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ON SELF-SIMILARITIES OF CUT-AND-PROJECT SETS

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Abstract. Among the commonly used mathematical models of quasicrystals are Delone sets constructed using a cut-and-project scheme, the so-called cut-and-project sets. A cut-and-project scheme $(\mathcal{L}, \pi_1, \pi_2)$ is given by a lattice $\mathcal{L}$ in $\mathbb{R}^s$ and projections π_1, π_2 to suitable subspaces V_1, V_2. In this paper we derive several statements describing the connection between self-similarity transformations of the lattice $\mathcal{L}$ and transformations of its projections $\pi_1(\mathcal{L})$, $\pi_2(\mathcal{L})$. For a self-similarity of a set Σ we take any linear mapping A such that $A\Sigma \subset \Sigma$, which generalizes the notion of self-similarity usually restricted to scaled rotations. We describe a method of construction of cut-and-project schemes with required self-similarities and apply it to produce a cut-and-project scheme such that $\pi_1(\mathcal{L}) \subset \mathbb{R}^2$ is invariant under an isometry of order 5. We describe all linear self-similarities of this scheme and show that they form an 8-dimensional associative algebra over the ring $\mathbb{Z}$. We perform an example of a cut-and-project set with linear self-similarity which is not a scaled rotation.

Keywords: self-similarity; quasicrystal; cut-and-project scheme.

1. Introduction

Quasicrystals, their mathematical models and their physical properties stand in the front row of interest of scientists since 1984 when Shechtmann and his collegues [18] published his 1982 discovery of non-crystallographic materials with long-range order. Advances in the description of these materials obtained first as rapidly solidified intermetallic alloys have been since achieved both on the mathematical side and in the experiments. A number of overview books and survey papers were published, see for example [1]. A fresh impuls to the research on quasicrystals was given by awarding 2011 Nobel Prize for chemistry to Dan Shechtmann for his discovery.

While crystals are modeled by periodic lattices, as a suitable mathematical model of atomic positions in quasicrystals one recognizes the cut-and-project method that stems in projecting lattice points from a higher-dimensional space to suitable subspaces (called usually physical and inner spaces). A cut-and-project scheme is thus given by a lattice $\mathcal{L}$ and two projections π_1, π_2, see details in Section 2. Then, choosing properly a Delone subset $\Sigma(\Omega)$ of $\pi_1(\mathcal{L})$ one obtains a quasiperiodic structure in the physical space which allows symmetries forbidden for periodic sets by the crystallographic restriction theorem. The choice is directed by a suitable window Ω in the inner space. The set $\Sigma(\Omega)$ is then called a cut-and-project set. The origins of the idea can be stepped back to times long before quasicrystal discovery, to Bohr [6] who developed his theory of quasiperiodic functions, and then to Meyer in connection to harmonic analysis [15]. De Bruijn performed [8] this construction for obtaining the vertices of the famous Penrose tiling [17]. The utility of this method for constructing quasicrystal models was then recognized by Kramer and Neri [10].

When studying two-dimensional quasicrystal models, one is interested in those displaying 5-, 8-, 10- and 12-fold rotational symmetry which corresponds to experimentally observed cases [19]. The family of symmetries is however much more rich; besides rotations/reflections it contains scalings by irrational factors and other affine symmetries.

It follows from the results of Lagarias [11] that if $\Sigma(\Omega)$ is a cut-and-project set and $\eta > 1$ is such that $\eta\Sigma(\Omega) \subset \Sigma(\Omega)$, than η can only be a Pisot or Salem number. Some authors [2] have considered self-similarities of quasilattices in the form of scaled rotations, i.e., mappings ηR, where $\eta > 1$ and R is an orthogonal map. According to our knowledge, no systematic study of general affine self-similarities, i.e., affine mappings A such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$, is found in the literature.

In this paper we focus on two main problems about general linear self-similarities. First, given a linear map A, we are interested in what are the cut-and-project schemes that allow A as a self-similarity, i.e., such that $A\pi_1(\mathcal{L}) \subset \pi_1(\mathcal{L})$. Then, we may want to fix the cut-and-project scheme and ask about all linear self-similarities allowed by this specific scheme. To this aim we present a matrix formalism for the study of cut-and-project schemes and derive several general statements (Theorem 3.1 and Proposition 3.3). These statements we then apply in the context of quasicrystal models with 5, resp. 10-fold symmetry.

We derive what is the necessary form of the cut-and-project scheme $(\mathcal{L} \subset \mathbb{R}^4, \pi_1, \pi_2)$ if one aims to obtain a quasicrystal model with 10-fold symmetry. It turns out that the requirement of the symmetry alone leads to a construction equivalent to the classical one [3] which uses preliminary knowledge of space group description, provided by Fedorov and Schönflies in dimension ≤ 3 and then generalized by Bieberbach to any dimension [5]. We demonstrate the comparison of the two constructions, see Section 5.

Next, given the cut-and-project scheme allowing 5-fold symmetry, we study its linear self-similarities in Section 6. We show that these mappings form an 8-dimensional $\mathbb{Z}$-algebra $\mathfrak{Z}$ and we provide explicit description of its elements. The $\mathbb{Z}$-algebra $\mathfrak{Z}$ has a 4-dimensional commutative subalgebra, whose elements give scaled rotations. This subalgebra is ring-isomorphic to the ring $\mathbb{Z}[\omega]$ of cyclotomic integers, where $\omega = e^{\frac{2\pi i}{5}}$.

Not all self-similarities of the set $\pi_1(\mathcal{L})$ are self-similarities of some cut-and-project set $\Sigma(\Omega)$. General statements providing a necessary and some sufficient conditions for existence of a suitable window Ω are given in Section 4. Application of this theory is then performed in Section 7. We focus on the commutative subalgebra of the algebra $\mathfrak{Z}$ and describe the scaled rotations S for which a window Ω such that $S\big(\Sigma(\Omega)\big) \subset \Sigma(\Omega)$ exists. We also provide an example illustrating that a cut-and-project set may have a linear self-similarity which is not a scaled rotation.

2. Preliminaries

It is commonly understood that a mathematical model for quasicrystals should satisfy the so-called Delone property.

Definition 2.1. *We say that $X \subset \mathbb{R}^n$ is Delone, if the following conditions are satisfied:*

(1.) *There exists $r > 0$ such that every open ball of radius r contains at most one point of X (uniform discreteness).*

(2.) *There exists $R > 0$ such that every closed ball of radius R contains at least one point of X (relative density).*

Note that the supremum of the values r from uniform discreteness bounds the distances between points in the Delone set X from below. If the supremum is achieved, then it is the minimal distance in X. The infimum of the values R from relative density is the so-called covering radius of the set X.

The essential idea behind the cut-and-project scheme is to project elements of a higher-dimensional lattice to suitable subspaces. There are two basic approaches when doing this: Either one takes the standard lattice $\mathbb{Z}^{n+m}$ and projects to general irrationally oriented subspaces V_1, V_2 of dimensions n, m, respectively, whose direct sum is equal to $\mathbb{R}^{n+m}$. Or one chooses a general lattice $\mathcal{L}$ and projects to subspaces spanned by vectors of the standard basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+m}$. Both methods are equivalent in principal. For formal reasons, it is suitable for us to choose the second approach.

Definition 2.2. *Let $\mathcal{L} \subset \mathbb{R}^{n+m}$ be a $(n+m)$-dimensional lattice. Let further $V_1 = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ and $V_2 = \operatorname{span}_{\mathbb{R}}\{\boldsymbol{e}_{n+1}, \boldsymbol{e}_{n+2}, \ldots, \boldsymbol{e}_{n+m}\}$. Let $\pi_1 : \mathbb{R}^{n+m} \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be projections to V_1 and V_2, respectively. The triple $(\mathcal{L}, \pi_1, \pi_2)$ is called a cut-and-project scheme.*

We say that the cut-and-project scheme is non-degenerated, if $\pi_1\big|_{\mathcal{L}}$ is injective. We say that the cut-and-project scheme is irreducible, if $\pi_2(\mathcal{L})$ is dense in $\mathbb{R}^m$.

It can be easily seen that the set $\pi_1(\mathcal{L})$ is a $\mathbb{Z}$-module in $\mathbb{R}^n$ and in case that the cut-and-project scheme is non-degenerated, it is not a discrete set. A quasicrystal model is constructed as a suitable subset of the $\mathbb{Z}$-module $\pi_1(\mathcal{L})$. In order to choose a Delone subset of $\pi_1(\mathcal{L})$, one puts a condition on the second projection of lattice points.

Definition 2.3. *Let $(\mathcal{L}, \pi_1, \pi_2)$ be a non-degenerated irreducible cut-and-project scheme. Given a bounded set Ω with non-empty interior, we define the cut-and-project set $\Sigma(\Omega)$ with acceptance window Ω by*

$$\Sigma(\Omega) := \{\, \pi_1(\boldsymbol{x}) \,:\, \boldsymbol{x} \in \mathcal{L}, \pi_2(\boldsymbol{x}) \in \Omega \}. \tag{1}$$

In the literature, one sometimes puts different requirements on the acceptance window Ω, for example Lagarias [11] asks it to be bounded and open. On the other hand, Cotfas [7] requires compactness and $\overline{\operatorname{int}(\Omega)} \neq \emptyset$. Moody [16] sets that the bouded set Ω satisfies $\Omega \subset \overline{\operatorname{int}(\Omega)}$. Some additional conditions, such as empty intersection of the boundary with $\pi_1(\mathcal{L})$, or convexity, may influence some specific properties of the cut-and-project set, namely repetitivity [12], or closedness under quasiaddition [4]. Here we stick to the two basic requirements which ensure the Delone property of $\Sigma(\Omega)$, see [16].

In the study of quasicrystal models with the observed rotational symmetries, one necessarily encounters certain numbertheoretic notions. An algebraic number α is a root of a polynomial with rational coefficients. If this polynomial is monic and irreducible over the rationals, it is called the minimal polynomial of α and its degree is the degree of α. Algebraic numbers with the same minimal polynomial are called algebraic conjugates.

If the minimal polynomial of α has integer coefficients, then α is said to be an algebraic integer. A special class of algebraic integers is given by the so-called Pisot numbers. A Pisot number is an algebraic integer $\beta > 1$ whose algebraic conjugates lie in the interior of the unit disk. The most prominent example of a Pisot number is the golden ratio $\tau = \frac{1}{2}(1+\sqrt{5})$, with minimal polynomial $x^2 - x - 1$. The golden ratio is strongly linked to the 5-fold symmetry, namely by the equality $2\cos\frac{2\pi}{5} = \tau^{-1}$. Pisot numbers appear as self-similarity factors of cut-and-project sets. Another class of important numbers are Salem numbers, algebraic integers > 1 with conjugates in the unit disk with at least one being on the unit circle. The notion of Pisot numbers is transferred to the complex plane by the term complex Pisot number – a complex algebraic integer β such that all algebraic conjugates but β and its complex conjugate $\overline{\beta}$ belong to the interior of the unit disk. These will play important role in Section 7.

3. Matrix formalism for the cut-and-project method

In what will follow, we use a matrix formalism for describing the cut-and-project sets. On its basis, we will set the conditions on the vectors generating the lattice $\mathcal{L} = \left\{ \sum_{i=1}^{s} a_i \boldsymbol{l}_i \,:\, a_i \in \mathbb{Z} \right\}$, so that the scheme is self-similar. Denote by V the $s \times s$ matrix formed by the vectors $\boldsymbol{l}_1, \ldots, \boldsymbol{l}_s$ written in columns. Every lattice vector can be then written as $\boldsymbol{l} = V\boldsymbol{x}$ with $\boldsymbol{x} \in \mathbb{Z}^s$. The projections π_1, π_2 then act on a lattice vector $\boldsymbol{l} \in \mathcal{L}$ as

$$\pi_1(\boldsymbol{l}) = (I_n, O)\boldsymbol{l}, \quad \pi_2(\boldsymbol{l}) = (O, I_{s-n})\boldsymbol{l},$$

where I_k is the identity matrix of order k and O stands for the zero matrix of the size $n \times (s-n)$ or $(s-n) \times n$, respectively. Assume having a window $\Omega \subset B(0,r) \subset \mathbb{R}^{s-n}$. An n-dimensional cut-and-project set with window Ω can be expressed as

$$\Sigma(\Omega) = \left\{ (I_n, O)V\boldsymbol{x} \,:\, \boldsymbol{x} \in \mathbb{Z}^s, (O, I_{s-n})V\boldsymbol{x} \in \Omega \right\}.$$

Denoting $(I_n, O)V\boldsymbol{x} = \pi_1(V\boldsymbol{x}) = \boldsymbol{b} \in \mathbb{R}^n$, $(O, I_{s-n})V\boldsymbol{x} = \pi_2(V\boldsymbol{x}) = \boldsymbol{b}^* \in \mathbb{R}^{s-n}$, we obtain the cut-and-project set in the form

$$\Sigma(\Omega) = \left\{ \boldsymbol{b} \in \pi_1(\mathcal{L}) \,:\, \boldsymbol{b}^* \in \Omega \right\},$$

which fully corresponds to the definition (1).

We further use the above matrix formalism for deriving several statements about self-similarities of the cut-and-project scheme and cut-and-project sets.

Theorem 3.1. *Let $(\mathcal{L}, \pi_1, \pi_2)$ be a non-degenerate irreducible cut-and-project scheme with $\mathcal{L} \subset \mathbb{R}^{s\times s}$. Let $A \in \mathbb{R}^{n\times n}$ satisfy $A\pi_1(\mathcal{L}) \subset \pi_1(\mathcal{L})$. Then there exists a matrix $C \in \mathbb{Z}^{s\times s}$, similar to a matrix*

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

where $B \in \mathbb{R}^{(s-n)\times(s-n)}$. In particular

$$C = V^{-1} \begin{pmatrix} A & O \\ O & B \end{pmatrix} V,$$

where $V \in \mathbb{R}^{s\times s}$ is the matrix formed by the vector generators of the lattice $\mathcal{L}$ written in columns.

Proof. Let $\boldsymbol{l}_1, \boldsymbol{l}_2, \ldots, \boldsymbol{l}_s$ be the linearly independent vectors in $\mathbb{R}^s$ generating the lattice $\mathcal{L}$. Since A is a self-similarity of the set $\pi_1(\mathcal{L})$, for every $\boldsymbol{l} \in \mathcal{L}$ there exists $\boldsymbol{l}' \in \mathcal{L}$ such that

$$A\pi_1(\boldsymbol{l}) = \pi_1(\boldsymbol{l}').$$

Correspondingly, to every $\boldsymbol{x} \in \mathbb{Z}^s$ there exists $\boldsymbol{x}' \in \mathbb{Z}^s$ such that $A\pi_1(V\boldsymbol{x}) = \pi_1(V\boldsymbol{x}')$. The mapping $\boldsymbol{x} \mapsto \boldsymbol{x}'$ is linear over $\mathbb{Z}$, and thus there exists a matrix $C \in \mathbb{Z}^{s\times s}$ such that $C\boldsymbol{x} = \boldsymbol{x}'$. We further define a linear map B by

$$B\pi_2(V\boldsymbol{x}) = \pi_2(V\boldsymbol{x}'), \qquad \text{for } \boldsymbol{x} \in \mathbb{Z}^s.$$

Together, rewritten in the matrix formalism, we have

$$A(I_n, O)V\boldsymbol{x} = (I_n, O)VC\boldsymbol{x},$$
$$B(O, I_{s-n})V\boldsymbol{x} = (O, I_{s-n})VC\boldsymbol{x},$$

which can be put together into

$$\begin{pmatrix} A & O \\ O & B \end{pmatrix} V\boldsymbol{x} = VC\boldsymbol{x}.$$

Since this holds for every $\boldsymbol{x} \in \mathbb{Z}^s$, we derive that

$$C = V^{-1} \begin{pmatrix} A & O \\ O & B \end{pmatrix} V, \tag{2}$$

which we aimed to show. ∎

We have an obvious corollary to the above theorem.

Corollary 3.2. *Let $(\mathcal{L}, \pi_1, \pi_2)$ be a non-degenerate irreducible cut-and-project scheme with $\mathcal{L} \subset \mathbb{R}^{s\times s}$. Let $A \in \mathbb{R}^{n\times n}$ satisfy $A\pi_1(\mathcal{L}) \subset \pi_1(\mathcal{L})$. Then the eigenvalues of the matrix A are algebraic integers and their minimal polynomial divides the characteristic polynomial of the matrix C over $\mathbb{Q}$.*

The matrix framework also enables us to find a cut-and-project scheme displaying a self-similarity defined by a given integer matrix C.

Proposition 3.3. *Let for $C \in \mathbb{Z}^{s\times s}$ there exist a matrix $V \in \mathbb{R}^{s\times s}$ of rank s such that VCV^{-1} is block diagonal, i.e.,*

$$VCV^{-1} = \begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{(s-n)\times(s-n)}$. Denote $\mathcal{L} = \left\{ \sum_{i=1}^{s} a_i \boldsymbol{l}_i \,:\, a_i \in \mathbb{Z} \right\}$ the lattice generated by the linearly independent columns $\boldsymbol{l}_i$ of V and for a lattice vector $\boldsymbol{l} \in \mathcal{L}$ set the projections $\pi_1 : \mathbb{R}^s \to \mathbb{R}^n$, $\pi_2 : \mathbb{R}^s \to \mathbb{R}^{s-n}$ to

$$\pi_1(\boldsymbol{l}) = (I_n, O)\boldsymbol{l}, \qquad \pi_2(\boldsymbol{l}) = (O, I_{s-n})\boldsymbol{l}.$$

(1.) *Then $A\pi_1(\mathcal{L}) \subset \pi_1(\mathcal{L})$.*

(2.) *If $Z \in \mathbb{Z}^{s\times s}$ is another matrix satisfying*

$$VZV^{-1} = \begin{pmatrix} S & O \\ O & T \end{pmatrix},$$

for some $S \in \mathbb{R}^{n\times n}$, $T \in \mathbb{R}^{(s-n)\times(s-n)}$, then $S\pi_1(\mathcal{L}) \subset \pi_1(\mathcal{L})$.

The above proposition follows from Theorem 3.1. Note that the proposition does not state anything about non-degeneracy or irreducibility of the cut-and-project scheme $(\mathcal{L}, \pi_1, \pi_2)$ obtained as shown. Examples of degenerate or reducible schemes may be constructed.

When studying the structure of the set of all self-similarities of a given cut-and-project scheme, one easily realizes that they form an associative algebra over the ring $\mathbb{Z}$. This follows from the fact that $\pi_1(\mathcal{L})$ is a $\mathbb{Z}$-module.

Proposition 3.4. *Let $R \subset \mathbb{R}^n$ be a $\mathbb{Z}$-module. Denote by $\mathcal{R}$ the set of all linear mappings S on $\mathbb{R}^n$ such that $SR \subset R$. Then $\mathcal{R}$ is an associative $\mathbb{Z}$-algebra.*

Proof. Let $S_1, S_2 \in \mathcal{R}$, i.e., $S_1, S_2 \in \mathbb{R}^{n\times n}$ such that $S_iR \subset R$. Then clearly

$$(S_1 + S_2)R = S_1R + S_2R \subset R + R = R, \qquad (S_1S_2)R = S_1(S_2R) \subset S_1R \subset R,$$

where we have used that for a $\mathbb{Z}$-module R we have $R + R = R$. This means that $S_1 + S_2, S_1S_2 \in \mathcal{R}$, and necessarily also $kS_1 \in \mathcal{R}$ for any $k \in \mathbb{Z}$. Associativity is obvious. ∎

4. Self-similarities of a cut-and-project set

The statements in the previous section concerned self-similarities of the $\mathbb{Z}$-module $\pi_1(\mathcal{L})$. Let us now concentrate on what can be said in general about the self-similarities of cut-and-project sets, given a cut-and-project scheme $(\mathcal{L}, \pi_1, \pi_2)$ with a self-similarity A.

Theorem 4.1. *Let $(\mathcal{L}, \pi_1, \pi_2)$ be a non-degenerated irreducible cut-and-project scheme with a self-similarity A. If there exists a window $\Omega \subset \mathbb{R}^m$, such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$, then the eigenvalues of the matrix B from Theorem 3.1 are in modulus smaller or equal to 1.*

Proof. Assume that for some $\Omega \subset \mathbb{R}^m$ we have $A\Sigma(\Omega) \subset \Sigma(\Omega)$. This means that $A^k\boldsymbol{z} \in \Sigma(\Omega)$ for any $\boldsymbol{z} \in \Sigma(\Omega)$ and $k \in \mathbb{N}$. For the integer matrix $C \in \mathbb{Z}^{s\times s}$ and the real matrix $B \in \mathbb{R}^{(s-n)\times(s-n)}$ from Theorem 3.1 we have, by iterating (2), that for any $k \in \mathbb{N}$

$$\begin{pmatrix} A^k & O \\ O & B^k \end{pmatrix} = VC^kV^{-1}.$$

Realize that if $\boldsymbol{z} \in \Sigma(\Omega)$, then $\boldsymbol{z} = \pi_1(\boldsymbol{l})$ for some $\boldsymbol{l} \in \mathcal{L}$ and $\pi_2(\boldsymbol{l}) \in \Omega$. Thus for any k there exist $\boldsymbol{l}' \in \mathcal{L}$ such that $A^k\boldsymbol{z} = \pi_1(\boldsymbol{l}')$ and $\pi_2(\boldsymbol{l}') \in \Omega$. Rewriting in the matrix formalism,

$$A^k\pi_1(\boldsymbol{l}) = A^k(I_n, O)\boldsymbol{l} = (I_n, O)\begin{pmatrix} A^k & O \\ O & B^k \end{pmatrix}\boldsymbol{l} = (I_n, O)VC^kV^{-1}\boldsymbol{l} = \pi_1(\boldsymbol{l}').$$

Since the scheme is non-degenerate, the projection π_1 is injective, and thus we can derive that $\boldsymbol{l}' = VC^kV^{-1}\boldsymbol{l}$. The condition $\pi_2(\boldsymbol{l}') \in \Omega$ is therefore equivalent to

$$\pi_2(\boldsymbol{l}') = (O, I_{s-n})\boldsymbol{l}' = (O, I_{s-n})VC^kV^{-1}\boldsymbol{l} = (O, I_{s-n}) \begin{pmatrix} A^k & O \\ O & B^k \end{pmatrix} \boldsymbol{l} = B^k(O, I_{s-n})\boldsymbol{l} = B^k\pi_2(\boldsymbol{l}). \qquad (3)$$

Now realize that by irreducibility the set $\{\, \pi_2(\boldsymbol{l}) : \boldsymbol{l} \in \mathcal{L},\ \pi_1(\boldsymbol{l}) \in \Sigma(\Omega) \,\}$ is dense in the bounded window Ω. By linearity of B, we must have for the closure of the window that $B^k\overline{\Omega} \subset \overline{\Omega}$. Suppose that B has a real eigenvalue λ of modulus strictly exceeding 1. As $B \in \mathbb{R}^{(s-n)\times(s-n)}$, we have a real eigenvector $\boldsymbol{w}$ of B corresponding to the eigenvalue λ. Iterating, we obtain a contradiction $B^k\boldsymbol{w} = \lambda^k\boldsymbol{w} \notin \Omega$ for sufficiently large $k \in \mathbb{N}$. If λ is a non-real eigenvalue of B with $|\lambda| > 1$ with a non-real eigenvector $\boldsymbol{w}$, then $\overline{\lambda}$ is an eigenvalue of B corresponding to the eigenvector $\overline{\boldsymbol{w}}$. Over the real space of dimension 2, spanned by the vectors $\boldsymbol{w} + \overline{\boldsymbol{w}}$, $i(\boldsymbol{w} - \overline{\boldsymbol{w}})$, the mapping B acts as multiplication by $|\lambda|$ and rotation by the argument of λ. We obtain a similar contradiction with boundedness of the window Ω as before. ■

Theorem 4.2. *Let $(\mathcal{L}, \pi_1, \pi_2)$ be a non-degenerated irreducible cut-and-project scheme with a self-similarity A. If the matrix B from Theorem 3.1 has all eigenvalues in modulus strictly smaller than 1, then there exists a window Ω such that A is a self-similarity of the cut-and-project set $\Sigma(\Omega)$.*

Proof. Since the eigenvalues of the matrix B are strictly smaller than 1, by [9, Corollary 1.2.3] there exists a metric ρ in $\mathbb{R}^{s-n}$ such that the mapping B is in that metric contracting, i.e., there exists $\delta < 1$ such that for every $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{s-n}$ we have $\delta\rho(\boldsymbol{x}, \boldsymbol{y}) > \rho(B\boldsymbol{x}, B\boldsymbol{y})$. Choosing for the window the set $\Omega = \{\, \boldsymbol{x} \in \mathbb{R}^m : \rho(\boldsymbol{x}, 0) \le 1 \,\}$, we have for any $\boldsymbol{l} \in \mathcal{L}$ with $\pi_2(\boldsymbol{l}) \in \Omega$, that $B\pi_2(\boldsymbol{l}) \in \Omega$. Therefore $A\pi_1(\boldsymbol{l}) \in \Sigma(\Omega)$ for any $l \in \mathcal{L}$ such that $\pi_1(\boldsymbol{l}) \in \Sigma(\Omega)$. Thus $A\Sigma(\Omega) \subset \Sigma(\Omega)$. ■

When the matrix B is diagonalizable we can weaken the assumption on its eigenvalues.

Theorem 4.3. *Let $(\mathcal{L}, \pi_1, \pi_2)$ be a non-degenerated irreducible cut-and-project scheme with a self-similarity A. If the matrix B from Theorem 3.1 is diagonalizable and all its eigenvalues in modulus smaller than or equal to 1, then there exists a window Ω such that A is a self-similarity of the cut-and-project set $\Sigma(\Omega)$.*

Proof. We will construct a positive semi-definite matrix which induces an inner product on $\mathbb{R}^{s-n}$ (and consequently a metric) in which the mapping B is non-expanding, i.e., does not enlarge the distances. First we define an inner product in which the eigenvectors of B form an orthonormal basis of $\mathbb{R}^{s-n}$. Denote by $\eta_1, \eta_2, \dots, \eta_m$, $m := s - n$, the eigenvalues of B and the corresponding eigenvectors by $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m$. The inner product is defined using the hermitian matrix $H = G^*G$ where G^* stands for conjugate transpose. For G we take the matrix transferring the eigenvectors $\boldsymbol{w}_1, \boldsymbol{w}_2, \dots, \boldsymbol{w}_m$ into the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_m$. Then the inner product for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$ is defined

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_H := \boldsymbol{x}^* G^* G \boldsymbol{y}.$$

Consider a general vector $\boldsymbol{x} = \sum_{i=1}^{m} \alpha_i \boldsymbol{w}_i$. Then

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle_H = \left\langle \sum_{i=1}^{m} \alpha_i \boldsymbol{w}_i, \sum_{j=1}^{m} \alpha_j \boldsymbol{w}_j \right\rangle_H = \sum_{i,j=1}^{m} \overline{\alpha_i}\alpha_j \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle_H = \sum_{i=1}^{m} |\alpha_i|^2,$$

$$\langle B\boldsymbol{x}, B\boldsymbol{x} \rangle_H = \left\langle \sum_{i=1}^{m} \alpha_i B\boldsymbol{w}_i, \sum_{j=1}^{m} \alpha_j B\boldsymbol{w}_j \right\rangle_H = \sum_{i,j=1}^{m} \overline{\alpha_i\eta_i}\alpha_j\eta_j \langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle_H = \sum_{i=1}^{m} |\eta_i|^2|\alpha_i|^2.$$

As for all eigenvalues $|\eta| \le 1$, we thus have

$$\langle \boldsymbol{x}, \boldsymbol{x} \rangle_H \ge \langle B\boldsymbol{x}, B\boldsymbol{x} \rangle_H$$

for any $\boldsymbol{x} \in \mathbb{R}^m$, and therefore the mapping B is not expanding. Setting for the acceptance window Ω a ball in the metric induced by this inner product, i.e.,

$$\Omega = \{\, \boldsymbol{x} \in \mathbb{R}^m : \langle \boldsymbol{x}, \boldsymbol{x} \rangle_H \le 1 \,\},$$

it can be again easily derived that the cut-and-project $\Sigma(\Omega)$ has self-similarity A. ■

5. Construction of a cut-and-project scheme with 5-fold symmetry

In the following, we shall apply Proposition 3.3 in order to construct a cut-and-project scheme allowing 5-fold symmetry, and subsequently, to describe all its self-similarities. The desired cut-and-project scheme must admit a cut-and-project set closed under an isometry of order 5, i.e., a mapping satisfying $A^5 = I$. The minimal polynomial of the matrix A over $\mathbb{Z}$ (monic polynomial $\mu_A \in \mathbb{Z}[X]$ of lowest degree satisfying $\mu(A) = O$) must divide the polynomial $X^5 - 1$, which over $\mathbb{Z}$ factors as $X^5 - 1 = (X-1)(X^4 + X^3 + X^2 + X + 1)$. The smallest non-trivial example is thus the cyclotomic polynomial $\mu_A(X) = \Phi_5(X) := X^4 + X^3 + X^2 + X + 1$. The minimal polynomial of the integer matrix C obtained in Theorem 3.1 should be divisible by μ_A. Thus, as the simplest example, we consider for C the companion matrix of the polynomial $\Phi_5(X)$, namely

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}. \tag{4}$$

Let $\omega = e^{2\pi i/5}$. Then the eigenvalues of C are ω, ω^2, $\omega^3 = \overline{\omega^2}$ and $\omega^4 = \overline{\omega}$, the four roots of the polynomial $\Phi_5(X) = X^4 + X^3 + X^2 + X + 1$ which is irreducible over the rationals. Note that these are precisely the primitive 5th roots of unity and they generate the cyclotomic field $\mathbb{Q}(\omega)$. Since the minimal polynomial of ω is Φ_5 and is of degree 4, the cyclotomic field is expressed as

$$\mathbb{Q}(\omega) = \left\{ a + b\omega + c\omega^2 + d\omega^3 \;:\; a,b,c,d \in \mathbb{Q} \right\}.$$

The field $\mathbb{Q}(\omega)$ has four automorphisms, induced by

$$\sigma_j(\omega) = \omega^j \qquad \text{for } j = 1,2,3,4.$$

Recall that the isomorphisms are identical over the rationals.

We can diagonalize the matrix C using a matrix Y composed of the corresponding eigenvectors written in columns, and its inverse. Denote

$$\boldsymbol{y}_1 = \frac{1}{5}\begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix}, \quad \boldsymbol{y}_2 = \frac{1}{5}\begin{pmatrix} 1 \\ \omega^4 \\ \omega^3 \\ \omega^2 \end{pmatrix}, \quad \boldsymbol{y}_3 = \frac{1}{5}\begin{pmatrix} 1 \\ \omega^2 \\ \omega^4 \\ \omega \end{pmatrix}, \quad \boldsymbol{y}_4 = \frac{1}{5}\begin{pmatrix} 1 \\ \omega^3 \\ \omega \\ \omega^4 \end{pmatrix}, \tag{5}$$

where the scaling factor $\frac{1}{5}$ is chosen just for convenience. Then Y and its inverse Y^{-1} are given by

$$Y = \frac{1}{5}\begin{pmatrix} 1 & 1 & 1 & 1 \\ \omega & \omega^4 & \omega^2 & \omega^3 \\ \omega^2 & \omega^3 & \omega^4 & \omega \\ \omega^3 & \omega^2 & \omega & \omega^4 \end{pmatrix}, \qquad Y^{-1} = \begin{pmatrix} 1-\omega & \omega^4-\omega & \omega^3-\omega & \omega^2-\omega \\ 1-\omega^4 & \omega-\omega^4 & \omega^2-\omega^4 & \omega^3-\omega^4 \\ 1-\omega^2 & \omega^3-\omega^2 & \omega-\omega^2 & \omega^4-\omega^2 \\ 1-\omega^3 & \omega^2-\omega^3 & \omega^4-\omega^3 & \omega-\omega^3 \end{pmatrix}. \tag{6}$$

Note that we grouped the columns into pairs that are complex conjugates. Then $Y^{-1}CY = \operatorname{diag}(\omega, \omega^4, \omega^2, \omega^3)$ is a diagonal matrix over $\mathbb{C}$. In order to obtain a block diagonal matrix over $\mathbb{R}$ we use matrices

$$P = \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{pmatrix}, \qquad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix}. \tag{7}$$

We thus have

$$P^{-1}Y^{-1}CYP = \begin{pmatrix} \cos\frac{2\pi}{5} & \sin\frac{2\pi}{5} & 0 & 0 \\ -\sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} & 0 & 0 \\ 0 & 0 & \cos\frac{4\pi}{5} & \sin\frac{4\pi}{5} \\ 0 & 0 & -\sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} \end{pmatrix}. \tag{8}$$

Therefore, by Proposition 3.3, as the matrix composed of vectors generating the lattice $\mathcal{L}$ we can take

$$V = P^{-1}Y^{-1} = \begin{pmatrix} 1-\cos\frac{2\pi}{5} & 0 & \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} & \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & 2\sin\frac{2\pi}{5} & \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} & \sin\frac{2\pi}{5}-\sin\frac{4\pi}{5} \\ 1-\cos\frac{4\pi}{5} & 0 & \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} & \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5} & 2\sin\frac{4\pi}{5} & \sin\frac{4\pi}{5}-\sin\frac{2\pi}{5} & \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} \end{pmatrix}$$
$$= \frac{1}{2}\begin{pmatrix} 2-\omega-\omega^4 & 0 & \omega^3+\omega^2-(\omega+\omega^4) & \omega^3+\omega^2-(\omega+\omega^4) \\ i(\omega^4-\omega) & 2i(\omega^4-\omega) & i(\omega^3-\omega^2+\omega^4-\omega) & i(\omega^2-\omega^3+\omega^4-\omega) \\ 2-\omega^2-\omega^3 & 0 & \omega+\omega^4-(\omega^2+\omega^3) & \omega+\omega^4-(\omega^2+\omega^3) \\ i(\omega^3-\omega^2) & 2i(\omega^3-\omega^2) & i(\omega-\omega^4+\omega^3-\omega^2) & i(\omega^4-\omega+\omega^3-\omega^2) \end{pmatrix}. \tag{9}$$

The lattice $\mathcal{L}$ is then of the form

$$\mathcal{L} = \mathbb{Z}\underbrace{\begin{pmatrix} 1-\cos\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} \\ 1-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_1} + \mathbb{Z}\underbrace{\begin{pmatrix} 0 \\ 2\sin\frac{2\pi}{5} \\ 0 \\ 2\sin\frac{4\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_2} + \mathbb{Z}\underbrace{\begin{pmatrix} \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} \\ \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} \\ \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5}-\sin\frac{2\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_3} + \mathbb{Z}\underbrace{\begin{pmatrix} \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} \\ \sin\frac{2\pi}{5}-\sin\frac{4\pi}{5} \\ \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_4}. \tag{10}$$

The relation (8) is thus expression of the matrix C in the block diagonal form $VCV^{-1} = \begin{pmatrix} A & O \\ O & B \end{pmatrix}$, where the matrices

$$A = \begin{pmatrix} \cos\frac{2\pi}{5} & \sin\frac{2\pi}{5} \\ -\sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\frac{4\pi}{5} & \sin\frac{4\pi}{5} \\ -\sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} \end{pmatrix} \tag{11}$$

correspond to rotations by angle $-\frac{2\pi}{5}$ and $-\frac{4\pi}{5}$, respectively.

Notation 5.1. *For further reference we denote by* Λ *the cut-and-project scheme* $\Lambda := (\mathcal{L}, \pi_1, \pi_2)$,

$$\mathbb{R}^2 \xleftarrow{\pi_1} \mathcal{L} \subset \mathbb{R}^4 \xrightarrow{\pi_2} \mathbb{R}^2$$

composed of the lattice $\mathcal{L}$ *defined in* (10), *with the projections* $\pi_1, \pi_2 : \mathbb{R}^4 \to \mathbb{R}^2$ *given as before in our formalism, i.e.,*

$$\pi_1(\boldsymbol{l}) = (I_2, O)\boldsymbol{l}, \qquad \pi_2(\boldsymbol{l}) = (O, I_2)\boldsymbol{l}.$$

In order to understand completely the structure of cut-and-project sets defined by the cut-and-project scheme Λ, let us apply the projections to the generating vectors $\boldsymbol{l}_1, \ldots, \boldsymbol{l}_4$. We obtain for the π_1 projection

$$\underbrace{\begin{pmatrix} 1-\cos\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_1^{\parallel}}, \quad \underbrace{\begin{pmatrix} 0 \\ 2\sin\frac{2\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_2^{\parallel}}, \quad \underbrace{\begin{pmatrix} \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} \\ \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_3^{\parallel}}, \quad \underbrace{\begin{pmatrix} \cos\frac{4\pi}{5}-\cos\frac{2\pi}{5} \\ \sin\frac{2\pi}{5}-\sin\frac{4\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_4^{\parallel}} \tag{12}$$

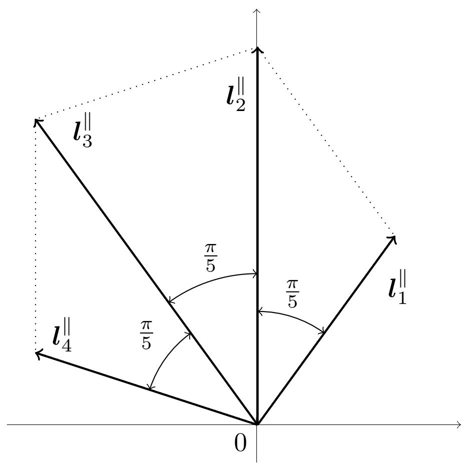
and for the π_2 projection

$$\underbrace{\begin{pmatrix} 1-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_1^{\perp}}, \quad \underbrace{\begin{pmatrix} 0 \\ 2\sin\frac{4\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_2^{\perp}}, \quad \underbrace{\begin{pmatrix} \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5}-\sin\frac{2\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_3^{\perp}}, \quad \underbrace{\begin{pmatrix} \cos\frac{2\pi}{5}-\cos\frac{4\pi}{5} \\ \sin\frac{4\pi}{5}+\sin\frac{2\pi}{5} \end{pmatrix}}_{\boldsymbol{l}_4^{\perp}}. \tag{13}$$

Let us consider the projection π_1. Rewritten in another form, we have for the projected lattice vectors

$$\boldsymbol{l}_1^{\parallel} = 2\sin\frac{\pi}{5}\begin{pmatrix} \sin\frac{\pi}{5} \\ \cos\frac{\pi}{5} \end{pmatrix}, \quad \boldsymbol{l}_2^{\parallel} = 2\sin\frac{2\pi}{5}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{l}_3^{\parallel} = 2\sin\frac{3\pi}{5}\begin{pmatrix} -\sin\frac{\pi}{5} \\ \cos\frac{\pi}{5} \end{pmatrix}, \quad \boldsymbol{l}_4^{\parallel} = 2\sin\frac{\pi}{5}\begin{pmatrix} -\sin\frac{3\pi}{5} \\ -\cos\frac{3\pi}{5} \end{pmatrix}.$$

In this form, it is easily seen how one can draw the vectors $\pi_1(\boldsymbol{l}_j)$ into the plane.



As the vectors $\boldsymbol{l}_i^{\parallel}$ together with the origin form the vertices of a regular pentagon, we can rewrite

$$\boldsymbol{l}_2^{\parallel} = \boldsymbol{l}_4^{\parallel} + \tau \boldsymbol{l}_1^{\parallel}, \qquad \boldsymbol{l}_3^{\parallel} = \boldsymbol{l}_1^{\parallel} + \tau \boldsymbol{l}_4^{\parallel},$$

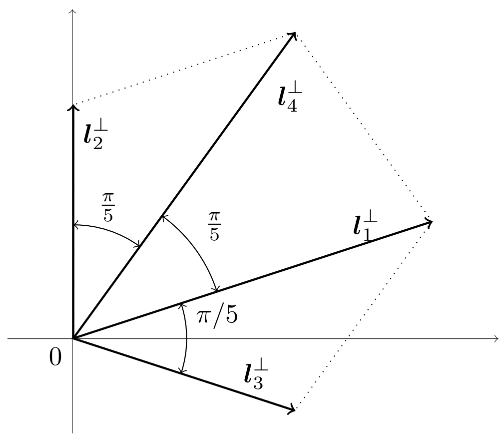
where τ is the golden ratio. These relations can be verified with the use of $2\cos\frac{2\pi}{5} = \tau^{-1}$, $2\cos\frac{4\pi}{5} = -\tau$. For any integer $a, b, c, d \in \mathbb{Z}$ we have

$$a\boldsymbol{l}_1^{\parallel} + b\boldsymbol{l}_2^{\parallel} + c\boldsymbol{l}_3^{\parallel} + d\boldsymbol{l}_4^{\parallel} = (a + c + b\tau)\boldsymbol{l}_1^{\parallel} + (b + d + c\tau)\boldsymbol{l}_4^{\parallel},$$

and thus

$$\pi_1(\mathcal{L}) = \left\{ a\boldsymbol{l}_1^{\parallel} + b\boldsymbol{l}_2^{\parallel} + c\boldsymbol{l}_3^{\parallel} + d\boldsymbol{l}_4^{\parallel} \,:\, a, b, c, d \in \mathbb{Z} \right\} = \mathbb{Z}[\tau]\boldsymbol{l}_1^{\parallel} + \mathbb{Z}[\tau]\boldsymbol{l}_4^{\parallel}. \tag{14}$$

Similarly, we have for the second projection



and we can derive that

$$\pi_2(\mathcal{L}) = \boldsymbol{l}_1^{\perp}\mathbb{Z}[\tau] + \boldsymbol{l}_4^{\perp}\mathbb{Z}[\tau]\,. \tag{15}$$

We will now show that the constructed cut-and-project scheme is non-degenerate and irreducible, which is obligatory, in order that it allows constructing cut-and-project sets. First we show non-degeneracy, i.e., that the projection π_1 restricted to $\mathcal{L}$ is a one-to-one mapping.

Lemma 5.2. *Let $\mathcal{L}$ be given in* (10) *and $\pi_1 : \mathbb{R}^4 \to \mathbb{R}^2$ by $\pi_1(\boldsymbol{l}) = (I_2, O)\boldsymbol{l}$. Then π_1 restricted to $\mathcal{L}$ is injective.*

Proof. In order to verify injectivity of $\pi_1\big|_{\mathcal{L}}$ it suffices to show that the preimage of the zero vector is $\mathbf{0}$. This amounts to showing that the vectors $\pi_1(\boldsymbol{l}_i) = \boldsymbol{l}_i^{\parallel}$, $i = 1 \dots 4$, are linearly independent over $\mathbb{Q}$. Recalling (12), this can be verified with the use of the equality

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}, \tag{16}$$

which follows from the obvious relation

$$0 = \omega^4 + \omega + \omega^3 + \omega^2 + 1 = 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} + 1.$$

■

It remains to show that the second projection of the lattice $\pi_2(\mathcal{L})$ is dense in $\mathbb{R}^2$.

Lemma 5.3. *Let $\mathcal{L}$ be given in* (10) *and $\pi_2 : \mathbb{R}^4 \to \mathbb{R}^2$ by $\pi_2(\boldsymbol{l}) = (O, I_2)\boldsymbol{l}$. Then $\pi_2(\mathcal{L})$ is dense in $\mathbb{R}^2$.*

Proof. Recall (15), where vectors $\boldsymbol{l}_1^\perp, \boldsymbol{l}_4^\perp$ are linearly independent and that, due to irrationality of the golden ratio τ, the set $\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$ is dense in $\mathbb{R}$. Thus $\pi_2(\mathcal{L})$ is a cartesian product of two sets, each of them dense in the subspace it generates. Whence, $\pi_2(\mathcal{L})$ is dense in $\mathbb{R}^2$. ■

Lemmas 5.2 and 5.3 can be summarized as follows.

Corollary 5.4. *The cut-and-project scheme $\Lambda = (\mathcal{L}, \pi_1, \pi_2)$ defined above is non-degenerate and irreducible. If $\Omega \subset \mathbb{R}^2$ is bounded and such that $\overline{\operatorname{int}(\Omega)} \neq \emptyset$, then the cut-and-project set $\Sigma(\Omega)$ can be written as*

$$\Sigma(\Omega) = \left\{ (a + b\tau)\boldsymbol{l}_1^{\parallel} + (c + d\tau)\boldsymbol{l}_4^{\parallel} \;:\; a, b, c, d \in \mathbb{Z}, (a + b\tau')\boldsymbol{l}_1^{\perp} + (c + d\tau')\boldsymbol{l}_4^{\perp} \in \Omega \right\}.$$

One can review the results of Barache et al. [3] to see that our method leads to the same model as the classical method of defining decagonal cut-and-project set based on Coxeter groups, where one projects the crystallographic root system A_4 to the non-crystallographic system H_2.

6. Self-similarities of the constructed scheme

Let us now study in according to item (ii) of Proposition 3.3 what other self-similarities are present in the cut-and-project scheme Λ constructed in Section 5. We thus need to find all integer matrices Z which by similarity transformation VZV^{-1} (with the matrix V defined in (9)) becomes a block diagonal matrix with blocks of the size 2. This means that Z has the same two eigenspaces of dimension 2.

Let us first consider those integer matrices Z which have the same eigenvectors $\boldsymbol{y}_i$, $i = 1, \dots, 4$, as C defined in (4). Rewriting this requirement into matrix equation for a general integer matrix Z, we obtain

$$Z\boldsymbol{y}_1 = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ j & k & l & m \\ n & o & p & q \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix} = \rho \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix}, \qquad \text{for some } \rho \in \mathbb{R}.$$

From the first row, we obtain $\rho = a + b\omega + c\omega^2 + d\omega^3$. Using this, and the expression for ω^4 in terms of lower powers of ω, namely $\omega^4 = -1 - \omega - \omega^2 - \omega^3$, we get from the remaining rows the following relations between the integer coefficients $a, \dots, q$,

$$\begin{aligned} e &= -d, & j &= d - c, & n &= c - b, \\ f &= a - d, & k &= -c, & o &= d - b, \\ g &= b - d, & l &= a - c, & p &= -b, \\ h &= c - d, & m &= b - c, & q &= a - b. \end{aligned}$$

One can check that, not surprisingly, a matrix Z satisfying such conditions, i.e.,

$$Z = \begin{pmatrix} a & b & c & d \\ -d & a-d & b-d & c-d \\ d-c & -c & a-c & b-c \\ c-b & d-b & -b & a-b \end{pmatrix} =: Z_{a,b,c,d} \tag{17}$$

is nothing else then an integer combination $Z = aI + bC + cC^2 + dC^3$ of the powers of the matrix C given in (4), namely

$$I = C^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad C^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

Note that we do not use higher than third powers of the matrix C, as by Hamilton-Cayley theorem, $C^4 = -C_1^3 - C_1^2 - C_1 - I$. Note also, that we do not need to use $Z\boldsymbol{y}_i = \rho_i\boldsymbol{y}_i$, for $i = 2, 3, 4$, since the result is obtainable applying the Galois automorphisms of the field $\mathbb{Q}(\omega)$, and it would not provide any new information. It follows that the matrix Z of (17) can be diagonalized using the matrix Y (of (6)) composed of eigenvectors $\boldsymbol{y}_i$, and on the diagonal we find the numbers $\sigma_j(a + b\omega + c\omega^2 + d\omega^4)$. These are thus the eigenvalues of Z.

Corollary 6.1. *The set*

$$\mathfrak{Z}_{com} := \{\, Z_{a,b,c,d} \,:\, a,b,c,d \in \mathbb{Z} \,\}$$

with standard matrix addition and multiplication is a commutative ring isomorphic to the ring $\mathbb{Z}[\omega]$ *of cyclotomic integers.*

In order to transform the matrix Z into the real block diagonal form, we use the similarity transformation by the matrix P of (7). This yields $P^{-1}Y^{-1}ZYP = \begin{pmatrix} S & O \\ O & T \end{pmatrix}$, where

$$S = \begin{pmatrix} a + b\cos\frac{2\pi}{5} + (c+d)\cos\frac{4\pi}{5} & b\sin\frac{2\pi}{5} + (c-d)\sin\frac{4\pi}{5} \\ -b\sin\frac{2\pi}{5} + (d-c)\sin\frac{4\pi}{5} & a + b\cos\frac{2\pi}{5} + (c+d)\cos\frac{4\pi}{5} \end{pmatrix},$$

$$T = \begin{pmatrix} a + b\cos\frac{4\pi}{5} + (c+d)\cos\frac{2\pi}{5} & b\sin\frac{4\pi}{5} + (c-d)\sin\frac{2\pi}{5} \\ -b\sin\frac{4\pi}{5} + (d-c)\sin\frac{2\pi}{5} & a + b\cos\frac{4\pi}{5} + (c+d)\cos\frac{2\pi}{5} \end{pmatrix}.$$

The matrices S, T are in the form λR, where $\lambda > 1$ and R is an orthogonal matrix. Indeed, denote η the cyclotomic integer $\eta = a + b\omega + c\omega^2 + d\omega^4$ and find its goniometric form $\eta = |\eta|(\cos\varphi + i\sin\varphi)$. Then by construction, S satisfies

$$S = |\eta| \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{18}$$

Similarly,

$$T = |\nu| \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}, \tag{19}$$

where $\nu = \sigma_3(\eta) = |\nu|(\cos\psi + i\sin\psi)$. We thus see that our original assumption on the integer matrix Z having the same eigenvectors as C leads to self-similarities S of the cut-and-project scheme in the form of scaled rotations.

As we will see, these are not the only self-similarities of the constructed cut-and-project scheme. In order to find all of the self-similarities, we relax the condition on eigenvectors of Z and require only that C and Z have the same invariant subspaces of dimension 2. With this, we obtain the following proposition. In order to formulate the statement, define

$$\mathfrak{Z} := \left\{ Z \in \mathbb{Z}^{4\times 4} \,:\, \exists S, T \in \mathbb{R}^{2\times 2},\ VZV^{-1} = \begin{pmatrix} S & O \\ O & T \end{pmatrix} \right\}, \tag{20}$$

where V is the matrix defining the lattice $\mathcal{L}$ of the cut-and-project scheme Λ.

Proposition 6.2. *For integer* $a, b, \ldots, h \in \mathbb{Z}$ *denote*

$$Z_{a,b,\ldots,h} := \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ -a-e+f-h & -b+g-h & -c-e+f & -d-e+g-h \\ a-b+d+e-f+h & -c+d-g+h & a-b+e-f & a-c+d+e-g+h \end{pmatrix}.$$

Then $\mathfrak{Z} = \{\, Z_{a,b,\ldots,h} \,:\, a,b,\ldots,h \in \mathbb{Z} \,\}$.

Proof. Recalling the definition of V in (9), we rewrite the requirement

$$\begin{pmatrix} S & O \\ O & T \end{pmatrix} = VZV^{-1},$$

for some matrices $S, T \in \mathbb{R}^{2\times 2}$ by

$$P \begin{pmatrix} S & O \\ O & T \end{pmatrix} P^{-1} = Y^{-1}ZY \iff YP \begin{pmatrix} S & O \\ O & T \end{pmatrix} P^{-1} = ZY. \tag{21}$$

Since also P and P^{-1} are block diagonal, the latter represents the requirement that the matrix C' has two invariant subspaces, namely $\mathrm{span}_{\mathbb{R}}\{\boldsymbol{y}_1, \boldsymbol{y}_2\}$ and $\mathrm{span}_{\mathbb{R}}\{\boldsymbol{y}_3, \boldsymbol{y}_4\}$. Stated otherwise, we require existence of complex coefficients $\mu, \mu', \nu, \nu', \zeta, \zeta', \eta, \eta'$ such that

$$Z\boldsymbol{y}_1 = \mu\boldsymbol{y}_1 + \nu\boldsymbol{y}_2, \qquad Z\boldsymbol{y}_2 = \mu'\boldsymbol{y}_1 + \nu'\boldsymbol{y}_2, \qquad Z\boldsymbol{y}_3 = \zeta\boldsymbol{y}_3 + \eta\boldsymbol{y}_4, \qquad Z\boldsymbol{y}_4 = \zeta'\boldsymbol{y}_3 + \eta'\boldsymbol{y}_4. \tag{22}$$

Since $\boldsymbol{y}_1 = \overline{\boldsymbol{y}_2}$ and $\boldsymbol{y}_3 = \overline{\boldsymbol{y}_4}$, it follows that

$$\mu' = \overline{\nu}, \qquad \zeta' = \overline{\eta}, \qquad \nu = \overline{\mu}, \qquad \eta' = \overline{\zeta}.$$

Consider a general integer matrix $Z \in \mathbb{Z}^{4\times 4}$:

$$Z = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ j & k & l & m \\ n & o & p & q \end{pmatrix}.$$

Conditions (22) are conveniently rewritten as

$$\underbrace{\begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix}}_{Z_u} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ \omega & \omega^4 \end{pmatrix}}_{Y_u^{(1)}} \begin{pmatrix} \mu \\ \nu \end{pmatrix}, \qquad \underbrace{\begin{pmatrix} j & k & l & m \\ n & o & p & q \end{pmatrix}}_{Z_d} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix} = \underbrace{\begin{pmatrix} \omega^2 & \omega^3 \\ \omega^3 & \omega^2 \end{pmatrix}}_{Y_d^{(1)}} \begin{pmatrix} \mu \\ \nu \end{pmatrix}. \tag{23}$$

Excluding parameters μ, ν, we obtain relation between coefficients of matrices F_h and F_d,

$$Y_d^{(1)} {Y_u^{(1)}}^{-1} Z_u \boldsymbol{y}_1 = Z_d \boldsymbol{y}_1,$$

$$\frac{1}{\omega^4 - \omega} \begin{pmatrix} \omega^2 & \omega^3 \\ \omega^3 & \omega^2 \end{pmatrix} \begin{pmatrix} \omega^4 & -1 \\ -\omega & 1 \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix} = \begin{pmatrix} j & k & l & m \\ n & o & p & q \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix},$$

$$\begin{pmatrix} -1 & \omega + \omega^4 \\ -\omega - \omega^4 & -\omega - \omega^4 \end{pmatrix} \begin{pmatrix} a + b\omega + c\omega^2 + d\omega^3 \\ e + f\omega + g\omega^2 + h\omega^3 \end{pmatrix} = \begin{pmatrix} j + k\omega + l\omega^2 + m\omega^3 \\ n + o\omega + p\omega^2 + q\omega^3 \end{pmatrix},$$

$$\begin{pmatrix} -a - e + f - h + \omega(-b + g - h) \\ \qquad - a - e + f - h + \omega(-b + g - h) \\ a - b + d + e - f + h + \omega(-c + d - g + h) \\ \qquad a - b + d + e - f + h + \omega(-c + d - g + h) \end{pmatrix} = \begin{pmatrix} j + k\omega + l\omega^2 + m\omega^3 \\ n + o\omega + p\omega^2 + q\omega^3 \end{pmatrix}.$$

Since the entries are – as elements of the cyclotomic field $\mathbb{Q}(\omega)$ – uniquely written as a rational combination of $1, \omega, \omega^2, \omega^3$, we find expression for $j, \dots, q$ in terms of $a, \dots, h$. The matrix Z thus has only eight independent integer parameters,

$$Z = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ -a-e+f-h & -b+g-h & -c-e+f & -d-e+g-h \\ a-b+d+e-f+h & -c+d-g+h & a-b+e-f & a-c+d+e-g+h \end{pmatrix}. \tag{24}$$

As $\mu, \nu \in \mathbb{Q}(\omega)$, relations (22) are obtained from the first of them by application of the field automorphisms. Therefore

$$\mu' = \sigma_4(\mu), \qquad \nu' = \sigma_4(\nu), \qquad \zeta = \sigma_2(\mu), \qquad \eta = \sigma_2(\nu), \qquad \zeta' = \sigma_3(\mu), \qquad \eta' = \sigma_3(\nu). \qquad \blacksquare$$

Remark 6.3. *Note that setting*

$$e = -d, \qquad f = a - d, \qquad g = b - d, \qquad h = c - d,$$

we obtain the matrix (17)*. Therefore the set $\mathfrak{Z}_{\mathrm{com}}$ is a commutative $\mathbb{Z}$-subalgebra of the $\mathbb{Z}$-algebra $\mathfrak{Z}$.*

In order to describe self-similarities od the module $\pi_1(\mathcal{L})$ corresponding to the matrices $Z_{a,\dots,h}$ let us determine the values of μ, ν from the relations (23),

$$\begin{aligned} \begin{pmatrix} \mu \\ \nu \end{pmatrix} &= \frac{1}{\omega^4 - \omega} \begin{pmatrix} \omega^4 & -1 \\ -\omega & 1 \end{pmatrix} \begin{pmatrix} a & b & c & d \\ e & f & g & h \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ \omega^3 \end{pmatrix} \\ &= \frac{1}{5}(2 + 4\omega + \omega^2 + 3\omega^3) \begin{pmatrix} b - e + \omega(c - f) + \omega^2(d - g) - h\omega^3 + a\omega^4 \\ e + \omega(f - a) + \omega^2(g - b) + \omega^3(h - c) - d\omega^4 \end{pmatrix}, \end{aligned}$$

$$\mu = \frac{1}{5}\big(2a+2b-3c+2d-2e+3f-2g+3h+\omega(-a+4b-c-d-4e+f+g+h) \\ +\omega^2(a+b+c+d-e-f-g+4h)+\omega^3(-2a+3b-2c+3d-3e+2f-3g+2h)\big),$$

$$\nu = \frac{1}{5}\big(3a-2b+3c-2d+2e-3f+2g-3h+\omega(a+b+c+d+4e-f-g-h) \\ +\omega^2(-a-b+4c-d+e+f+g-4h)+\omega^3(2a-3b+2c+2d+3e-2f+3g-2h)\big).$$

In the matrix formalism, (22) rewrites as

$$ZY = Y\begin{pmatrix} \mu & \overline{\nu} & 0 & 0 \\ \nu & \overline{\mu} & 0 & 0 \\ 0 & 0 & \zeta & \overline{\eta} \\ 0 & 0 & \eta & \overline{\zeta} \end{pmatrix}.$$

Comparing to (21), we obtain the expression for S, T,

$$\begin{pmatrix} S & O \\ O & T \end{pmatrix} = P^{-1}\begin{pmatrix} \mu & \overline{\nu} & 0 & 0 \\ \nu & \overline{\mu} & 0 & 0 \\ 0 & 0 & \zeta & \overline{\eta} \\ 0 & 0 & \eta & \overline{\zeta} \end{pmatrix} P = \begin{pmatrix} \mathrm{Re}(\mu+\nu) & \mathrm{Im}(\mu+\nu) & 0 & 0 \\ \mathrm{Im}(\nu-\mu) & \mathrm{Re}(\mu-\nu) & 0 & 0 \\ 0 & 0 & \mathrm{Re}(\zeta+\eta) & \mathrm{Im}(\zeta+\eta) \\ 0 & 0 & \mathrm{Im}(\eta-\zeta) & \mathrm{Re}(\zeta-\eta) \end{pmatrix}, \tag{25}$$

where the coefficients are of the form

$$\begin{aligned}
\mathrm{Re}(\mu+\nu) &= a + b\cos\frac{2\pi}{5} + (c+d)\cos\frac{4\pi}{5}, \\
\mathrm{Im}(\mu+\nu) &= b\sin\frac{2\pi}{5} + (c-d)\sin\frac{4\pi}{5}, \\
\mathrm{Re}(\mu-\nu) &= -c+d+f+h+(2g-b)\cos\frac{2\pi}{5} + (-c+d+2h)\cos\frac{4\pi}{5}, \\
\mathrm{Im}(\nu-\mu) &= \frac{1}{5}\Big((2a-3b+2c+2d+8e-2f-2g-2h)\sin\frac{2\pi}{5} \\
&\qquad + (-6a+4b-c-d-4e+6f-4g-4h)\sin\frac{4\pi}{5}\Big), \\
\mathrm{Re}(\zeta+\eta) &= a + (c+d)\cos\frac{2\pi}{5} + b\cos\frac{4\pi}{5}, \\
\mathrm{Im}(\zeta+\eta) &= (d-c)\sin\frac{2\pi}{5} + b\sin\frac{4\pi}{5}, \\
\mathrm{Re}(\zeta-\eta) &= -c+d+f+h+(-c+d+2h)\cos\frac{2\pi}{5} + (2g-b)\cos\frac{4\pi}{5}, \\
\mathrm{Im}(\eta-\zeta) &= \frac{1}{5}\Big((6a-4b+c+d+4e-6f+4g+4h)\sin\frac{2\pi}{5} \\
&\qquad + (2a-3b+2c+2d+8e-2f-2g-2h)\sin\frac{4\pi}{5}\Big).
\end{aligned}$$

Remark 6.4. *The matrices S of (25) of all linear self-similarities of the cut-and-project scheme $\Lambda = (\mathcal{L}, \pi_1, \pi_2)$ of Notation 5.1 form an 8-dimensional associative $\mathbb{Z}$-algebra.*

7. Self-similarities of cut-and-project sets with 5-fold symmetry

The requirement of preserving the invariant subspaces alone is not sufficient for providing a complete description of linear mappings that are self-similarities of some cut-and-project set. In order that a matrix Z gives rise to a self-similarity of $\Sigma(\Omega)$ for some window Ω, it is necessary to set conditions on the eigenvalues of the matrix B, as specified in Proposition 4.2. We shall do that for the self-similarities given by matrices of the $\mathbb{Z}$-algebra $\mathfrak{Z}_{com}$ defined in Corollary 6.1.

Proposition 7.1. *Let $Z \in \mathfrak{Z}_{com}$ and let S correspond to Z by $VZV^{-1} = \begin{pmatrix} S & O \\ O & T \end{pmatrix}$. If S is a self-similarity of a cut-and-project set $\Sigma(\Omega)$, then there exists an algebraic integer $\eta = a+b\omega+c\omega^2+d\omega^3 = |\eta|(\cos\varphi + i\sin\varphi) \in \mathbb{Z}[\omega]$ such that $S = |\eta|R$, where R is a rotation by the angle φ. Moreover, η is a Pisot number, complex Pisot number or a tenth root of unity.*

Proof. Recall that a matrix $Z_{a,b,c,d}$ has as its eigenvalues the numbers $\sigma_j(\eta)$, $j = 1,2,3,4$, where $\eta = a + b\omega + c\omega^2 + d\omega^3$. The product $\prod_{j=1}^4 \sigma_j(\eta)$ is equal to the determinant of the integer matrix $Z_{a,b,c,d}$. The corresponding matrices S, T from (18), (19), have the same eigenvalues, namely $\sigma_1(\eta)$, $\sigma_4(\eta)$ for the matrix S and $\sigma_2(\eta)$, $\sigma_3(\eta)$ for the matrix T. Recall that

$$\sigma_1(\eta) = \overline{\sigma_4(\eta)} \quad \text{and} \quad \sigma_2(\eta) = \overline{\sigma_3(\eta)}.$$

If S is a self-similarity of a cut-and-project set $\Sigma(\Omega)$, then by Proposition 4.2, the eigenvalues of T must be in modulus smaller or equal to 1. As $\prod_{j=1}^4 \sigma_j(\eta) \in \mathbb{Z}$ we have $|\eta|^2 = \sigma_1(\eta)\sigma_4(\eta) \geq 1$.

Assume first that $|\sigma_2(\eta)| = 1$. Then necessarily $|\sigma_2(\eta)\sigma_3(\eta)| = 1$, i.e., $\sigma_2(\eta) = \sigma_3(\eta)^{-1}$. The characteristic polynomial of the matrix $Z_{a,b,c,d}$ is therefore reciprocal and its roots are algebraic units lying on the unit circle. By the well-known result of Kronecker, these must be roots of unity, but the only roots of unity lying in the field $\mathbb{Q}(\omega)$ are tenth roots of unity.

Secondly, let $|\sigma_2(\eta)| < 1$. Then $|\sigma_1(\eta)| > 1$. In this case either the characteristic polynomial of the matrix $Z_{a,b,c,d}$ is irreducible over $\mathbb{Q}$ and then η is a complex Pisot number of degree 4. On the other hand, if it is reducible, than this is possible only if $\eta \in \mathbb{R}$ is of degree 2. In this case $\eta = \sigma_1(\eta) = \sigma_4(\eta)$ and $\sigma_3(\eta) = \sigma_4(\eta)$ is its algebraic conjugate. It follows that η is a quadratic Pisot number. ∎

The above proposition states that non-trivial scaled rotations correspond to complex Pisot numbers in the cyclotomic integers ring $\mathbb{Z}[\omega]$. It is clear that any complex Pisot number in $\mathbb{Z}[\omega]$ gives a scaled rotation of some cut-and-project set.

Proposition 7.2. *Let $\eta \in \mathbb{Z}[\omega]$ be a complex Pisot number, $\eta = |\eta|(\cos\varphi + i\sin\varphi)$, and denote by S the mapping $S = |\eta|R$, where R is a rotation by the angle φ. Then there exists a window $\Omega \subset \mathbb{R}^2$ such that the mapping S is a self-similarity of the cut-and-project set $\Sigma(\Omega)$.*

However, scaled rotations are not the only linear self-similarities possible. The following example shows a non-trivial linear map S for which we construct a cut-and-project set $\Sigma(\Omega)$ with self-similarity S.

Consider the matrix $\mathbb{Z}_{a,\dots,h} \in \mathfrak{Z}$ where we choose

$$a = e = g = h = 0, \quad b = -1, \quad \text{and} \quad c = d = f = 1,$$

i.e.,

$$Z = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding mappings S, T related to Z by $VZV^{-1} = \begin{pmatrix} S & O \\ O & T \end{pmatrix}$ are of the form

$$S = \begin{pmatrix} -\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} & -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} + 1 \end{pmatrix} = \begin{pmatrix} -\tau - \frac{1}{2\tau} & -\frac{1}{2}\sqrt{\tau^2+1} \\ \frac{1}{2}\sqrt{\tau^2+1} & \frac{1}{2\tau} + 1 \end{pmatrix},$$

$$T = \begin{pmatrix} 2\cos\frac{2\pi}{5} - \cos\frac{4\pi}{5} & -\sin\frac{4\pi}{5} \\ \sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} + 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\tau} + \frac{\tau}{2} & -\frac{1}{2\tau}\sqrt{\tau^2+1} \\ \frac{1}{2\tau}\sqrt{\tau^2+1} & 1 - \frac{\tau}{2} \end{pmatrix}.$$

Let us study the action of the linear map S. Its eigenvalues are

$$\lambda_1 = -\tau, \qquad \lambda_2 = 1,$$

corresponding to the eigenvectors

$$\boldsymbol{w}_1 = \begin{pmatrix} -\sin\frac{2\pi}{5} \\ \cos\frac{2\pi}{5} \end{pmatrix}, \qquad \boldsymbol{w}_2 = \begin{pmatrix} \cos\frac{2\pi}{5} \\ -\sin\frac{2\pi}{5} \end{pmatrix}.$$

The mapping S thus acts in the direction of $\boldsymbol{w}_1$ as a scaling by the factor $-\tau$, and in the direction of $\boldsymbol{w}_2$ as the identity. Figure 1 shows the action of S on the regular decagon $\boldsymbol{v}_0, \dots \boldsymbol{v}_9$ centered in the origin.

Let us study the action of the mapping T. Its eigenvalues are

$$\eta_1 = \frac{1}{\tau}, \qquad \eta_2 = 1,$$

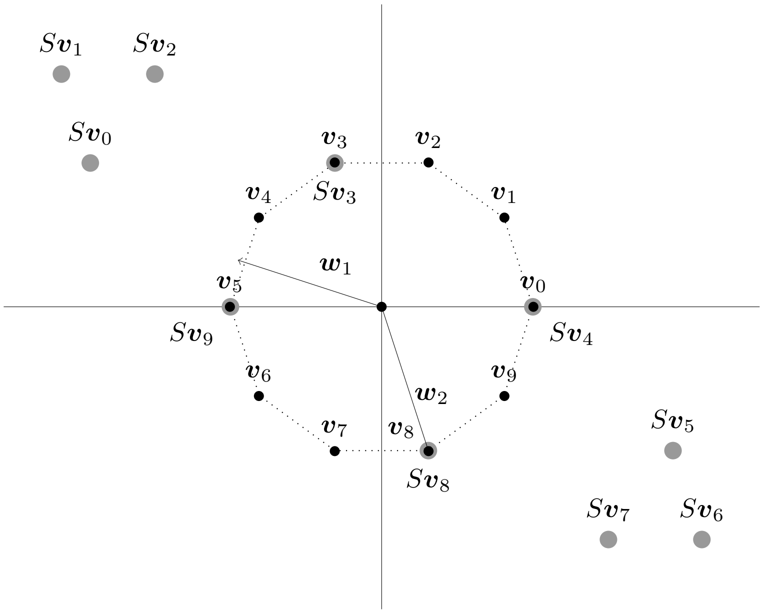


Figure 1. The action of the mapping S on the regular decagon. Its vertices, namely the points $\boldsymbol{v}_i$, $i = 0, \dots, 9$, are marked with black dots. The points $S\boldsymbol{v}_i$ are marked by larger grey dots.

with the corresponding eigenvectors

$$\boldsymbol{z}_1 = \begin{pmatrix} -\sin\frac{4\pi}{5} \\ \cos\frac{4\pi}{5} \end{pmatrix}, \qquad \boldsymbol{z}_2 = \begin{pmatrix} \cos\frac{4\pi}{5} \\ -\sin\frac{4\pi}{5} \end{pmatrix}.$$

Since the eigenvalues of the matrix T are in modulus ≤ 1, by Proposition 4.2, there exists a window Ω such that the cut-and-project set $\Sigma(\Omega)$ has self-similarity S. The window Ω must be chosen such that it is invariant under the action of T. Obviously, we could choose for example a parallelogram

$$\Omega = \{\, \alpha_1 \boldsymbol{z}_1 + \alpha_2 \boldsymbol{z}_2 \,:\, \alpha_i \in [-1, 1] \,\}.$$

For illustration, let us find a window according to the construction presented in the proof of Proposition 4.3, namely with the use of a special inner product. It can be easily shown that a matrix G transferring the eigenvectors $\boldsymbol{z}_i$ of T to the vectors of the standard basis is of the form

$$G = \frac{1}{\cos\frac{2\pi}{5}} \begin{pmatrix} \sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} \\ \cos\frac{4\pi}{5} & \sin\frac{4\pi}{5} \end{pmatrix}.$$

It is a symmetric real matrix, and thus $G = G^*$. Then the matrix $H = G^*G = G^2$ determining the inner product is given by

$$H = \frac{1}{\cos^2\frac{2\pi}{5}} \begin{pmatrix} 1 & -\sin\frac{2\pi}{5} \\ -\sin\frac{2\pi}{5} & 1 \end{pmatrix}.$$

For the window Ω we can choose a ball in the metric induced by the new inner product, namely

$$\Omega = \{\, \boldsymbol{x} \in \mathbb{R}^2 \,:\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle_H \leq \text{const.} \,\}.$$

In particular, we can have

$$\Omega = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,:\, x^2 - 2xy\sin\frac{2\pi}{5} + y^2 \leq 1 \right\}.$$

Such a window Ω satisfies $T\Omega \subset \Omega$ and thus $S\Sigma(\Omega) \subset \Sigma(\Omega)$. Figure 2 shows the window Ω and its transformation by T.

8. Comments

In this article we have studied affine self-similarities of quasicrystal models obtained by the cut-and-project method. It is a first step towards solving the general question: Given a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ under which

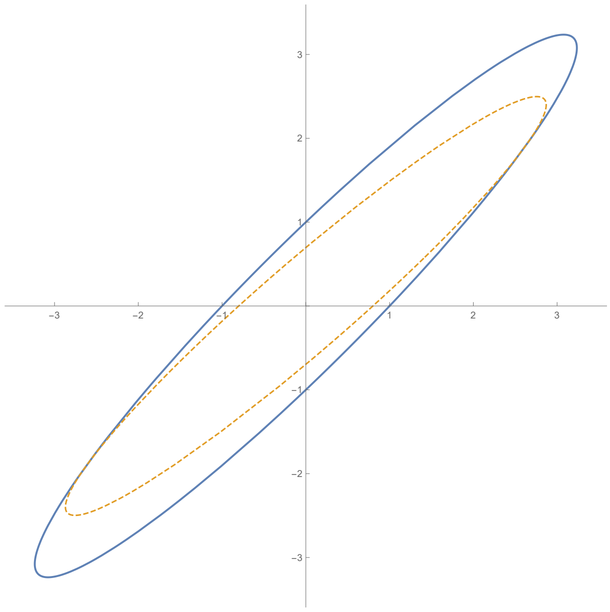


Figure 2. Acceptance window Ω is marked by full line and the action of the mapping T to Ω is marked by dashed line.

conditions there exist a cut-and-project scheme that admits a cut-and-project set $\Sigma(\Omega)$ such that $A\Sigma(\Omega) \subset \Sigma(\Omega)$? How to find such a cut-and-project scheme? Which other linear self-similarities such a cut-and-project set has?

Many problems that concern these questions remain, however, unsolved. For example, what are the conditions on the linear map A, or the corresponding integer matrix C, so that the constructed cut-and-project scheme with self-similarity A is non-degenerate and irreducible? A second important question is about the linear maps A that admit a window Ω, so that a cut-and-project set $\Sigma(\Omega)$ satisfies $A\Sigma(\Omega) \subset \Sigma(\Omega)$. The answer to such a question could be viewed as a generalization of Lagarias' result of [11].

One can also ask a further question, namely: Given a cut-and-project set $\Sigma(\Omega)$, what are its possible self-similarities? However, answer to such a question heavily depends on the form of the acceptance window Ω. Some research in this direction has been done for pentagonal quasicrystals in [4] and [13, 14], all of these however only for scaling symmetries.

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