Size Effect in Fracture of Concrete Specimens and Structures: New Problems and Progress

Z. P. Bažant, Q. Yu

Presented is a concise summary of recent Northwestern University studies of six new problems. First, the decrease of fracture energy during crack propagation through a boundary layer, documented by Hu and Wittmann, is shown to be captured by a cohesive crack model in which the softening tail slope depends on the distance from the boundary (which causes an apparent size effect on fracture energy and implies that the nonlocal damage model is more fundamental than the cohesive crack model). Second, an improved universal size effect law giving a smooth transition between failures at large cracks (or notches) and at crack initiation is presented. Third, a recent renewed proposal that the nominal strength variation as a function of notch depth be used for measuring fracture energy is critically examined. Fourth, numerical results and a formula describing the size effect of finite-angle notches are presented. Fifth, a new size effect law derivation from dimensional analysis coupled with asymptotic matching is given. Finally, an improved code-type formula for shear capacity of R.C. beams is proposed.

Keywords: size effect, scaling, fracture mechanics, cohesive cracks, quasibrittle materials, concrete.

Note: This paper is a republication, with minor modification of Bažant & Yu (2004), authorized by IA-FraMCoS

1 Introduction

In solid mechanics, unlike fluid mechanics, it is still not widely recognized that knowledge of the size effect, or scaling, is the means to obtain analytical predictions of quasibrittle failures *in general*, even if the size effect need not be calculated. For the actual size of interest, a direct analytical solution is hard, next to impossible. However, by scaling the structure down to vanishing size, or up to infinite size, one gets a ductile, or brittle, response, either of which is much easier to solve. Knowing these asymptotic solutions, an approximate failure prediction for the middle range of practical interest can then be obtained by *asymptotic matching* – "interpolation" between the opposite infinities. It is for this reason that the size effect is the key problem for all quasibrittle failures.

The purpose of this paper is to present a brief summary of the advances in six problems of size effect recently studied at Northwestern University.

2 Variation of cohesive softening law in boundary layer

By now it has been well established that the total fracture energy G_F of a heterogeneous material such as concrete, defined as the area under the cohesive softening curve, is not constant but varies during crack propagation across the ligament. The variation of G_F at the beginning of fracture growth, which is described by the R-curve, is only an apparent phenomenon which is perfectly consistent with the cohesive crack model (with a fixed softening law) and can be calculated from it. However, the variation during propagation through the boundary layer at the end of the ligament is not consistent with the cohesive crack model and implies that the softening curve of this model is not an invariant property.

The fact that the fracture energy representing the area under the softening curve should decrease to zero at the end of the ligament was pointed out in a paper by Bažant (1996), motivated by the experiments of Hu and Wittmann (1991 and 1992a), and was explained by a decrease of the fracture process zone (FPZ) size, as illustrated on the left of Fig. 1a reproduced from (1996) paper of Zdeněk Bažant. An experimental verification and detailed justification of this property was provided in the works of Hu (1997, 1998), Hu and Wittmann (1992b, 2000), Duan, Hu and Wittmann (2002, 2003), and Karihaloo, Abdalla, and Imjai (2003). As mentioned by Bažant (1996) as well as Hu and Wittmann, the consequence of these experimental observations is that:

$$\overline{G}_F(a) = \frac{1}{D-a_0}\int_0^\infty P\,\mathrm{d}u = \int_0^D \Gamma(x)\mathrm{d}x < G_F, \tag{1}$$

where $\overline{G}_F$ = average fracture energy in the ligament (Fig. 1b), D = specimen size (Fig. 1a), a_0 = notch depth, P = load, u = load-point deflection, $\Gamma(x)$ = local fracture energy as a function of coordinate x along the ligament (Fig. 1a left), $G_F = \int_0^\infty \sigma\mathrm{d}w = \Gamma(x)$ = value at points x remote from the boundary (= area under the complete $\sigma(w)$ diagram, Fig. 1c), σ = cohesive (crack-bridging) stress, and w = crack opening = = separation of crack faces.

Is this behavior compatible with the cohesive crack model? To check it, consider a decreasing FPZ attached to the boundary at the end of ligament, Fig. 1a. Exteding to this situation Rice's (1968) approach, which effectively launched the use of the cohesive (or fictitious) crack model, we calculate the J-integral along a path touching the crack faces as shown in Fig. 2a,b:

$$\begin{aligned} J &= \oint\left(\overline{U}\mathrm{d}y - t_i\frac{\partial u_i}{\partial x}\mathrm{d}x\right) = -2\int_{x_{tip}}^{D}\sigma\frac{\mathrm{d}v(x)}{\mathrm{d}x}\mathrm{d}x - [t_2\Delta w]_{\mathrm{end}} = \\ &= \int_0^\infty \sigma(x)\mathrm{d}w - [\sigma\Delta w]_{\mathrm{end}} = \text{area below } w_{\mathrm{end}} = \\ &= J_{\mathrm{end}}(D) \text{ or } J_{\mathrm{tail}}(x) \text{ or } J_{\mathrm{middle}}[(x+D)/2], \end{aligned} \tag{2}$$

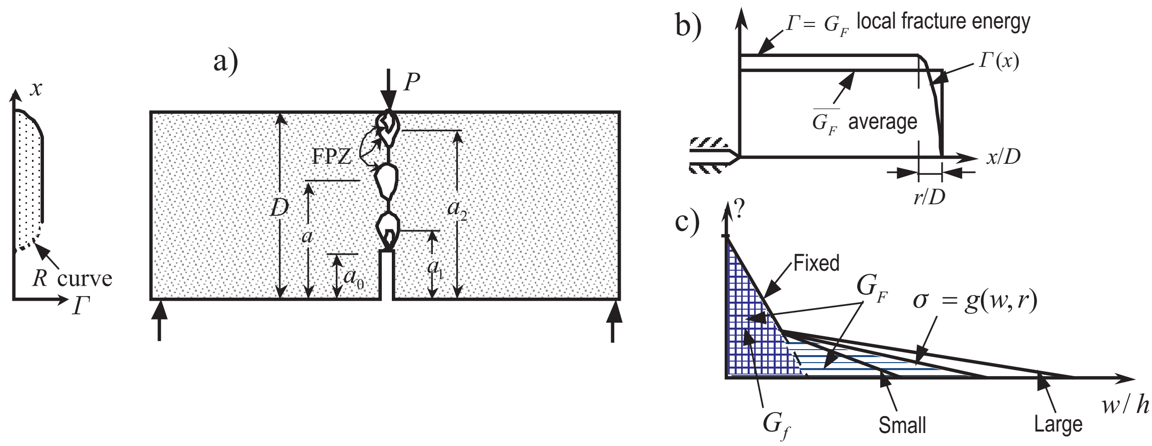

Fig. 1: (a) Variation of local fracture energy $\Gamma(x)$ across the ligament, decreasing in the boundary layer (reproduced from Bažant 1996); (b) Average fracture energy G_F; (c) Required modification of cohesive (or fictitious) crack model

in which σ = path length; $x = x_1$, $y = x_2$ are the Cartesian coordinates; u_i = displacements; t_i = tractions acting on the path from the outside; $\overline{U}$ = strain energy density; $v = w/2$; $x = D$ is the end point of the ligament; w_{end} is the opening at the end of ligament (Fig. 2b). From these equations, we see that the instantaneous flux of energy, J, into the shrinking FPZ attached to the end of ligament (Fig. 1a) represents the area below the line $\sigma = \sigma_{\mathrm{end}}$ in the softening diagram, cross-hatched in Fig. 2i.

It is, however, a matter of choice with which coordinate x in the FPZ this flux J should be associated. If we associate J with the front, the tail, or the middle (Fig. 2b) of the FPZ, we get widely different plots of $\Gamma = J_{\mathrm{end}}, J_{\mathrm{tail}}, J_{\mathrm{middle}}$, as shown in Fig. 2c, d, e, respectively (the first one terminating with Dirac delta function). This ambiguity means that the boundary layer effect experimentally documented by Hu and Wittmann cannot be represented by the standard cohesive crack model, with a fixed stress-separation diagram.

Can the cohesive crack model be adapted for this purpose? It follows from Eq. (4) and Fig. 2 (h, i) (and has been computationally verified) that Wittmann et al.'s (1990) and Elices et al.'s (1992) data can be matched (Fig. 2j) if the slope of the tail segment of the bilinear stress-separation diagram for concrete is assumed to decrease (Fig. 1c) in proportion to diminishing distance $r = D - x$ (Fig. 2b) from the end of the ligament. After such an adaptation, the cohesive (or fictitious) crack model has a general applicability, including the boundary layer.

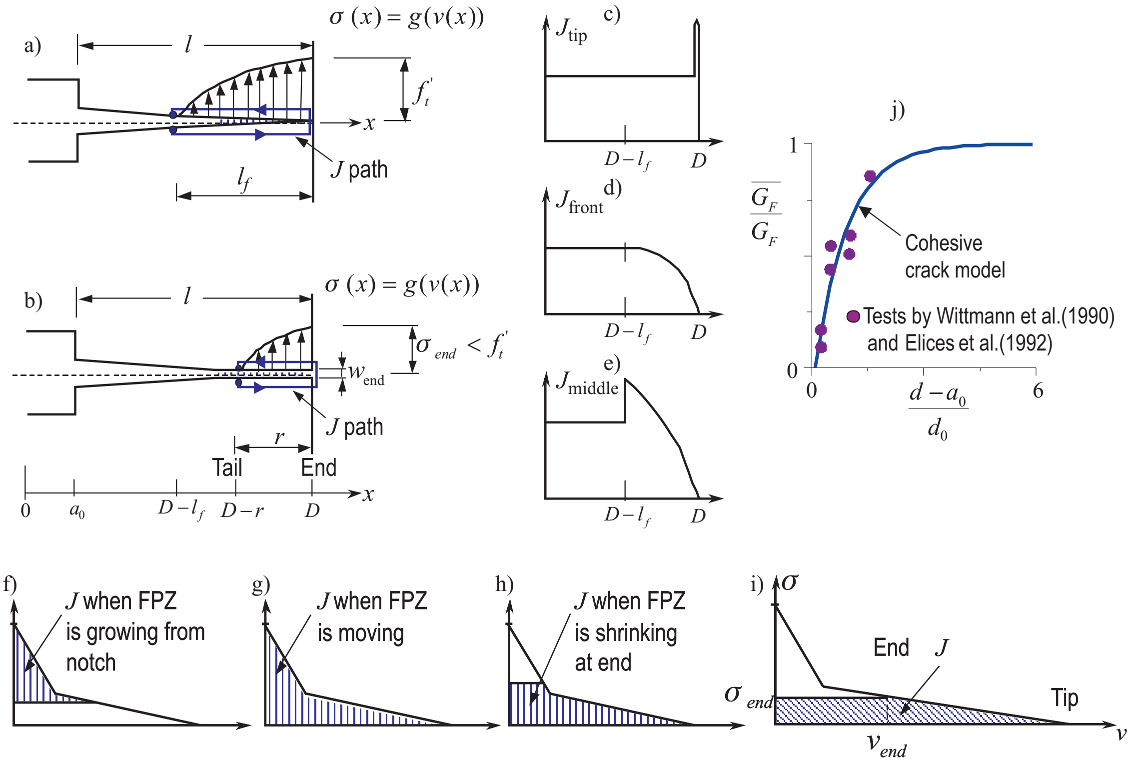

Fig. 2: (a, b) J-integral path; (c, d, e) Ambiguity in J-integral variation; (f, g, h, i) Fracture energies corresponding to J-integral; (j) Test data fitted by modified cohesive crack model with variable tail

However, the consequence is that the total fracture energy G_F (area under the complete stress-separation curve) is not constant. Noting that the larger the structure, the smaller is the length fraction of the boundary layer, one must conclude that the diminishing tail slope in Fig. 1c automatically implies a certain size effect on the apparent G_F, as given by Eq. (1).

It further follows from Fig. 2 (h, i), and has been computationally verified, that the initial tangent of the stress-separation diagram, the area under which represents the initial fracture energy G_f (Bažant 2002a, b, Bažant, Yu and Zi 2002), can be considered as fixed – in other words, G_f, unlike G_F, is a material constant. Aside from the fact that the maximum loads of specimens and structures are generally controlled by G_f, not G_F, this suggests that the standard fracture test that should be introduced is that which yields not G_F but G_f (the size effect method, as well as the method of Guinea, Planas and Elices, 1994a, b, serve this purpose, while the work-of-fracture method does not). This conclusion is not surprising in the light of abundant experimental data revealing that G_F is statistically much more variable than G_f (Bažant and Becq-Giraudon 2002, Bažant, Yu and Zi 2002).

Jirásek (2003) showed that Hu and Wittmann's data can be matched by a nonlocal continuum damage model in which the characteristic softening curve is kept fixed. Consequently the nonlocal model is a more general, and thus more fundamental, characterization of fracture than the cohesive (or fictitious) crack model. This finding should be taken into account in fracture testing. It appears that G_F would better be defined by the area under the softening curve of the nonlocal model, multiplied by the characteristic length of material corresponding to the effective width of FPZ, which equals the minimum possible spacing of parallel cracks (Bažant 1985, Bažant and Jirásek 2002), to be distinguished from $l_0 = EG_f \big/ f_t'^2$.

3 Universal size effect law

As is now well known, the size effect for crack initiation from a smooth surface ($a_0 = 0$) is very different from the size effect for large notches or large stress-free (fatigued) cracks at maximum load (a_0/D not to small). As far as the *mean* nominal strength of structure, σ_N, is concerned, the former is always energetic (i.e., purely deterministic), while the latter is purely energetic only for small enough sizes and becomes statistical for large enough sizes. It is of interest to find a universal size effect law that includes both of these size effects and spans the transition between them. A formula for this purpose was proposed in Bažant and Li (1996) (also Bažant and Chen 1997, Bažant 2002b). However, that formula was not smooth and did not include the statistical (Weibull) part for crack initiation failures.

$$\sigma_N = \left(\frac{E' G_f}{g_0' c_f + g_0 D}\right)^{1/2} \left(1 - \frac{r c_f^2 g_0'' e^{-k\alpha^2}}{4(l_0 + D)(g_0 D + g_0' c_f)}\right)^{1/r} \qquad \sigma_N = \left(\frac{E' G_f}{g_0' c_f + g_0 D}\right)^{1/2} \left(\left(\frac{l_s}{l_s + D e^{-(a_0/D)^2}}\right)^{\frac{rn}{m}} - \frac{r c_f^2 g_0'' e^{-k\alpha^2}}{4(l_0 + D)(g_0 D + g_0' c_f)}\right)$$

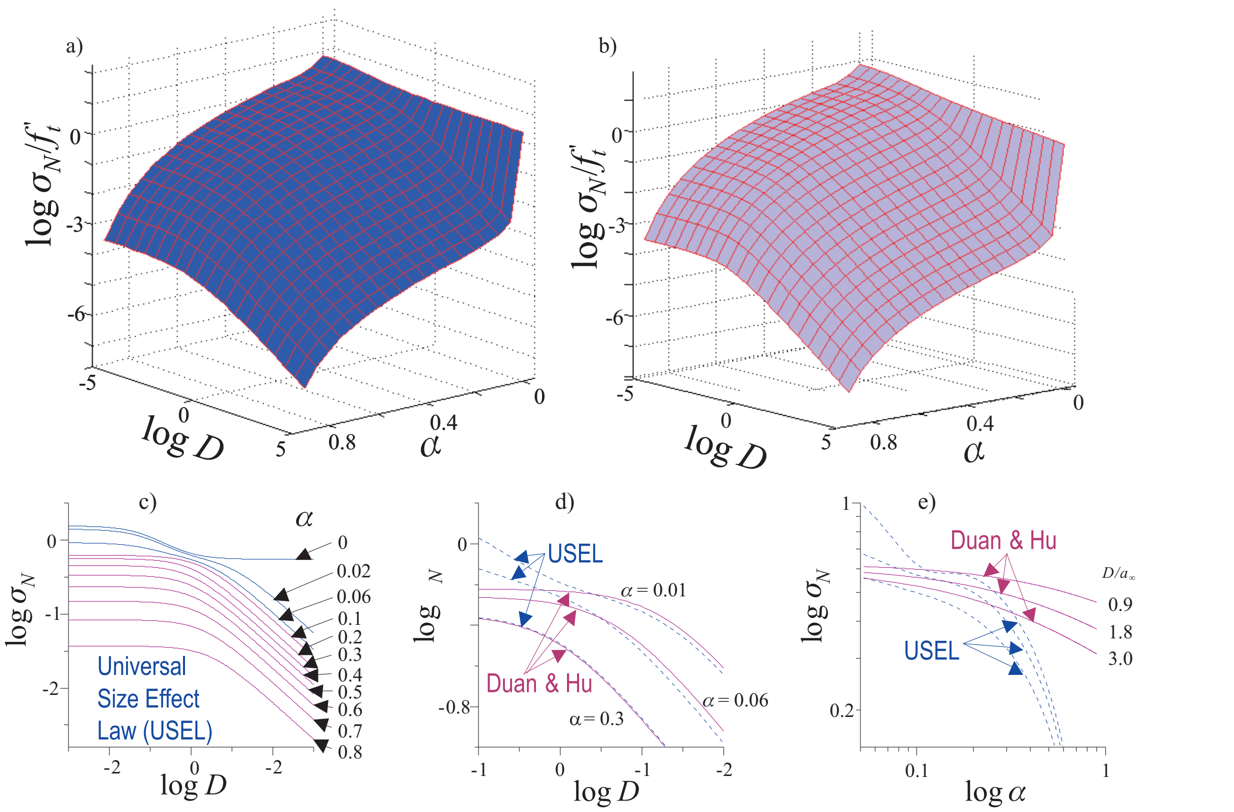

Fig. 3: (a, b) Improved universal size effect law USEL (left–without, right –with Weibull statistics), and (c, d, e) profiles obtained from USEL, compared to Duan and Hu's (2003) approximation

A better formula has now been obtained; see Figs. 3, the left (Fig. 3a) without, and the right (Fig. 3b) with, the statistical (Weibull) part, in which E' = effective Young modulus; f_t' = local tensile strength of material; g_0, g_0', g_0'' are values of dimensionless energy release function $g(\alpha)$ and its derivatives at $\alpha_0 = a_0/D$; $l_0 = E'G_f \Big/ f_t'^2$ = Irwin's characteristic length (corresponding to the initial fracture energy); $g(\alpha) = k^2(\alpha)$, $k(\alpha)$ is the dimensionless stress intensity factor; m = Weibull modulus of concrete (about 24), n = number of dimensions for scaling; r, k = empirical positive constants; and c_f = constant (the ratio c_f/l_0 depends on the softening curve shape, and $c_f \approx l_0/2$ for triangular softening). The formulas in Figs. 3a, b were derived by asymptotic matching of 6 cases: the small-size and large-size asymptotic behaviors (first two terms of expansion for each), of the large-notch and vanishing-notch behaviors, and of the energetic and statistical parts of size effect.

4 Can fracture energy be measured on one-size specimens with different notch lenghts?

The fact that specimens of different sizes are needed for the size effect method of measuring G_f is considered by some as a disadvantage. For this reason, Bažant and Kazemi (1990), Bažant and Li (1996) and Tang et al. (1996) generalized the size effect method to dissimilar specimens, the dissimilarity being caused by the use of different notch lengths a_0 in specimens of one size. If the random scatter of test data were small (coefficient of variation CoV < 4 %), this approach would work. However, for the typical scatter of maximum loads of concrete specimens (CoV = 8%), the range of brittleness numbers attainable by variation of notch length in a specimen of any geometry (about 1:3, Bažant and Li 1996) does not suffice to get a sharp trend in size effect regression of test data, and thus prevents determining G_f accurately.

Recently, this problem was considered independently by Duan and Hu (2003). They proposed the semi-empirical formula:

$$\sigma_n = \sigma_0 \left(1 + \frac{a}{a_\infty^*}\right)^{-\frac{1}{2}}, \tag{6}$$

where $\sigma_0 = f_t'$ for small 3PB specimens; a_∞^* is a certain constant; and σ_n represents the maximum tensile stress in the ligament based on a linear stress distribution over the ligament, $\sigma_n = \sigma_N / A(\alpha_0)$.

This alternative formula, intended for specimens of the same size when the notch length a_0 is varied, in effect attempts to replace the profile of the universal size effect law (Fig. 3) at constant size D, scaled by the ratio

$$\sigma_n/\sigma_N = 1/A(\alpha_0),$$

where $A(\alpha_0)$ depends on specimen geometry; $A(\alpha_0) = (1-\alpha_0)^2$ for notched three-point bend beams. However, the curve of the proposed formula has, for $a_0 \to 0$, a size independent limit approached, in $\log a_0$ scale, with a horizontal asymptote, while the correct curve, amply justified by tests of modulus of rupture (flexural strength) of unnotched beams (Bažant and Li 1995, Bažant 1998, 2001, 2002b, Bažant and Novák 2000a, b, c), terminates with a steep slope for $a_0 \to 0$ and has a size dependent limit, as seen in the aforementioned profiles in Fig. 3a, b, and better in Fig. 3c. Fig. 3d, e shows the profiles in D and in α, and it is seen that they cannot be matched well by Duan and Hu's approximation converted from σ_n to σ_N. The test data on the dependence of σ_n on a_0, which Duan and Hu fitted by their formula, should better be fitted with the size effect law (Bažant and Kazemi 1990, Bažant and Planas 1998, and Bažant 1997, 2002b):

$$\sigma_n = \frac{\sigma_N}{A(\alpha_0)}, \quad \sigma_N = \sqrt{\frac{E' G_f}{g'(\alpha_0) c_f + g(\alpha_0) D}}, \tag{7}$$

in which D is constant and $\alpha_0 = a_0/D$ is varied (and function $g(\alpha_0)$ is available from handbooks). However, very short or zero notches ($a_0 < 0.15\ D$) must be excluded, which means that the value of strength f_t' cannot be used with Duan and Hu's approach. To use it, it is necessary to adopt either the approach of Guinea et al. (1994a, b) or the zero brittleness method (Bažant, Yu and Zi 2002).

5 Size effect of finite-angle notches

In elastic bodies, a sharp notch of a finite angle (Fig. 4) causes stress singularity $\sigma \propto r^{\lambda-1}$ that is weaker than the crack singularity ($\lambda > 0.5$) and is given by Williams' (1952) formulas (a)–(e) shown in Fig. 4, in which r, φ = polar coordinates, σ_{rr}, $\sigma_{\varphi\varphi}$, $\sigma_{r\varphi}$ = near-tip stresses. If the structure has a positive geometry, it will fail as soon as an FPZ of a certain characteristic length $2c_f$ is fully formed at the notch tip. In the limit of $D \to \infty$, the structure will fail as soon as a crack can start propagating from the notch tip, which requires a certain critical energy release rate equal to G_f. Experiments show that the load (or nominal stress σ_N) at which this occurs increases with angle γ. In previous studies (e.g., Carpinteri 1987, Dunn et al. 1997a,b), some arguments in terms of a non-standard 'stress intensity factor' K_γ corresponding to singularity exponent $1-\lambda < 0.5$ were used to propose that the nominal stress $\sigma_N \propto D^{\lambda-1}$.

A notch of finite angle cannot propagate. So, a realistic approach requires considering that a cohesive crack must propagate from the notch tip (Fig. 4 left). Circular bodies with notches of various angles 2γ (and ligament dimension D, Fig. 4a) were simulated by finite elements with a mesh progressively refined as $r \to 0$ (the first and second rings of elements were bounded by $r = D/6000$ and $D/3000$). The circular boundary was loaded by normal and tangential surface tractions equal to stresses σ_{rr} and $\sigma_{r\varphi}$ taken from Williams' symmetric (Mode I) solution; P = load parameter representing the resultant of these tractions; and $\sigma_N = P/bD$ = nominal stress ($b = 1$).

First, ligament D was considered to be so large that the length of the FPZ, l_c, was less than 0.01 D. In that case, the angular distribution of stresses along each circle with $r \geq 0.1\,D$ ought to match Williams' functions f_{rr}, $f_{r\varphi}$, $f_{\varphi\varphi}$. Indeed, the numerical results could not be visually distinguished from these functions. The logarithmic plots of the calculated stress versus r for any fixed φ (and any γ) were straight lines, and

(a) $\sigma_{rr} = Ar^{\lambda-1} f_{rr}(\varphi,\gamma)$
$= Ar^{\lambda-1}\left[(\lambda+1)f(\varphi,\gamma) + f''(\varphi,\gamma)\right]$

(b) $\sigma_{00} = Ar^{\lambda-1} f_{\varphi\varphi}(\varphi,\gamma)$
$= Ar^{\lambda-1}\left[\lambda(\lambda+1)f(\varphi,\gamma)\right]$

(c) $\sigma_{r0} = Ar^{\lambda-1} f_{r\varphi}(\varphi,\gamma)$
$= Ar^{\lambda-1}\left[-\lambda f'(\varphi,\gamma)\right]$

(d) Mode I: $f(\varphi,\gamma) = \cos(\lambda+1)\,\varphi - \dfrac{(\lambda+1)\sin(\lambda+1)(\pi-\gamma)}{(\lambda-1)\sin(\lambda-1)(\pi-\gamma)}\cos(\lambda-1)\,\varphi$

(e) Eq. for λ: $\lambda\sin 2(\pi-\gamma) = -\sin 2\lambda(\pi-\gamma)$

(f) $A = \dfrac{\sigma_N D^{1-\lambda}}{\psi(\gamma)}$ $\qquad \sigma_N = \dfrac{P}{D}$

(g) $\psi(\gamma) = \int_0^{(\pi-\gamma)}\left[f_{rr}(\varphi,\gamma)\sin\varphi + f_{r\varphi}(\varphi,\gamma)\cos\varphi\right]\mathrm{d}\varphi$

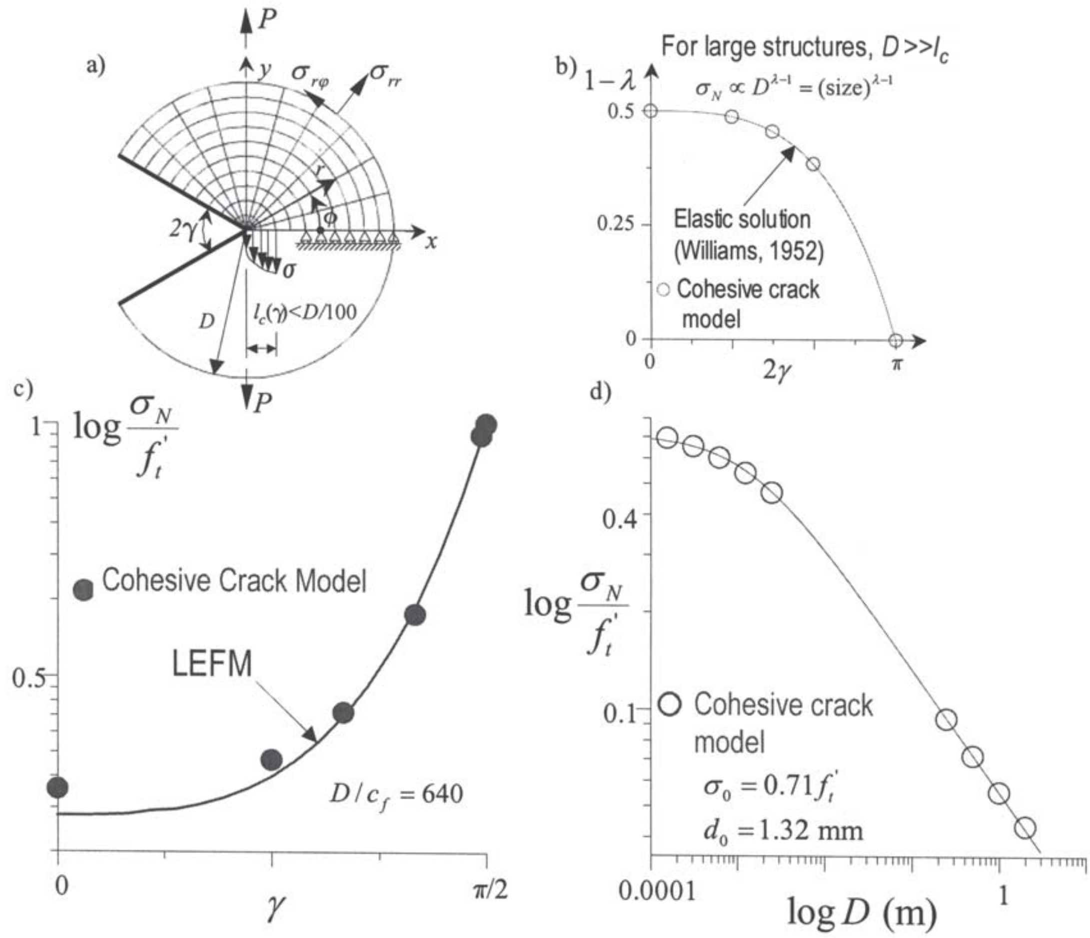

Fig. 4: (a) Angle-notched circular specimen considered for analysis, (b) Numerically computed singularity exponent compared to Williams' elastic solution, (c) Computed variation of nominal strength with notch angle, (d) Analytical size effect law for angular notches compared to cohesive crack model

their slopes agreed with the exponent $\lambda - 1$ required by Williams solution, see Fig. 4 (right). Thus the correctness of the cohesive finite element simulation was confirmed.

From Eqs. (b), (f) and (g) of Williams' LEFM solution in Fig. 4,

$$\sigma_N = \frac{(r/D)^{1-\lambda}\sigma_{\varphi\varphi}\,\psi(\gamma)}{f_{\varphi\varphi}(0,\gamma)}. \tag{8}$$

According to the equivalent LEFM approximation of cohesive fracture, $\sigma_{\varphi\varphi}$ for $r = c_f$ (the middle of FPZ), should be approximately equal to material tensile strength f_t'. This condition yields,

$$\sigma_N = f_t'(D/H_r)^{\lambda(\gamma)-1} \tag{9}$$

$$H_r = c_f\left[\frac{f_{\varphi\varphi}(0,\gamma)}{\psi(\gamma)}\right] \tag{10}$$

valid for $D \ge 250\, l_0$; $\lambda(\gamma)$ is the λ value for angle γ. To check this equation, geometrically similar scaled circular bodies of different ligament dimensions D (Fig. 4) were analyzed by finite elements for various angles γ using the same linear softening stress-separation diagram of cohesive crack. The numerically obtained values of log σ_N for various fixed D/c_f are plotted in Fig. 4c as a function of angle γ. We see that this size effect curve matches perfectly the curve of Eq. (9) and (10) for $D/c_f > 500$, confirming that the equivalent LEFM approximation obtained for $r = c_f$ is good enough.

A general approximate formula for the size effect of notches of any angle, applicable to any size D, may be written, as proposed by Bažant (2003), as follows:

$$\sigma_N = \sigma_0 \left(1 + \frac{H_0}{H_\gamma} \frac{D}{D_0}\right)^{\lambda(\gamma)-1} \tag{11}$$

where $H_0 = h_\gamma$ value $\gamma = 0$, and D_0 is given in terms of $g(\alpha)$, and is the same as for a crack ($\gamma = 0$). Eq. (11), which is of course valid only for large enough notches penetrating through the boundary layer of concrete, has been derived by asymptotic matching of the following asymptotic conditions:

1) for $\gamma \to 0$, the classical size effect law $\sigma_N = \sigma_0 (1 + D/D_0)^{-1/2}$ must be recovered;
2) for $D/l_0 \to 0$, there must be no size effect;
3) for $D/l_0 \to \infty$, Eqs. (11) and (9) must coincide;
4) for $\gamma = \pi/2$ (flat surface), the formula must give no size effect when $D \to \infty$.

In reality, there is of course a size effect in the last case, but it requires a further generalization of Eq. (11) (which will be presented separately). Therefore, Eqs. (11) and (9) can be applied only when the notch is deeper than the boundary layer, which is at least one aggregate size. Complete generality will require amalgamating Eq. (11) with the universal size effect law in Fig. 3.

The plot of log σ_N versus log D for $\gamma = \pi/3$ according to Eq. (11) is compared to the finite element results for notched circular bodies with cohesive cracks in Fig. 4d. The agreement is seen to be excellent.

6 New derivation of size effect law from asymptotic dimensional analysis

From Buckingham's (1914) Π–theorem of dimensional analysis, two size effects can be easily proven: (1) if the failure depends only on material strength f_t' (dimension N/m^2), then there is no size effect, i.e., the nominal strength of geometrically similar specimens does not depend on their size, and (2) if the failure depends only on material facture energy G_f (dimension N/m), then of course the size effect is $\sigma_N \propto D^{-1/2}$ (e.g. Bažant 1993). Nothing more can be deduced.

Knowing that Irwin's characteristic length $l_0 = E\,G_f / f_t'^2$ has the physical meaning of fracture process zone length, one can deduce more. When $D/l_0 \to 0$, the body is much smaller than the fracture process zone, and so G_f cannot matter. It follows that case (1) corresponds to the small-size asymptotic limit, i.e., a horizontal asymptote in the plot of log σ_N versus log D. When $D/l_0 \to \infty$, the fracture process zone is a point compared D, and so there is a stress singularity, which means that the local material strength cannot matter. It follows that case (2) corresponds to the large-size asymptotic limit, i.e. an inclined asymptote of slope −1/2 in the same plot. Hence, for the intermediate sizes, the size effect must be a gradual transition from the horizontal to the inclined asymptote. To deduce the form of this transition, one needs to further take into account the known asymptotic properties of the cohesive crack model (Bažant 2001, 2002b). They may be satisfied as follows (Bažant 2003).

From the governing parameters of the failure problem, σ_N, D, f_t', G_f and E, we may form, according to Buckingham's Π-theorem, two and only two independent dimensionless parameters, which we choose as

$$\Pi_1 = \frac{\sigma_N^2}{E' G_f}, \qquad \Pi_2 = \frac{\sigma_N^2}{f_t'^2}, \tag{12}$$

(note that, for size effect, the ratios of the structural dimensions characterizing the geometry may be ignored because they remain constant when the structure is scaled up or down). The equation governing failure may, therefore, be assumed in the form: $F(\Pi_1, \Pi_2) = 0$. If function F is assumed to be smooth, we can approximate it by Taylor series truncated after the linear terms:

$$\Delta F = F^* + F_1 \Pi_1 + F_2 \Pi_2, \tag{13}$$

where:

$F_i = \partial F / \partial \Pi_i$ $(i = 1, 2)$, $\Pi_1 = \sigma_N^2 / E' G_f$ and $\Pi_2 = \sigma_N^2 / f_t'^2$ (Bažant 2003).

Substituting these values into Eq. (13) and solving for σ_N, one gets an equation of the form of Bažant's classical size effect law (1984):

$$\sigma_N = \sigma_0 (1 + D/D_0)^{-1/2}. \tag{14}$$

This function satisfies the first two asymptotic terms of both the large-size and the small-size asymptotic expansions of the cohesive crack model (Bažant 2001, 2002). If we chose for Π_1 and Π_2 any other dimensionless monomial, these asymptotic conditions could not be satisfied. This proves that (14) is the proper asymptotic matching formula.

For a cohesive crack with its FPZ attached to either a notch or a stress-free (fatigued) crack, the first two terms of the small-size asymptotic series expansion in terms of powers of D, as well as the first two terms of the large-size asymptotic series expansion in terms of powers of $1/D$ (Bažant 2001), were previously shown to be matched by the asymptotic expansions of this law.

If different dimensionless variables Π_1 and Π_2 were chosen, different size effect laws would result by using the same logical procedure. However, these laws either would not match the asymptotic properties of the cohesive crack model, or would lead to more complex size effect formulas differing from (14) only by third- and higher-order terms of the asymptotic expansions (Bažant 2003). Thus, if the dimensional analysis is combined with the known asymptotic requirements, the resulting size effect law is unique (except for complex formulas with higher than second-order deviations).

It may be noted that a size effect formula recently promulgated by Karihaloo (1999), and Karihaloo et al. (2003), $\sigma_N \propto (1 - D/D_0)^{1/2}$, is not an asymptotic matching formula because it matches only the first two terms of the large-size asymptotic expansion in terms of powers of $1/D$, but not the terms of the small-size expansion in powers of D.

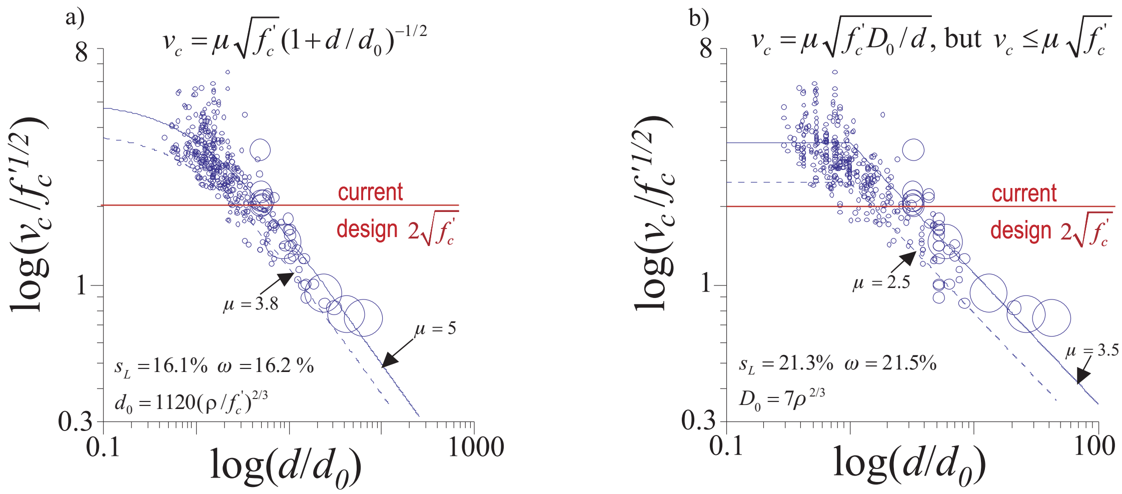

Fig. 5: Improved formulas for shear failure of reinforced concrete beams without stirrups: (a) 2nd order match; (b) 1st order match (circle area represents the assigned weights in fitting)

The foregoing argument is not valid for failure at crack initiation because the energy release rate at crack initiation vanishes. The size effect is here governed by the derivatives of the energy release rate, for which G_f is immaterial.

7 Design formula with size effect for shear capacity of RC beams

The time is now ripe for adding size effect to all the design specifications dealing with brittle failures of concrete structures (shear and torsion of R.C. beams with or without stirrups, slab punching, column failure, bar embedment length, splices, bearing strength, plain concrete flexure, etc.). In ACI Standard 318, only the new formula for anchor pullout includes the size effect (of LEFM type). By analysis of the latest ACI 445 database with 398 data (Reineck et al. 2003), representing an update of 1984 and 1986 Northwestern University databases (with 296 data), two improved formulas for shear failure of reinforced concrete beams without stirrups, having the simplicity desired in ACI, have recently been developed by asymptotic matching of first or second order (Bažant and Yu 2003); see Fig. 5 ($v_c = \sigma_N$ = mean shear stress in the cross sections at failure, d = beam depth up to reinforcement centroid, f_c' = standard compression strength of concrete). The formula in Fig. 5a is more accurate (2nd order match), the other is simpler (1st order match). They are shown in Fig. 5, where they are also compared to the database (the solid line is the mean formula, the dashed line is a formula scaled down to achieve additional safety, as practiced in ACI). The coefficients of variation of the vertical deviations from data points, ω, are shown in the Fig. 5.

In the derivation of these formulas, the following three principles were adhered to:

1) Only theoretically justified formulas must be used in data fitting because the size effect (which is of main interest for d ranging from 1 m to 10 m) requires enormous extrapolation of the ACI 445 database (in which 86 % of data pertain to $d < 0.6$ m, 99 % to $d < 1.1$ m and 100 % to $d < 1.89$ m).
2) The validity of the formula must be assessed by comparing it only to (nearly) geometrically scaled beams of broad enough size range (only 11 such test series exist).
3) The entire database must be used only for the final calibration of the chosen formula (and not for choosing the best formula, because data for many different concretes and geometries are mixed in the database, and only 2 % of the data have a non-negligible size range).

8 Closing remarks

Although the size effect in fracture of concrete structures has been studied for over quarter a century, there are still significant issues to be resolved. Among them, the introduction of size effect into the specifications in concrete design codes is of the greatest practical importance.

The present paper is a mere summary of six recent investigations at Northwestern University which will be fully presented in forthcoming papers.

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Zdeněk P. Bažant

Qiang Yu

McCormick School of Engineering and Applied Science, CEE
2145 Sheridan Road, Northwestern University,
Evanston, Illinois 60208, U.S.A.