R}$ with $A + \min\{1, B\} > 3$. Let $F : \Omega \mapsto \mathbb{R}^3$ be measurable with $F|\Omega_{S_1} \in L^{6/5}(\Omega_{S_1})^3$ and $|F(z)| \le \gamma\, |z|^{-A} s(z)^{-B}$ for $z \in B^c_{S_1}$.*

Let $b \in W^{7/6,6/5}(\partial\Omega)^3$, $u \in W^{1,1}_{loc}(\Omega)^3 \cap L^6(\Omega)^3$ such that $\nabla u \in L^2(\Omega)^9$, $\operatorname{div} u = 0$, $u|\partial\Omega = b$ and equation (3.3) is satisfied.

For $R \in [2S_1,\, \infty)$, put $V_R := \widetilde{V}(R,F,b) + \mathfrak{E}(b)$, with $\mathfrak{E}(b)$ from Theorem 5.6, and $\widetilde{V}(R,F,b)$ from Theorem 5.7. Then

$$|u|_{\Omega_R} - V_R|^{(R)} \le C\, R^{-1} \quad \text{for } R \in [2S, \infty).$$

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