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MODELING OF FLOWS THROUGH A CHANNEL BY THE NAVIER–STOKES VARIATIONAL INEQUALITIES

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Abstract. We deal with a mathematical model of a flow of an incompressible Newtonian fluid through a channel with an artificial boundary condition on the outflow. We explain how several artificial boundary conditions formally follow from appropriate variational formulations and the way one expresses the dynamic stress tensor. As the boundary condition of the "do nothing"-type, that is predominantly considered to be the most appropriate from the physical point of view, does not enable one to derive an energy inequality, we explain how this problem can be overcome by using variational inequalities. We derive a priori estimates, which are the core of the proofs, and present theorems on the existence of solutions in the unsteady and steady cases.

Keywords: Variational inequality, Navier-Stokes equation, "do nothing" outflow boundary condition.

1. Introduction

1.1. The considered initial–boundary value problem

We denote by Ω a Lipschitzian domain in $\mathbb{R}^3$, which represents a channel. An incompressible Newtonian fluid is supposed to flow into the channel through the part Γ_1 of the boundary $\partial\Omega$ and to flow essentially out of the channel through the part Γ_2 of $\partial\Omega$. (See Fig. 1.) By "essentially" we mean that we do not exclude possible backward flows on Γ_2. A fixed wall of the channel is denoted by Γ_0. The flow is described by the equations of motion

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} - \operatorname{div}\mathbb{S}_{\mathrm{d}} + \nabla p \;=\; \boldsymbol{f}, \tag{1}$$

$$\operatorname{div}\boldsymbol{v} \;=\; 0, \tag{2}$$

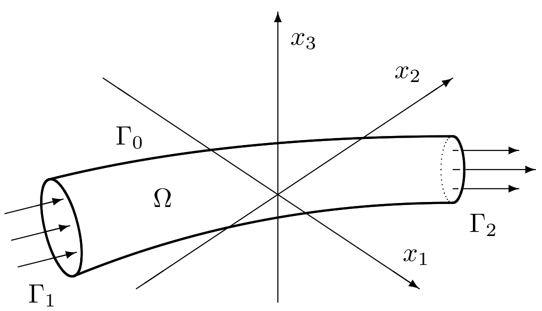

Fig. 1 The channel.

where $\boldsymbol{v}$ denotes the velocity, p is the pressure, $\mathbb{S}_{\mathrm{d}}$ is the dynamic stress tensor and $\boldsymbol{f}$ represents an external body force. For simplicity, we assume that the density of the fluid is equal to one. We use the homogeneous Dirichlet boundary condition

$$\boldsymbol{v} \;=\; \boldsymbol{0} \qquad \text{on } \Gamma_0\times(0,T), \tag{3}$$

where $(0,T)$ is a time interval. The velocity on Γ_1 can be naturally assumed to be known, which yields the inhomogeneous Dirichlet boundary condition

$$\boldsymbol{v} \;=\; \boldsymbol{v}^* \qquad \text{on } \Gamma_1\times(0,T). \tag{4}$$

On the other hand, since the velocity profile on Γ_2 cannot be predicted in advance, it is logical to apply some "artificial" boundary condition. There appear various artificial boundary conditions in the literature, see e.g. [1–8]. Boundary conditions, that follow automatically from an appropriate weak formulation of the considered problem if one a priori assumes a sufficient regularity of asolution, are usually called the "do nothing" conditions. (See e.g. [1, 6, 9] for more details.) An example, and probably the most often used artificial boundary condition is

$$-\,p\boldsymbol{n} + \nu\,\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}} \;=\; \boldsymbol{g} \qquad \text{on } \Gamma_2\times(0,T), \tag{5}$$

where $\boldsymbol{n}$ denotes the outer normal vector field on $\partial\Omega$, ν is the coefficient of viscosity and $\boldsymbol{g}$ is a given function. The non–steady problem also contains the initial condition

$$\boldsymbol{v} \;=\; \boldsymbol{v}_0 \qquad \text{in } \Omega\times\{0\}. \tag{6}$$

1.2. On some previous related existential results

The existential theory for the system (1)–(6) is based on appropriate a priori estimates. As the boundary condition (5) admits a possible reverse flow on Γ_2, which may bring to Ω an arbitrarily large amount of kinetic energy from the outside, the usual energy inequality does not hold. This does not matter if the given data of the problem are in an appropriate sense "sufficiently small" or if the number T is "sufficiently small" in the non–steady case. (See [1, 9, 10].) The existence of a weak solution of the problem (1)–(6) on an arbitrarily large time interval for "large" data (which is well known for the Navier–Stokes equations with the Dirichlet or Navier boundary conditions on $\partial\Omega$), is an open problem. A similar problem arises if one studies a flow through a 2D turbine cascade, see [3, 4]. Some authors use boundary conditions on Γ_2, modified in such a way that it enables one to estimate the kinetic energy of the fluid flowing to Ω through Γ_2. Examples of such modifications can be found in [2, 5, 7, 11, 12]. Another modification, used in connection with the heat transfer, can be found in [8].

A different approach has been suggested in papers [13–15]. There the authors have considered the stationary and non-stationary problems and impose an additional condition on Γ_2, that enables one to estimate the kinetic energy of a possible reverse flow and obtain an a priori estimate of the energy type. However, the new additional condition implies that the solution cannot lie in the whole Sobolev space $\boldsymbol{W}^{1,2}(\Omega)$, but only in a certain closed convex subset of this space. Since one does not know in advance whether the solution of the problem (1)–(5) falls into this convex set, one must consider a variational inequality instead of the momentum equation.

Note that artificial boundary conditions on a part of the boundary are also being used, if one approximates a problem in an exterior domain D by a problem in a bounded domain $D \cap B_R(\mathbf{0})$ (for "large" R) and prescribes an artificial boundary condition on $\partial B_R(\mathbf{0})$. (See e.g. [16] and [17].)

2. Several boundary conditions of the "do nothing" type

2.1. Three equivalent forms of the dynamic stress tensor in equation (1)

Since the difficulties, caused by the artificial boundary conditions on Γ_2, are of the same nature in stationary and non-stationary problems, here we consider, for simplicity, only the stationary problem. The dynamic stress tensor $\mathbb{S}_\mathrm{d}$, in an incompressible Newtonian fluid, equals $2\nu\mathbb{D}$, where $\mathbb{D}$ is the rate of the deformation tensor. (It coincides with the symmetrized gradient of velocity.) The term $\operatorname{div}\mathbb{S}_\mathrm{d}$, which appears in equation (1), can be expressed by any of these formulas:

$$\left.\begin{aligned}
&\text{a)} && \operatorname{div}\mathbb{S}_\mathrm{d} = \nu\Delta\boldsymbol{v},\\
&\text{b)} && \operatorname{div}\mathbb{S}_\mathrm{d} = \nu \operatorname{div}\left[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right],\\
&\text{c)} && \operatorname{div}\mathbb{S}_\mathrm{d} = -\nu\,\mathbf{curl}^2\boldsymbol{v}.
\end{aligned}\right\} \tag{7}$$

2.2. Variational formulations of the initial–boundary value problem

A variational formulation of the system (1), (2) with the boundary conditions (3), (4) formally follows from the classical formulation if one multiplies equation (1) by a "smooth" test function $\boldsymbol{\phi}$, such that $\operatorname{div}\boldsymbol{\phi} = 0$, and integrates in Ω. As $\boldsymbol{v}$ should satisfy the Dirichlet boundary conditions (3) and (4) on Γ_0 and Γ_1, respectively, it is logical to assume that $\boldsymbol{\phi} = \mathbf{0}$ on $\Gamma_0 \cup \Gamma_1$. On the other hand, one imposes no boundary condition on $\boldsymbol{\phi}$ on Γ_2. Applying the integration by parts, using the forms a) – c) of the dynamic stress tensor, we successively obtain the equations

$$\begin{aligned}
\text{a)}\ &\int_\Omega \big[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\phi} + \nu\nabla\boldsymbol{v} : \nabla\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\Gamma_2} [-p\boldsymbol{n} + \nu\nabla\boldsymbol{v}\cdot\boldsymbol{n}]\cdot\boldsymbol{\phi}\,\mathrm{d}S + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},\\
\text{b)}\ &\int_\Omega \big[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\phi} + \nu(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T) : \nabla\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\Gamma_2} \big[-p\boldsymbol{n} + \nu\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right)\cdot\boldsymbol{n}\big]\cdot\boldsymbol{\phi}\,\mathrm{d}S\\
&\quad + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},\\
\text{c)}\ &\int_\Omega \big[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\phi} + \nu\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\Gamma_2} [-p\boldsymbol{n} - \nu\,\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}]\cdot\boldsymbol{\phi}\,\mathrm{d}S\\
&\quad + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}$$

However, the integrals on Γ_2 cannot be involved by the weak formulation because the integrands are generally not integrable if one stays in the usual level of weak solutions. This is why it is logical to neglect these integrals or to replace them just by $\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S$, where $\boldsymbol{g}$ is an arbitrarily given function on Γ_2. Then the variational forms of the system (1), (2) are

$$\left.\begin{aligned}
\text{a)}\ &\int_\Omega \big[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\phi} + \nu\,\nabla\boldsymbol{v} : \nabla\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},\\
\text{b)}\ &\int_\Omega \big[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\phi} + \nu\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right) : \nabla\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x},\\
\text{c)}\ &\int_\Omega \big[\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{\phi} + \nu\,\mathbf{curl}\,\boldsymbol{v}\cdot\mathbf{curl}\,\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\\
&= \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{\phi}\,\mathrm{d}S + \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}.
\end{aligned}\right\} \tag{8}$$

The equations should satisfy all functions $\boldsymbol{\phi}$ with the aforementioned properties and $\boldsymbol{v}$ should also satisfy the boundary conditions (3) and (4). If a weak solution $\boldsymbol{v}$ exists and is sufficiently smooth then, by a reverse integration by parts, one can prove that there exists an appropriate associated pressure p and show that $\boldsymbol{v}$ and p satisfy the boundary conditions

$$\left.\begin{aligned}
&\text{a)} && -p\boldsymbol{n} + \nu\nabla\boldsymbol{v}\cdot\boldsymbol{n} = \boldsymbol{g},\\
&\text{b)} && -p\boldsymbol{n} + \nu\,[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T]\cdot\boldsymbol{n} = \boldsymbol{g},\\
&\text{c)} && -p\boldsymbol{n} - \nu\,\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{g},
\end{aligned}\right\} \tag{9}$$

respectively, on Γ_2. It is well known that the pressure p in equation (1) is not unique, because $p+c$, where c is an arbitrary additional constant (or a function of time), also satisfies the same equation. However, the same consideration is not possible in the boundary conditions a) – c) in (9). Here, it only follows from the variational formulation that , one can choose only one pressure p from all associated pressures, that satisfies the boundary condition.

2.3. The momentum equation with the Bernoulli pressure

None of the conditions a) – c) in (9) excludes a possible reverse flow on Γ_2 that could hypothetically bring an arbitrarily large amount of the kinetic energy back to Ω. One usually derives an a priori energy inequality so that the momentum equation (1) is multiplied by $\boldsymbol{v}$ and integrated over Ω. Then the flow of the kinetic energy through Γ_2 comes from the integral of $(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{v}$. Applying the integration by parts, we can express this integral as follows:

$$\begin{aligned}\int_\Omega(\boldsymbol{v}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{v}\;\mathrm{d}\boldsymbol{x} \;&=\; \int_{\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\,\tfrac{1}{2}|\boldsymbol{v}|^2\;\mathrm{d}S\\ &= \int_{\Gamma_1}(\boldsymbol{v}^*\cdot\boldsymbol{n})\,\tfrac{1}{2}|\boldsymbol{v}^*|^2\;\mathrm{d}S+\int_{\Gamma_2}(\boldsymbol{v}\cdot\boldsymbol{n})\,\tfrac{1}{2}|\boldsymbol{v}|^2\;\mathrm{d}S.\end{aligned} \tag{10}$$

The last integral can hypothetically take an arbitrarily large value if $\boldsymbol{v}\cdot\boldsymbol{n}<0$ on the part of Γ_2, i.e. in the case of a reverse flow.

The situation is different if the nonlinear term in equation (1) is considered in the form $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{v}+\nabla\frac{1}{2}|\boldsymbol{v}|^2$ and one involves $\nabla\frac{1}{2}|\boldsymbol{v}|^2$ and p to the so called Bernoulli pressure $q:=p+\frac{1}{2}|\boldsymbol{v}|^2$. Then, instead of (9), one obtains from the integral equations (8) the boundary conditions

$$\left.\begin{array}{ll}\text{a)} & -q\boldsymbol{n}+\nu\nabla\boldsymbol{v}\cdot\boldsymbol{n}=\boldsymbol{g},\\ \text{b)} & -q\boldsymbol{n}+\nu\,[\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T]\cdot\boldsymbol{n}=\boldsymbol{g},\\ \text{c)} & -q\boldsymbol{n}-\nu\,\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{g}.\end{array}\right\} \tag{11}$$

Now, the nonlinear term in equation (1) is just $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{v}$ (instead of $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$), which yields the term $\int_\Omega\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{v}\cdot\boldsymbol{\phi}\;\mathrm{d}\boldsymbol{x}$ in the variational formulation. If one formally multiplies equation (1) by the velocity $\boldsymbol{v}$ then the nonlinear term vanishes, because $(\mathbf{curl}\boldsymbol{v}\times\boldsymbol{v})\cdot\boldsymbol{v}=0$. This has the following consequences:

1) the nonlinear term $\mathbf{curl}\,\boldsymbol{v}\times\boldsymbol{v}$ generates no backward inflow of kinetic energy to Ω through the surface Γ_2,

2) the usual energy–type inequality can be derived,

3) the existence of a weak solution on an arbitrarily long time interval can be proven by similar methods, as if one considers the homogeneous or inhomogeneous Dirichlet boundary condition on the whole boundary $\partial\Omega$ (see e.g. [18]).

2.4. Which artificial boundary condition is the best?

There arises a natural question: which of the formulated boundary conditions a) – c) in (9) and a) – c) in (11) on Γ_2 is most appropriate? The advantage of all the conditions in (11) is that, in contrast to the conditions from (9), they enable one to prove the existence of a weak solution. On the other hand, the condition a) from (9) is fulfilled (with $\boldsymbol{g}=\mathbf{0}$) by the Poiseuille flow in a circular pipe. This is mainly why this condition is usually considered as the best from a physical point of view. On the other hand, in any of the formulated boundary conditions, one can always calculate an appropriate function $\boldsymbol{g}$ so that the Poiseuille flow satisfies the considered condition with this concrete function $\boldsymbol{g}$. Thus, the suitability of the chosen boundary condition probably depends only on a particular situation. Moreover, in our opinion, it would be very useful to perform numerical calculations with various boundary conditions so that one could compare the results among themselves and also with physical measurements.

3. The Navier–Stokes inequality – the non-steady case

In this section, we deal with the Navier–Stokes problem (1)–(4) in Ω with the boundary condition (5) on Γ_2. Due to the reasons, explained in Sections 1 and 2, we study the problem in the form of a variational inequality.

3.1. Notation

(i) We use the usual notation of the norms in the Lebesgue spaces: $\|\,.\,\|_r$ is the norm in $L^r(\Omega)$ or in $\boldsymbol{L}^r(\Omega)$ (the space of vector functions) or in $L^r(\Omega)^{3\times3}$ (the space of tensorial functions). By analogy, $\|\,.\,\|_{k,r}$ denotes the norm in the Sobolev space $W^{k,r}(\Omega)$ or $\boldsymbol{W}^{k,r}(\Omega)$ or $W^{k,r}(\Omega)^{3\times3}$. If the norm is related to another set than Ω then we denote it e.g. by $\|\,.\,\|_{r;\,\Gamma_2}$, etc.

(ii) We assume that $\boldsymbol{v}^*$ is a given function on $\Gamma_1\times(0,T)$, such that $\frac{1}{2}\int_{\Gamma_1}[\boldsymbol{v}^*\cdot(-\boldsymbol{n})]\,|\boldsymbol{v}^*|^2\;\mathrm{d}S$ (the inflow of the kinetic energy to Ω through Γ_1) is bounded, as a function of t, for $t\in(0,T)$.

(iii) Furthermore, we assume that $\boldsymbol{v}^*$ can be extended to $\Omega\times(0,T)$ so that the extended function (which is denoted by $\boldsymbol{v}^*_{\mathrm{ext}}$) satisfies the boundary condition (3) on $\Gamma_0\times(0,T)$ and a) $\boldsymbol{v}^*_{\mathrm{ext}}\in L^\infty\big(0,T;\,\boldsymbol{W}^{1,2}(\Omega)\big)$ and $\partial_t\boldsymbol{v}^*_{\mathrm{ext}}\in L^2\big(0,T;\,\boldsymbol{W}^{-1,2}(\Omega)\big)$, b) $\boldsymbol{v}^*_{\mathrm{ext}}$ is divergence–free. (Here, we denote by $\boldsymbol{W}^{-1,2}(\Omega)$ the dual space to $\boldsymbol{W}^{1,2}(\Omega)$. The duality pairing between $\boldsymbol{W}^{-1,2}(\Omega)$ and $\boldsymbol{W}^{1,2}(\Omega)$ is denoted by $\langle\,.\,,\,.\,\rangle$.) Due to [19, Theorem I.3.1] function $\boldsymbol{v}^*_{\mathrm{ext}}$ belongs to $C^0\big([0,T];\,\boldsymbol{L}^2(\Omega)\big)$.

(iv) We denote by $\boldsymbol{V}$ the linear space of all divergence–free vector functions $\boldsymbol{\phi}\in\boldsymbol{W}^{1,2}(\Omega)$, such that $\boldsymbol{\phi}=\mathbf{0}$ on $\Gamma_0\cup\Gamma_1$. Then $\boldsymbol{v}^*_{\mathrm{ext}}(t)+\boldsymbol{V}$ (for a.a. $t\in(0,T)$)

is the set of all functions $\boldsymbol{\phi}$ from $\boldsymbol{W}^{1,2}(\Omega)$, such that $\operatorname{div}\boldsymbol{\phi} = 0$, $\boldsymbol{\phi} = \boldsymbol{0}$ on Γ_0 and $\boldsymbol{\phi} = \boldsymbol{v}^*_{\text{ext}}(t)$ on Γ_1.

(v) Let $\epsilon_1 > 0$ and $\zeta \in L^\infty(0,T)$ satisfy the inequality

$$\left\| \left(\boldsymbol{v}^*_{\text{ext}}(t)\cdot\boldsymbol{n}\right)_- |\boldsymbol{v}^*_{\text{ext}}(t)|^2 \right\|_{1;\,\Gamma_2} + \epsilon_1 \;<\; \zeta(t) \quad (12)$$

for a.a. $t \in (0,T)$. (The subscript "−" denotes the negative part.) Such number ϵ_1 and function ζ exist, because the trace of $\boldsymbol{v}^*$ on $\Gamma_2 \times (0,T)$ is in $L^\infty\big(0,T;\, \boldsymbol{W}^{1/2,2}(\Gamma_2)\big)$ and this space is continuously imbedded to $L^\infty\big(0,T;\, \boldsymbol{L}^4(\Gamma_2)\big)$. We denote by $\boldsymbol{K}_t$ the set of all functions $\boldsymbol{\phi} \in \boldsymbol{v}^*_{\text{ext}}(t) + \boldsymbol{V}$ such that

$$\left\|(\boldsymbol{\phi}\cdot\boldsymbol{n})_- |\boldsymbol{\phi}|^2\right\|_{1;\,\Gamma_2} \;\leq\; \zeta(t) \quad \text{for a.a. } t \in (0,T), \quad (13)$$

by $\boldsymbol{K}^c_t$ the convex hull of $\boldsymbol{K}_t$ and define $\overline{\boldsymbol{K}^c_t}$ to be the closure of $\boldsymbol{K}^c_t$. Set $\overline{\boldsymbol{K}^c_t}$ is the so called closed convex hull of $\boldsymbol{K}_t$. (See [20] for more properties of the convex hull and the closed convex hull. Note that $\overline{\boldsymbol{K}^c_t}$ can also be defined as the intersection of all closed convex sets in $\boldsymbol{v}^*_{\text{ext}}(t) + \boldsymbol{V}$, containing $\boldsymbol{K}_t$.)

We assume that the number ϵ_1, and the functions $\boldsymbol{v}^*_{\text{ext}}$ and ζ are fixed throughout the whole paper. Using the presence of $\epsilon_1 > 0$ in inequality (12), one can also show that there exists $\epsilon_2 > 0$ such that $\overline{\boldsymbol{K}^c_t}$ contains the ϵ_2–neighborhood of $\boldsymbol{v}^*_{\text{ext}}(t)$, independently of t.

(vi) Denote by $\mathscr{W}(0,T)$ the space $\big\{\boldsymbol{w} \in L^2(0,T;\, \boldsymbol{W}^{1,2}(\Omega));\ \partial_t\boldsymbol{w} \in L^2(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))\big\}$ with the norm

$$|\!|\!|\boldsymbol{w}|\!|\!| := \left(\int_0^T \|\boldsymbol{w}\|^2_{1,2}\,\mathrm{d}t + \int_0^T \|\partial_t\boldsymbol{w}\|^2_{-1,2}\,\mathrm{d}t\right)^{1/2}.$$

Using [19, Theorem I.3.1], one can show that $\mathscr{W}(0,T) \subset C^0\big([0,T];\, \boldsymbol{L}^2(\Omega)\big)$.

(vii) Put $\overline{\mathscr{K}^c}(0,T) := \big\{\boldsymbol{w} \in \mathscr{W}(0,T);\ \boldsymbol{w}(t) \in \overline{\boldsymbol{K}^c_t}$ for a.a. $t \in (0,T)\big\}$.

3.2. A formal derivation of the variational inequality

Suppose that $\boldsymbol{v}$, p is a sufficiently smooth solution of the problem (1)–(6) and $\boldsymbol{w}$ is a sufficiently smooth function from $[0,T]$ such that $\boldsymbol{w}(t) \in \overline{\boldsymbol{K}^c_t}$ for a.a. $t \in [0,T]$. Using the form a) of the divergence of the dynamic stress tensor in (7), multiplying equation (1) by the difference $\boldsymbol{w} - \boldsymbol{v}$, integrating in $\Omega \times (0,T)$, applying the integration by parts and using the equality $\boldsymbol{w} - \boldsymbol{v} = \boldsymbol{0}$ on $\Gamma_0 \cup \Gamma_1 \times (0,T)$ and the boundary condition (5), we get

$$\begin{aligned}
&\int_0^T \int_\Omega [\partial_t\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}]\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \nu\nabla\boldsymbol{v} : \nabla(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + \int_0^T \int_{\Gamma_2} \boldsymbol{g}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}S\,\mathrm{d}t. \qquad (14)
\end{aligned}$$

The term, which contains the derivative $\partial_t\boldsymbol{v}$, satisfies

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t\boldsymbol{v}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \int_0^T \int_\Omega \partial_t(\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \partial_t\boldsymbol{w}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&= \frac{1}{2}\|\boldsymbol{w}(0)-\boldsymbol{v}(0)\|_2^2 - \frac{1}{2}\|\boldsymbol{w}(T)-\boldsymbol{v}(T)\|_2^2 \\
&\qquad + \int_0^T \int_\Omega \partial_t\boldsymbol{w}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\leq \frac{1}{2}\|\boldsymbol{w}(0)-\boldsymbol{v}_0\|_2^2 \\
&\qquad + \int_0^T \int_\Omega \partial_t\boldsymbol{w}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \qquad (15)
\end{aligned}$$

Let $\boldsymbol{w} \in \overline{\mathscr{K}^c}(0,T)$ further on. Since

$$\int_\Omega \partial_t\boldsymbol{w}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \;=\; \langle\partial_t\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{v}\rangle,$$

$$\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \;=\; \langle\boldsymbol{f}, \boldsymbol{w}-\boldsymbol{v}\rangle,$$

(15) and (14) yield

$$\begin{aligned}
&\int_0^T \big\langle\partial_t\boldsymbol{w}, \boldsymbol{w}-\boldsymbol{v}\big\rangle\,\mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega \nu\nabla\boldsymbol{v} : \nabla(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&\geq \int_0^T \langle\boldsymbol{f}, \boldsymbol{w}-\boldsymbol{v}\rangle\,\mathrm{d}t + \int_0^T \int_{\Gamma_2} \boldsymbol{g}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}S\,\mathrm{d}t \\
&\qquad - \frac{1}{2}\|\boldsymbol{w}(0)-\boldsymbol{v}_0\|_2^2. \qquad (16)
\end{aligned}$$

Since (16) is an inequality, we have the possibility to choose another condition and to impose it on the solution $\boldsymbol{v}$: we require that the inclusion $\boldsymbol{v}(t) \in \overline{\boldsymbol{K}^c_t}$ holds for a.a. $t \in (0,T)$.

3.3. Definition of the initial–boundary value problem $(\mathcal{P})$

Let $\boldsymbol{v}^*_{\text{ext}}$ be the extension of function $\boldsymbol{v}^*$ with the properties (i) and (ii) from paragraph 3.2. Let $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\Omega)$, $\operatorname{div}\boldsymbol{v}_0 = 0$ in Ω in the sense of distributions, $\boldsymbol{v}_0\cdot\boldsymbol{n} = 0$ on Γ_0 and $\boldsymbol{v}_0\cdot\boldsymbol{n} = \boldsymbol{v}^*(0)\cdot\boldsymbol{n}$ on Γ_1 in the sense of traces. Let $\boldsymbol{f} \in L^2(0,T;\, \boldsymbol{W}^{-1,2}(\Omega))$ and $\boldsymbol{g} \in L^2(0,T;\, \boldsymbol{L}^{4/3}(\Gamma_2))$. One looks for $\boldsymbol{v} \in L^\infty(0,T;\, \boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\, \boldsymbol{W}^{1,2}(\Omega))$ such that $\boldsymbol{v}(t) \in \overline{\boldsymbol{K}^c_t}$ for a.a. $t \in (0,T)$ and $\boldsymbol{v}$ satisfies inequality (16) for all $\boldsymbol{w} \in \overline{\mathscr{K}^c}(0,T)$.

It is well known that for $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\Omega)$, such that $\operatorname{div}\boldsymbol{v}_0 \in L^2(\Omega)$ (which it definitely satisfies if $\boldsymbol{v}_0$ is divergence–free in the sense of distributions), the scalar product $\boldsymbol{v}\cdot\boldsymbol{n}$ makes sense on $\partial\Omega$, as an element of $W^{-1/2,2}(\partial\Omega)$. (This follows e.g. from [21, Theorem III.2.2].) Thus, the conditions $\boldsymbol{v}_0\cdot\boldsymbol{n}=\boldsymbol{0}$ on Γ_0 and $\boldsymbol{v}_0\cdot\boldsymbol{n}=\boldsymbol{v}^*(0)\cdot\boldsymbol{n}$ are assumed to hold on Γ_0 and Γ_1 as equalities in $W^{-1/2,2}(\Gamma_0)$ and $W^{-1/2,2}(\Gamma_1)$, respectively.

The next theorem provides the information on the existence of a solution of a problem $(\mathcal{P})$ on an arbitrarily long time interval $(0,T)$.

Theorem 1. *Let $\boldsymbol{v}_0$, $\boldsymbol{v}^*_{\mathrm{ext}}$, $\boldsymbol{f}$ and $\boldsymbol{g}$ be the functions with the aforementioned properties. Then problem $(\mathcal{P})$ is solvable. The solution can be expressed in the form $\boldsymbol{v}=\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u}$, where $\boldsymbol{u}\in L^\infty(0,T;\,\boldsymbol{L}^2(\Omega))\cap L^2(0,T;\,\boldsymbol{V})$ satisfies the inequality*

$$\begin{aligned}
&\|\boldsymbol{u}(t)\|_2^2+\nu\int_0^t\|\nabla\boldsymbol{u}(s)\|_2^2\,\mathrm{d}s\\
&\quad\le\ \|\boldsymbol{u}(0)\|_2^2-\int_0^{t^*}\int_{\Gamma_1}(\boldsymbol{v}^*\cdot\boldsymbol{n})\,|\boldsymbol{v}^*|^2\,\mathrm{d}S\,\mathrm{d}t\\
&\qquad+c_1\int_0^t\|\boldsymbol{u}(s)\|_2^2\,\mathrm{d}s+\int_0^t\big[c_2\,\|\boldsymbol{f}(s)\|_{-1,2}^2\\
&\qquad+c_3\,\|\boldsymbol{v}^*_{\mathrm{ext}}(s)\|_{1,2}^2+c_4\,\|\partial_t\boldsymbol{v}^*_{\mathrm{ext}}(s)\|_{-1,2}^2\\
&\qquad+c_5\,\|\boldsymbol{g}(s)\|_{4/3;\,\Gamma_2}^2\big]\,\mathrm{d}s
\end{aligned}\tag{17}$$

for all t $in(0,T)$, where all the constants c_1–c_5 are independent of $\boldsymbol{v}_0$, $\boldsymbol{v}^*$, $\boldsymbol{v}^*_{\mathrm{ext}}$, $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{u}$.

Note that there is a minus sign in front of the first integral on the right hand side, because $\boldsymbol{n}$ is the outer normal vector and therefore $-\frac{1}{2}\int_0^{t^*}\int_{\Gamma_1}(\boldsymbol{v}^*\cdot\boldsymbol{n})\,|\boldsymbol{v}^*|^2\,\mathrm{d}S\,\mathrm{d}t$ represents the inflow of the kinetic energy to Ω through Γ_1 in the time interval $(0,t^*)$.

An analogous theorem, with a different convex set $\overline{\boldsymbol{K}^c_t}$, has been proven in [15].

3.4. The principle of the proof and a priori estimates

The complete way Theorem 1 can be proven consists of these main steps: 1) construction of appropriate approximations $\boldsymbol{v}_n$ (for $n\in\mathbb{N}$) of solution $\boldsymbol{v}$, 2) derivation of a series of estimates of the approximations, 3) derivation of various types of convergence of a subsequence of $\{\boldsymbol{v}_n\}$ in various spaces, 4) verification that the limit is the solution $\boldsymbol{v}$. Among others, one also needs the strong convergence in $L^2(0,T;\,\boldsymbol{W}^{1,2}(\Omega))$, which follows from an estimate of a fractional derivative with respect to t of $\boldsymbol{v}_n$ and from the Lions–Aubin lemma, see e.g. [22]. Since the complete estimates of the approximations are laborious, technically complicated and necessarily influenced by the technique, used just for the construction of the approximations, we show below on an a priori level how one can directly obtain from the variational inequality the estimates of $\boldsymbol{u}$ in $L^\infty(0,T;\,\boldsymbol{L}^2(\Omega))$ and in $L^2(0,T;\,\boldsymbol{W}^{1,2}(\Omega))$. The advantage of a priori estimates is that they enable one to abstract from the whole machinery, which is necessary in the proof of existence of the approximations. On the other hand, we assume, just inside the procedure, that $\boldsymbol{u}$ is smooth. (This formal assumption is naturally satisfied on the level of approximations.)

Thus, let $t^*\in(0,T)$, $\alpha\in(0,1)$ and $\delta>0$ be so small that $t^*+\delta<T$. Define function η of one variable t by the formulas

$$\eta(t)=\begin{cases}\alpha & \text{for } 0<t\le t^*,\\[4pt] \alpha+\dfrac{(1-\alpha)}{\delta}\,(t-t^*) & \text{for } t^*<t<t^*+\delta,\\[4pt] 1 & \text{for } t^*+\delta\le t<T.\end{cases}$$

(Function η is continuous on $(0,T)$, constant on $(0,t^*]$ and on $[t^*+\delta,T)$ and linear on $[t^*,t^*+\delta]$.) Solution $\boldsymbol{v}$ can be expressed in the form $\boldsymbol{v}=\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u}$, where $\boldsymbol{u}\in\boldsymbol{V}$. Put $\boldsymbol{w}:=\boldsymbol{v}^*_{\mathrm{ext}}+\eta\boldsymbol{u}=(1-\eta)\boldsymbol{v}^*_{\mathrm{ext}}+\eta\boldsymbol{v}$. (As set $\overline{\mathscr{K}^c}(0,T)$ is convex and $0<\eta\le1$, $\boldsymbol{w}$ belongs to $\overline{\mathscr{K}^c}(0,T)$.) Then $\boldsymbol{w}-\boldsymbol{v}=(\eta-1)\boldsymbol{u}$ (which equals $\boldsymbol{0}$ on the interval $[t^*+\delta,T)$). Substituting this to the first term in (16), we obtain

$$\begin{aligned}
&\int_0^T\big\langle\partial_t\boldsymbol{w},\boldsymbol{w}-\boldsymbol{v}\big\rangle\,\mathrm{d}t\\
&\quad=\int_0^{t^*}\big\langle\partial_t(\boldsymbol{v}^*_{\mathrm{ext}}+\alpha\boldsymbol{u}),(\alpha-1)\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\qquad+\int_{t^*}^{t^*+\delta}\big\langle\partial_t(\boldsymbol{v}^*_{\mathrm{ext}}+\eta\boldsymbol{u}),(\eta-1)\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\quad=(\alpha-1)\int_0^{t^*}\big\langle\partial_t\boldsymbol{v}^*_{\mathrm{ext}},\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\qquad+\alpha(\alpha-1)\int_0^{t^*}\big\langle\partial_t\boldsymbol{u},\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\qquad+\int_{t^*}^{t^*+\delta}\big\langle\partial_t\boldsymbol{v}^*_{\mathrm{ext}},(\eta-1)\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\qquad+\int_{t^*}^{t^*+\delta}\big\langle\partial_t(\eta\boldsymbol{u}),\eta\boldsymbol{u}\big\rangle\,\mathrm{d}t-\int_{t^*}^{t^*+\delta}\big\langle\dot\eta\boldsymbol{u},\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\qquad-\int_{t^*}^{t^*+\delta}\big\langle\eta\,\partial_t\boldsymbol{u},\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\quad=\int_0^{t^*+\delta}(\eta-1)\,\big\langle\partial_t\boldsymbol{v}^*_{\mathrm{ext}},\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\qquad+\frac{\alpha(\alpha-1)}{2}\big(\|\boldsymbol{u}(t^*)\|_2^2-\|\boldsymbol{u}(0)\|_2^2\big)\\
&\qquad+\frac{1}{2}\big(\|\eta(t^*+\delta)\boldsymbol{u}(t^*+\delta)\|_2^2-\|\eta(t^*)\boldsymbol{u}(t^*)\|_2^2\big)\\
&\qquad-\frac{1-\alpha}{\delta}\int_{t^*}^{t^*+\delta}\|\boldsymbol{u}\|_2^2\,\mathrm{d}t\\
&\qquad-\frac{1}{2}\int_{t^*}^{t^*+\delta}\eta\,\frac{\mathrm{d}}{\mathrm{d}t}\,\|\boldsymbol{u}\|_2^2\,\mathrm{d}t\\
&\quad=\int_0^{t^*+\delta}(\eta-1)\,\big\langle\partial_t\boldsymbol{v}^*_{\mathrm{ext}},\boldsymbol{u}\big\rangle\,\mathrm{d}t\\
&\qquad+\frac{\alpha(\alpha-1)}{2}\big(\|\boldsymbol{u}(t^*)\|_2^2-\|\boldsymbol{u}(0)\|_2^2\big)\\
&\qquad+\frac{1}{2}\big(\|\eta(t^*+\delta)\boldsymbol{u}(t^*+\delta)\|_2^2-\|\eta(t^*)\boldsymbol{u}(t^*)\|_2^2\big)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1-\alpha}{\delta} \int_{t^*}^{t^*+\delta} \|\boldsymbol{u}\|_2^2 \, \mathrm{d}t \\
&- \frac{1}{2}\big(\eta(t^*+\delta)\, \|\boldsymbol{u}(t^*+\delta)\|_2^2 - \eta(t^*)\, \|\boldsymbol{u}(t^*)\|_2^2\big) \\
&+ \frac{1}{2} \int_{t^*}^{t^*+\delta} \dot{\eta}\, \|\boldsymbol{u}\|_2^2 \, \mathrm{d}t \\
= &\int_0^{t^*+\delta} (\eta - 1)\, \langle \partial_t \boldsymbol{v}^*_{\mathrm{ext}}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&+ \frac{\alpha(\alpha-1)}{2} \big(\|\boldsymbol{u}(t^*)\|_2^2 - \|\boldsymbol{u}(0)\|_2^2\big) \\
&+ \frac{1}{2} \big(\|\eta(t^*+\delta)\boldsymbol{u}(t^*+\delta)\|_2^2 - \|\eta(t^*)\boldsymbol{u}(t^*)\|_2^2\big) \\
&- \frac{1-\alpha}{\delta} \int_{t^*}^{t^*+\delta} \|\boldsymbol{u}\|_2^2 \, \mathrm{d}t \\
&- \frac{1}{2}\big(\eta(t^*+\delta)\, \|\boldsymbol{u}(t^*+\delta)\|_2^2 - \eta(t^*)\, \|\boldsymbol{u}(t^*)\|_2^2\big) \\
&+ \frac{1-\alpha}{2\delta} \int_{t^*}^{t^*+\delta} \|\boldsymbol{u}\|_2^2 \, \mathrm{d}t.
\end{aligned}
$$

Considering $\delta \to 0+$, we get

$$
\begin{aligned}
&\int_0^T \big\langle \partial_t \boldsymbol{w}, \boldsymbol{w} - \boldsymbol{v}\big\rangle \, \mathrm{d}t \\
&\quad = (\alpha-1) \int_0^{t^*} \langle \partial_t \boldsymbol{v}^*_{\mathrm{ext}}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&\qquad + \frac{\alpha^2-1}{2} \|\boldsymbol{u}(t^*)\|_2^2 - \frac{\alpha(\alpha-1)}{2} \|\boldsymbol{u}(0\|_2^2.
\end{aligned}
$$

Substituting this to (16), using $\boldsymbol{v} = \boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}$ and $\boldsymbol{w} = \boldsymbol{v}^*_{\mathrm{ext}} + \eta \boldsymbol{u}$ in all other terms in (16), considering $\delta \to 0+$, dividing the whole inequality by $\alpha - 1$ (which is negative), and considering finally $\alpha \to 0+$, we obtain

$$
\begin{aligned}
&\int_0^{t^*} \langle \partial_t \boldsymbol{v}^*_{\mathrm{ext}}, \boldsymbol{u}\rangle \, \mathrm{d}t + \frac{1}{2} \|\boldsymbol{u}(t^*)\|_2^2 \\
&\quad + \int_0^{t^*}\!\!\int_\Omega (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot \nabla (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad + \int_0^{t^*}\!\!\int_\Omega \nu \nabla (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) : \nabla \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \\
&\leq \int_0^{t^*} \langle \boldsymbol{f}, \boldsymbol{u}\rangle \, \mathrm{d}t + \int_0^{t^*}\!\!\int_{\Gamma_2} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}S \, \mathrm{d}t + \frac{1}{2} \|\boldsymbol{u}(0)\|_2^2, \\
&\frac{1}{2} \|\boldsymbol{u}(t^*)\|_2^2 + \int_0^{t^*} \nu \|\nabla \boldsymbol{u}\|_2^2 \, \mathrm{d}t \\
&\quad + \int_0^{t^*}\!\!\int_\Omega (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot \nabla (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad + \int_0^{t^*} \langle \partial_t \boldsymbol{v}^*_{\mathrm{ext}}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&\leq \int_0^{t^*}\!\!\int_\Omega (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot \nabla (\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot \boldsymbol{v}^*_{\mathrm{ext}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad + \int_0^{t^*}\!\!\int_\Omega \nu \nabla \boldsymbol{v}^*_{\mathrm{ext}} : \nabla \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^{t^*} \langle \boldsymbol{f}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&\quad + \int_0^{t^*}\!\!\int_{\Gamma_2} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}S \, \mathrm{d}t + \frac{1}{2} \|\boldsymbol{u}(0)\|_2^2, \\
&\frac{1}{2} \|\boldsymbol{u}(t^*)\|_2^2 + \int_0^{t^*} \nu \|\nabla \boldsymbol{u}\|_2^2 \, \mathrm{d}t \\
&\quad + \frac{1}{2} \int_0^{t^*}\!\!\int_{\Gamma_1} (\boldsymbol{v}^* \cdot \boldsymbol{n})\, |\boldsymbol{v}^*|^2 \, \mathrm{d}S \, \mathrm{d}t \\
&\quad + \frac{1}{2} \int_0^{t^*}\!\!\int_{\Gamma_2} [(\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot \boldsymbol{n}]\, |\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}|^2 \, \mathrm{d}S \, \mathrm{d}t \\
&\quad + \int_0^{t^*} \langle \partial_t \boldsymbol{v}^*_{\mathrm{ext}}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&\leq \int_0^{t^*}\!\!\int_\Omega \big(\boldsymbol{v}^*_{\mathrm{ext}} \cdot \nabla \boldsymbol{v}^*_{\mathrm{ext}} \cdot \boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{v}^*_{\mathrm{ext}} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v}^*_{\mathrm{ext}} \\
&\qquad\qquad + \boldsymbol{u} \cdot \nabla \boldsymbol{v}^*_{\mathrm{ext}} \cdot \boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v}^*_{\mathrm{ext}}\big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad + \int_0^{t^*}\!\!\int_\Omega \nu \nabla \boldsymbol{v}^*_{\mathrm{ext}} : \nabla \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \int_0^{t^*} \langle \boldsymbol{f}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&\quad + \int_0^{t^*}\!\!\int_{\Gamma_2} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}S \, \mathrm{d}t + \frac{1}{2} \|\boldsymbol{u}(0)\|_2^2.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left| \int_0^{t^*}\!\!\int_\Omega \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v}^*_{\mathrm{ext}} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right| \\
&\quad \leq \int_0^{t^*} \|\boldsymbol{u}\|_3 \, \|\nabla \boldsymbol{u}\|_2 \, \|\boldsymbol{v}^*_{\mathrm{ext}}\|_6 \, \mathrm{d}t \\
&\quad \leq c \int_0^{t^*} \|\boldsymbol{u}\|_3 \, \|\nabla \boldsymbol{u}\|_2 \, \|\boldsymbol{v}^*_{\mathrm{ext}}\|_{1,2} \, \mathrm{d}t \\
&\quad \leq c \int_0^{t^*} \|\boldsymbol{u}\|_3 \, \|\nabla \boldsymbol{u}\|_2 \, \mathrm{d}t \\
&\quad \leq c \int_0^{t^*} \|\boldsymbol{u}\|_2^{1/2} \, \|\boldsymbol{u}\|_6^{1/2} \, \|\nabla \boldsymbol{u}\|_2 \, \mathrm{d}t \\
&\quad \leq c \int_0^{t^*} \|\boldsymbol{u}\|_2^{1/2} \, \|\nabla \boldsymbol{u}\|_2^{3/2} \, \mathrm{d}t \\
&\quad \leq \int_0^{t^*} \big(\xi \, \|\nabla \boldsymbol{u}\|_2^2 + c(\xi)\, \|\boldsymbol{u}\|^2\big) \, \mathrm{d}t,
\end{aligned}
$$

where c is a generic constant, we get

$$
\begin{aligned}
&\frac{1}{2} \|\boldsymbol{u}(t^*)\|_2^2 + (\nu - \xi) \int_0^{t^*} \|\nabla \boldsymbol{u}\|_2^2 \, \mathrm{d}t \\
&\quad + \frac{1}{2} \int_0^{t^*}\!\!\int_{\Gamma_1} (\boldsymbol{v}^* \cdot \boldsymbol{n})\, |\boldsymbol{v}^*|^2 \, \mathrm{d}S \, \mathrm{d}t + \int_0^T \langle \partial_t \boldsymbol{v}^*_{\mathrm{ext}}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&\leq \frac{1}{2} \int_0^{t^*}\!\!\int_{\Gamma_2} [(\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}) \cdot \boldsymbol{n}]_-\, |\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}|^2 \, \mathrm{d}S \, \mathrm{d}t \\
&\quad + \int_0^{t^*}\!\!\int_\Omega \big(\boldsymbol{v}^*_{\mathrm{ext}} \cdot \nabla \boldsymbol{v}^*_{\mathrm{ext}} \cdot \boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{v}^*_{\mathrm{ext}} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{v}^*_{\mathrm{ext}} \\
&\qquad\qquad + \boldsymbol{u} \cdot \nabla \boldsymbol{v}^*_{\mathrm{ext}} \cdot \boldsymbol{v}^*_{\mathrm{ext}}\big) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad + c(\xi) \int_0^{t^*} \|\boldsymbol{u}\|^2 \, \mathrm{d}t + \int_0^{t^*}\!\!\int_\Omega \nu \nabla \boldsymbol{v}^*_{\mathrm{ext}} : \nabla \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\
&\quad + \int_0^{t^*} \langle \boldsymbol{f}, \boldsymbol{u}\rangle \, \mathrm{d}t \\
&\quad + \int_0^T\!\!\int_{\Gamma_2} \boldsymbol{g} \cdot \boldsymbol{u} \, \mathrm{d}S \, \mathrm{d}t + \frac{1}{2} \|\boldsymbol{u}(0)\|_2^2. \qquad (18)
\end{aligned}
$$

(Note that $\xi > 0$ can be chosen arbitrarily small.) The first integral on the right hand side satisfies the inequality

$$\int_{\Gamma_2} [(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u})\cdot\boldsymbol{n}]_-\,|\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}|^2\,\mathrm{d}S \le \left(\int_{\Gamma_2} [(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u})\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}} \cdot \left(\int_{\Gamma_2} |\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}|^3\,\mathrm{d}S\right)^{\frac{2}{3}}. \tag{19}$$

Since $\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\in\overline{\boldsymbol{K}^c_t}$, there exists a sequence $\{\boldsymbol{u}_k\}$ in $\boldsymbol{K}^c_t$, such that $\boldsymbol{u}_k\to\boldsymbol{u}$ (for $k\to\infty$) in the norm of $\boldsymbol{W}^{1,2}(\Omega)$. Then we also have

$$\left(\int_{\Gamma_2} [(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u})\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}} = \lim_{k\to\infty}\left(\int_{\Gamma_2} [(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}_k)\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}}.$$

To each function $\boldsymbol{u}_k$, there exist finite families $\{\theta_{ki}\}_{i=1}^{N_k}$ and $\{\boldsymbol{u}_{ki}\}_{i=1}^{N_k}$ in $[0,1]$ and $\boldsymbol{K}_t$, respectively, such that

$$\sum_{i=1}^{N_k}\theta_{ki}=1 \qquad \text{and} \qquad \boldsymbol{u}_k = \sum_{i=1}^{N_k}\theta_{ki}\boldsymbol{u}_{ki}.$$

Then, applying Minkowski's inequality, we get

$$\begin{aligned}
&\left(\int_{\Gamma_2} [(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}_k)\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}}\\
&\quad= \left(\int_{\Gamma_2}\Big[\sum_{i=1}^{N_k}\theta_{ki}(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}_{ki})\cdot\boldsymbol{n}\Big]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}}\\
&\quad\le \left(\int_{\Gamma_2}\sum_{i=1}^{N_k}[\theta_{ki}(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}_{ki})\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}}\\
&\quad\le \sum_{i=1}^{N_k}\left(\int_{\Gamma_2}[\theta_{ki}(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}_{ki})\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}}\\
&\quad= \sum_{i=1}^{N_k}\theta_{ki}\left(\int_{\Gamma_2}[(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}_{ki})\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}}\\
&\quad\le \sum_{i=1}^{N_k}\theta_{ki}\zeta = \zeta.
\end{aligned}$$

Hence

$$\left(\int_{\Gamma_2} [(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u})\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}} \le \zeta, \tag{20}$$

too. Note that this is a crucial part, where we use the fact that $\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}$ lies in $\overline{\boldsymbol{K}^c_t}$. The estimates, following from this information, are not available if one deals with the Navier–Stokes equation instead of the Navier–Stokes variational inequality (16). As there exists a continuous operator of traces from the Sobolev–Slobodetski space $\boldsymbol{W}^{5/6,2}(\Omega)$ to $\boldsymbol{L}^3(\Gamma_2)$, which can be deduced e.g. by means of [23], we have

$$\begin{aligned}
&\left(\int_{\Gamma_2} [(\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u})\cdot\boldsymbol{n}]_-^3\,\mathrm{d}S\right)^{\frac{1}{3}}\left(\int_{\Gamma_2} |\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}|^3\,\mathrm{d}S\right)^{\frac{2}{3}}\\
&\quad\le \zeta\,\|\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\|^2_{3;\,\Gamma_2}\\
&\quad\le c\,\|\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\|^2_{5/6,2}\\
&\quad\le c\,\zeta\,\|\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\|_2^{\frac{1}{3}}\,\|\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\|_{1,2}^{\frac{5}{3}}\\
&\quad\le c\,\zeta\,\|\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\|_2^{\frac{1}{3}}\left(\|\boldsymbol{v}^*_{\text{ext}}\|_{1,2}^{\frac{5}{3}}+\|\nabla\boldsymbol{u}\|_2^{\frac{5}{3}}\right)\\
&\quad\le \xi\,\|\nabla\boldsymbol{u}\|_2^2+c(\xi)\,\zeta^6\,\|\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\|_2^2+\|\boldsymbol{v}^*_{\text{ext}}\|^2_{1,2}.
\end{aligned}$$

Recall that $\zeta\in L^\infty(0,T)$. Substituting to (18), and using also the estimates

$$\begin{aligned}
\int_0^{t^*}\!\!\int_\Omega \nu\nabla\boldsymbol{v}^*_{\text{ext}}:\nabla\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
&\le \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi)\,\nu^2\int_0^{t^*}\|\nabla\boldsymbol{v}^*_{\text{ext}}\|_2^2\,\mathrm{d}t\\
&= \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi,\boldsymbol{v}^*_{\text{ext}}),
\end{aligned}$$

$$\begin{aligned}
\left|\int_0^{t^*}\langle\partial_t\boldsymbol{v}^*_{\text{ext}},\boldsymbol{u}\rangle\,\mathrm{d}t\right| &\le \int_0^{t^*}\|\partial_t\boldsymbol{v}^*_{\text{ext}}\|_{-1,2}\,\|\boldsymbol{u}\|_{1,2}\,\mathrm{d}\boldsymbol{t}\\
&\le c\int_0^{t^*}\|\partial_t\boldsymbol{v}^*_{\text{ext}}\|_{-1,2}\,\|\nabla\boldsymbol{u}\|_2\,\mathrm{d}\boldsymbol{t}\\
&\le \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi)\int_0^{t^*}\|\partial_t\boldsymbol{v}^*_{\text{ext}}\|^2_{-1,2}\,\mathrm{d}t\\
&= \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi,\boldsymbol{v}^*_{\text{ext}}),
\end{aligned}$$

$$\begin{aligned}
\int_0^{t^*}\!\!\int_\Omega \boldsymbol{v}^*_{\text{ext}}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}^*_{\text{ext}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
&\le \int_0^{t^*}\|\nabla\boldsymbol{u}\|_2\,\|\boldsymbol{v}^*_{\text{ext}}\|_4^2\,\mathrm{d}t\\
&\le \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi)\int_0^{t^*}\|\boldsymbol{v}^*_{\text{ext}}\|_4^4\,\mathrm{d}t\\
&= \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi,\boldsymbol{v}^*_{\text{ext}}),
\end{aligned}$$

$$\begin{aligned}
\int_0^{t^*}\!\!\int_\Omega\big(\boldsymbol{v}^*_{\text{ext}}\cdot\nabla\boldsymbol{v}^*_{\text{ext}}\cdot\boldsymbol{v}^*_{\text{ext}}+\boldsymbol{u}\cdot\nabla\boldsymbol{v}^*_{\text{ext}}\cdot\boldsymbol{v}^*_{\text{ext}}\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t
&\le c(\boldsymbol{v}^*_{\text{ext}})+\int_0^{t^*}\!\!\int_\Omega\|\boldsymbol{u}\|_4\,\|\nabla\boldsymbol{v}^*_{\text{ext}}\|_2\,\|\boldsymbol{v}^*_{\text{ext}}\|_4\,\mathrm{d}t\\
&\le \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi,\boldsymbol{v}^*_{\text{ext}}),
\end{aligned}$$

$$\begin{aligned}
\int_0^{t^*}\langle\boldsymbol{f},\boldsymbol{u}\rangle\,\mathrm{d}t &\le \int_0^{t^*}\|\boldsymbol{f}\|_{-1,2}\,\|\boldsymbol{u}\|_{1,2}\,\mathrm{d}t\\
&\le \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t+c(\xi,\boldsymbol{f}),
\end{aligned}$$

$$\begin{aligned}
\int_0^{t^*}\!\!\int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}S\,\mathrm{d}t &\le \int_0^{t^*}\|\boldsymbol{g}\|_{4/3;\,\Gamma_2}\,\|\boldsymbol{u}\|_{4;\,\Gamma_2}\,\mathrm{d}t\\
&\le \int_0^{t^*}\|\boldsymbol{g}\|_{4/3;\,\Gamma_2}\,\|\boldsymbol{u}\|_{1,2}\,\mathrm{d}t\\
&\le c\int_0^{t^*}\|\boldsymbol{g}\|_{4/3;\,\Gamma_2}\,\|\nabla\boldsymbol{u}\|_2\,\mathrm{d}t\\
&\le \int_0^{t^*}\xi\,\|\nabla\boldsymbol{u}\|_2^2+c(\xi,\boldsymbol{g}),
\end{aligned}$$

where c is independent of t^*, we obtain

$$
\begin{aligned}
&\frac{1}{2}\,\|\boldsymbol{u}(t^*)\|_2^2 + (\nu - 9\xi)\int_0^{t^*} \|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t \\
&\qquad + \frac{1}{2}\int_0^{t^*}\int_{\Gamma_1} (\boldsymbol{v}^*\cdot\boldsymbol{n})\,|\boldsymbol{v}^*|^2\,\mathrm{d}S\,\mathrm{d}t \\
&\quad\le\; c(\xi)\int_0^{t^*} \|\boldsymbol{u}\|^2\,\mathrm{d}t + c(\xi,\boldsymbol{v}^*_{\mathrm{ext}},\boldsymbol{f},\boldsymbol{g}) + \\
&\qquad + c(\xi)\int_0^{t^*} \zeta^6(t)\left(\|\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u}\|_2^2 + \|\boldsymbol{v}^*_{\mathrm{ext}}\|_{1,2}^2\right)\mathrm{d}t \\
&\qquad + \frac{1}{2}\,\|\boldsymbol{u}(0)\|_2^2. \qquad (21)
\end{aligned}
$$

Choose ξ so small that $\xi < \frac{1}{18}\nu$. Evaluating precisely the right hand side (which concerns especially $c(\xi,\boldsymbol{v}^*_{\mathrm{ext}},\boldsymbol{f},\boldsymbol{g})$), we can rewrite the inequality in the form (17). Omitting at first the second term on the left hand side (i.e. the integral of $\|\nabla\boldsymbol{u}\|_2^2$) and applying the generalized Gronwall inequality, we obtain the estimate of $\boldsymbol{u}$ in $L^\infty(0,T;\,\boldsymbol{L}^2(\Omega))$ in terms of the norms of $\zeta(t)$, $\boldsymbol{v}^*_{\mathrm{ext}}$, $\boldsymbol{f}$ and $\boldsymbol{g}$ in appropriate spaces, which are all finite. Then, omitting the first term on the left hand side in (21) and considering $t^*\to T-$, we obtain the estimate of the norm of $\boldsymbol{u}$ in $L^2(0,T;\,\boldsymbol{W}^{1,2}(\Omega))$.

3.5. Remark

By analogy with [15], one can show that if $\boldsymbol{v}$ is a solution of a problem $(\mathcal{P})$ then there exists an associated pressure p as a distribution in $\Omega\times(0,T)$. The pair $(\boldsymbol{v},p)$ satisfies the equations (1), (2) in the sense of distributions in $\Omega\times(0,T)$. If, moreover, $\partial_t\boldsymbol{v}\in L^1(0,T;\,\boldsymbol{W}^{-1,2}(\Omega))$ and one prescribes $\overline{p}\in L^1(0,T)$ then the pressure p can be chosen so that $\int_\Omega p(t)\,\mathrm{d}\boldsymbol{x} = \overline{p}(t)$ for a.a. $t\in(0,T)$.

Suppose now that the solution $\boldsymbol{v}$ has these a posteriori properties: $\boldsymbol{v}\in L^2(0,T;\,\boldsymbol{W}^{2,2}(\Omega))$, $\partial_t\boldsymbol{v}$, $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$, $\boldsymbol{f}\in L^2(0,T;\,\boldsymbol{L}^2(\Omega))$ and there exists $\epsilon_3>0$ such that all $\boldsymbol{\phi}\in\boldsymbol{v}(t)+\boldsymbol{V}$, whose distance from $\boldsymbol{v}(t)$ in the $\boldsymbol{W}^{1,2}$–norm is less than ϵ_3, belong to $\overline{\boldsymbol{K}^c_t}$ for a.a. $t\in(0,T)$. (The last condition means that $\boldsymbol{v}(t)$ lies "uniformly" in the interior of $\overline{\boldsymbol{K}^c_t}$.) Then one can prove that the distribution p is regular and can be represented by a function from $L^2(0,T;\,W^{1,2}(\Omega))$. Moreover, one can also find a function $\vartheta\in L^2(0,T)$ so that

$$
\nu\,\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{n}} - (p+\vartheta)\boldsymbol{n} \;=\; \boldsymbol{g} \qquad (22)
$$

holds a.e. in $\Gamma_2\times(0,T)$. This shows that the concrete pressure, obtained from the variational inequality and satisfying the outflow boundary condition on $\Gamma_2\times(0,T)$, is unique in the sense that it cannot be changed by adding an arbitrary constant (or a function of t).

4. The Navier–Stokes inequality – the steady case

In both the non-steady and steady cases, the solution's proof of existence relies on the construction of appropriate approximations, the estimations of the approximations that in some sense copy a priori estimates, the deduction of various types of convergence of a sequence (or a subsequence) of approximations to some limit function, and the demonstration that the limit is a solution whose existence one wants to prove. As we have already mentioned in subsections 1.2 and 3.4, the crucial part is the derivation of a priori estimates. In order to obtain appropriate estimates, in the non–steady case, one can apply Gronwall–type inequalities in order to obtain a uniform (in time) estimate of the L^2–norm of the solution and the estimate of $\int_0^T \|\nabla\boldsymbol{u}\|_2^2\,\mathrm{d}t$ (see subsection 3.4). In the steady case, the estimates substantially depend on the properties of the extended function $\boldsymbol{v}^*_{\mathrm{ext}}$, introduced in subsection 3.1. Moreover, as follows from estimate (25), we are able to prove the existence of the steady solution only if ζ (which is now just a positive number) is "sufficiently small" in comparison to ν. (Recall the ζ estimates possible reverse flows on the outflow part Γ_2 of the boundary, see (13).)

The extended function $\boldsymbol{v}^*_{\mathrm{ext}}$ should now be naturally time–independent, and should be constructed so that the integral $\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}^*_{\mathrm{ext}}\,\mathrm{d}\boldsymbol{x}$ is "sufficiently small" in comparison with $\|\nabla\boldsymbol{u}\|_2^2$ for all $\boldsymbol{u}\in\boldsymbol{V}$. The reasons are the same as in the case of the steady Navier–Stokes problem with inhomogeneous Dirichlet–type boundary condition on the whole boundary of Ω, see e.g. [21, Chapter IX] for the detailed explanation. It follows from the paper [14] that this condition of "sufficient smallness" of the aforementioned integral is in fact not an obstacle. Concretely, it is shown in [14] that if $\boldsymbol{v}^*$ satisfies the condition

($\star$) $\boldsymbol{v}^*$ can be extended from Γ_1 onto the whole boundary $\partial\Omega$ so that the extended function belongs to $\boldsymbol{W}^{1/2,2}(\partial\Omega)$ is equal to zero on Γ_0 and its flux through $\partial\Omega$ is zero,

then the extension $\boldsymbol{v}^*_{\mathrm{ext}}$ can be constructed so that given $\delta>0$, $\boldsymbol{v}^*_{\mathrm{ext}}\in\boldsymbol{W}^{1,2}(\Omega)$, $\boldsymbol{v}^*_{\mathrm{ext}}$ is divergence–free and

$$
\int_\Omega \boldsymbol{u}_1\cdot\nabla\boldsymbol{u}_2\cdot\boldsymbol{v}^*_{\mathrm{ext}}\,\mathrm{d}\boldsymbol{x} \;\le\; \delta\,\|\nabla\boldsymbol{u}_1\|_2\,\|\nabla\boldsymbol{u}_2\|_2 \qquad (23)
$$

for all $\boldsymbol{u}_1$, $\boldsymbol{u}_2\in\boldsymbol{V}$. This is the analogue of the so called Leray–Hopf inequality, see [21].

Let us show how the a priori estimate looks. Obviously, in the steady case, ζ is just a number and set $\overline{\boldsymbol{K}^c_t}$ is independent of t. Hence we further on use the notation $\overline{\boldsymbol{K}^c}$ instead of $\overline{\boldsymbol{K}^c_t}$. The "steady state version" of inequality (16) is

$$
\begin{aligned}
&\int_\Omega \boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nu\nabla\boldsymbol{v}\cdot\nabla(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} \\
&\qquad \ge\; \langle\boldsymbol{f},\boldsymbol{w}-\boldsymbol{v}\rangle + \int_{\Gamma_2} \boldsymbol{g}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}S. \qquad (24)
\end{aligned}
$$

The solution $\boldsymbol{v}$ lies in $\overline{\boldsymbol{K}^c}$ and the inequality is required to be satisfied for all $\boldsymbol{w}\in\overline{\boldsymbol{K}^c}$. Writing $\boldsymbol{v}$ in the form $\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u}$, where $\boldsymbol{u}\in\boldsymbol{V}$, using inequality (24) with

$\boldsymbol{w} = \boldsymbol{v}^*_{\mathrm{ext}}$, and applying (19), (20) and (23), we obtain

$$\int_\Omega \boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x} + \int_\Omega \nu\nabla\boldsymbol{v}\cdot\nabla(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}\boldsymbol{x}$$
$$\geq \langle \boldsymbol{f}, \boldsymbol{w}-\boldsymbol{v}\rangle + \int_{\Gamma_2} \boldsymbol{g}\cdot(\boldsymbol{w}-\boldsymbol{v})\,\mathrm{d}S,$$

$$\nu\,\|\nabla\boldsymbol{u}\|_2^2 + \int_\Omega (\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u})\cdot\nabla(\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u})\cdot(\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}$$
$$\leq \int_\Omega \big[\nu\,\nabla\boldsymbol{v}^*_{\mathrm{ext}} : \nabla\boldsymbol{u}$$
$$+ (\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u})\cdot\nabla(\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u})\cdot\boldsymbol{v}^*_{\mathrm{ext}}\big]\,\mathrm{d}\boldsymbol{x}$$
$$+ \langle\boldsymbol{f},\boldsymbol{u}\rangle + \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}S,$$

$$\nu\,\|\nabla\boldsymbol{u}\|_2^2 + \frac{1}{2}\int_{\Gamma_1}(\boldsymbol{v}^*\cdot\boldsymbol{n})\,|\boldsymbol{v}^*|^2\,\mathrm{d}S$$
$$+ \frac{1}{2}\int_{\Gamma_2}\big[(\boldsymbol{v}^*+\boldsymbol{u})\cdot\boldsymbol{n}\big]\,|\boldsymbol{v}^*+\boldsymbol{u}|^2\,\mathrm{d}S$$
$$\leq \int_\Omega \big[\nu\,\nabla\boldsymbol{v}^*_{\mathrm{ext}} : \nabla\boldsymbol{u} + \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}$$
$$+ \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}$$
$$+ \boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}^*_{\mathrm{ext}}\big]\,\mathrm{d}\boldsymbol{x} + \langle\boldsymbol{f},\boldsymbol{u}\rangle + \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}S,$$

$$\nu\,\|\nabla\boldsymbol{u}\|_2^2 \;\leq\; -\frac{1}{2}\int_{\Gamma_1}(\boldsymbol{v}^*\cdot\boldsymbol{n})\,|\boldsymbol{v}^*|^2\,\mathrm{d}S$$
$$+ \frac{1}{2}\int_{\Gamma_2}\big[(\boldsymbol{v}^*+\boldsymbol{u})\cdot\boldsymbol{n}\big]_-\,|\boldsymbol{v}^*+\boldsymbol{u}|^2\,\mathrm{d}S$$
$$+ \int_\Omega \big[\nu\,\nabla\boldsymbol{v}^*_{\mathrm{ext}} : \nabla\boldsymbol{u} + \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}$$
$$+ \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}\big]\,\mathrm{d}\boldsymbol{x}$$
$$+ \delta\,\|\nabla\boldsymbol{u}\|_2^2 + \langle\boldsymbol{f},\boldsymbol{u}\rangle + \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}S,$$
$$\leq\; -\frac{1}{2}\int_{\Gamma_1}(\boldsymbol{v}^*\cdot\boldsymbol{n})\,|\boldsymbol{v}^*|^2\,\mathrm{d}S + \frac{1}{2}\,\zeta\,\|\boldsymbol{v}^*+\boldsymbol{u}\|_{3;\,\Gamma_2}^2$$
$$+ \int_\Omega \big[\nu\,\nabla\boldsymbol{v}^*_{\mathrm{ext}} : \nabla\boldsymbol{u} + \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}$$
$$+ \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}\big]\,\mathrm{d}\boldsymbol{x}$$
$$+ \delta\,\|\nabla\boldsymbol{u}\|_2^2 + \langle\boldsymbol{f},\boldsymbol{u}\rangle + \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}S,$$
$$\leq\; -\frac{1}{2}\int_{\Gamma_1}(\boldsymbol{v}^*\cdot\boldsymbol{n})\,|\boldsymbol{v}^*|^2\,\mathrm{d}S + \frac{\zeta}{2}\,c_6\,\|\boldsymbol{v}^*_{\mathrm{ext}}+\boldsymbol{u}\|_{1,2}^2$$
$$+ \int_\Omega \big[\nu\,\nabla\boldsymbol{v}^*_{\mathrm{ext}} : \nabla\boldsymbol{u} + \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}$$
$$+ \boldsymbol{v}^*_{\mathrm{ext}}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}\cdot\nabla\boldsymbol{v}^*_{\mathrm{ext}}\cdot\boldsymbol{v}^*_{\mathrm{ext}}\big]\,\mathrm{d}\boldsymbol{x}$$
$$+ \delta\,\|\nabla\boldsymbol{u}\|_2^2 + \langle\boldsymbol{f},\boldsymbol{u}\rangle + \int_{\Gamma_2}\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}S,$$

where $c_6 = c_6(\Omega)$. Writing only the terms with second powers of $\boldsymbol{u}$, which are decisive for the estimates, and involving all other terms to a generic constant c, we obtain

$$\nu\,\|\nabla\boldsymbol{u}\|_2^2 \;\leq\; \delta\,\|\nabla\boldsymbol{u}\|_2^2 + c_7\,\zeta\,\|\nabla\boldsymbol{u}\|_2^2 + c, \tag{25}$$

where $c_7 = c_7(\Omega)$. As $\delta > 0$ can be chosen to be arbitrarily small, we observe that these inequalities yield an a priori estimate of $\|\nabla\boldsymbol{u}\|_2$ in terms of $\boldsymbol{v}^*$, $\boldsymbol{f}$ and $\boldsymbol{g}$, provided that $\zeta > 0$ is so small that $c_7\zeta < \nu$. Obviously, in this case one also obtains an a priori estimate of $\|\boldsymbol{v}\|_{1,2} \equiv \|\boldsymbol{v}^*_{\mathrm{ext}} + \boldsymbol{u}\|_{1,2}$. Under the aforementioned condition on ζ, one can prove the existence of a weak solution $\boldsymbol{v} \in \overline{\boldsymbol{K}^c}$ of the variational inequality (24), applying the procedure sketched at the beginning of this section. (See also [14] for the construction of appropriate approximations and the detailed derivation of the estimates on the level of approximations. However, the convex set, used in paper [14], differs from $\overline{\boldsymbol{K}^c}$ used here.) Thus, we can formulate the theorem:

Theorem 2. *Let functions $\boldsymbol{v}^* \in \boldsymbol{W}^{1/2,2}(\Gamma_1)$ (satisfying condition $(\star)$), $\boldsymbol{f} \in \boldsymbol{W}^{-1,2}(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{L}^{4/3}(\Gamma_2)$ be given. Let number ζ be so small that $c_7\zeta < \nu$. Then there exists $\boldsymbol{v} \in \overline{\boldsymbol{K}^c}$, such that the variational inequality (24) is satisfied for all $\boldsymbol{w} \in \overline{\boldsymbol{K}^c}$.*

Recall that ζ is used in the definition of the convex set $\overline{\boldsymbol{K}^c}$, see (12) and (13). The smaller is ζ, the smaller is $\overline{\boldsymbol{K}^c}$ and the narrower space is left for possible reverse flows on Γ_2.

5. Conclusion

The paper provides a mathematical model of flows through a channel with an artificial boundary condition (5) on the outflow. Both unsteady and steady cases are considered. The core of the model is the variational inequalities (16) (in the unsteady case) and (24) (in the steady case). Solutions are sought in an appropriate closed convex subsets of relevant function spaces, defined by means of restrictions, imposed on possible reverse flows on the outflow. The restricting conditions bound the kinetic energy, brought back to Ω through Γ_2 by the reverse flows. Consequently, they enable one to derive energy–type a priori estimates. Then, applying a relatively standard technique (based e.g. on construction of appropriate approximations or some of the fixed point theorems), one can come to the conclusion on the existence of solutions. This confirms the sense of the used model and associated variational inequalities, in contrast to models based just on equations, where the existence of weak or strong solutions is generally an open problem.

Except for the discussion on various boundary conditions of the "do nothing" type (see paragraphs 2.2 and 2.3) and some a posteriori properties of solutions (paragraph 3.5), we present a detailed description of a priori estimates of solutions. These estimates clarify, on the formal level, how the information that the solutions belong to $L^\infty(0,T;\,\boldsymbol{L}^2(\Omega)) \cap L^2(0,T;\,\boldsymbol{W}^{1,2}(\Omega))$ (in the unsteady case) or $\boldsymbol{W}^{1,2}(\Omega)$ (in the steady case) directly follows from the used variational inequalities, regardless of other technicalities, connected e.g. with possible approximations. Analogous estimates have been obtained in a completely different and much more technical way (i.e. at first on the level of approximations and then considering an appropriate limit transition) in papers [14] and [15]. However, it must be noted that while the convex set, corresponding to our $\overline{\boldsymbol{K}_t^c}$, is defined in a rather artificial way in [14] and [15], our $\overline{\boldsymbol{K}_t^c}$ has a good physical sense. Naturally, the change of set $\overline{\boldsymbol{K}_t^c}$ requires a new technique in the derivation of approximations.

We do not present any numerical justification of our model. Nevertheless, we recall that corresponding numerical experiments, also involving comparison between various artificial boundary conditions on the outflow, suggested in paragraphs 2.2 and 2.3, would be very desirable and interesting.

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