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EXTREMAL VECTORS FOR VERMA TYPE REPRESENTATION OF B_2

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Abstract. Starting from the Verma modules of the algebra B_2 we explicitly construct factor representations of the algebra B_2 which are connected with the unitary representation of the group $\mathrm{SO}(3,2)$. We find a full set of extremal vectors for representations of this kind. So we can explicitly resolve the problem of the irreducibility of these representations.

Keywords: Verma modules, height-weight representation, reducibility, extremal vectors.

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1. Introduction

Representations of Lie algebras are important in many physical models. It is therefore useful to study various methods for constructing them.

The general method of construction of the highest-weight representation for the semisimple Lie algebra was developed in [1, 2]. The irreducibility of such representations (now called Verma modules) was studied by Gelfand in [3]. The theory of these representations is included in Dixmier's book [4].

In the 1970's prof. Havlíček with his coworkers dealt with the construction of realizations of the classical Lie algebras, see [5]. Our aim in this paper is to show how one can use realizations of the Lie algebra to construct so called extremal vectors of the Verma modules. To work with a specific Lie algebra, we choose Lie algebra so(3, 2), which plays an important role in physics, e.g. in AdS/CFT theory, see [6, 7].

In the construction of the Verma modules for B_2, the representations depend on parameters (λ_1, λ_2). For connection with irreducible unitary representations of $\mathrm{SO}(3,2)$ we take $\lambda_2 \in \mathbb{N}_0$, and in section 3 we explicitly construct the factor-Verma representation. Further, we construct a full set of extremal vectors. These vectors are called subsingular vectors in [8].

In this paper, we use an almost elementary partial differential equation approach to determine the extremal vectors in any factor-Verma module of B_2. It should be noted that our approach differs from a similar one used in [9]. First, we identify the factor-Verma modules with a space of polynomials, and the action of B_2 on the Verma module is identified with differential operators on the polynomials. Any extremal vector in the factor-Verma module becomes a polynomial solution of a system of variable-coefficient second-order linear partial differential equations.

2. The root system for Lie algebra B_2

In the Lie algebra $\mathfrak{g} = B_2$ we will take a basis composed by elements $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{E}_k$ and $\mathbf{F}_k$, where $k = 1, \dots, 4$, which fulfill the commutation relations

$$
\begin{aligned}
&[\mathbf{H}_1,\mathbf{E}_1] = 2\mathbf{E}_1, && [\mathbf{H}_1,\mathbf{E}_2] = -\mathbf{E}_2,\\
&[\mathbf{H}_1,\mathbf{E}_3] = \mathbf{E}_3, && [\mathbf{H}_1,\mathbf{E}_4] = 0,\\
&[\mathbf{H}_2,\mathbf{E}_1] = -2\mathbf{E}_1, && [\mathbf{H}_2,\mathbf{E}_2] = 2\mathbf{E}_2,\\
&[\mathbf{H}_2,\mathbf{E}_3] = 0, && [\mathbf{H}_2,\mathbf{E}_4] = 2\mathbf{E}_4,\\
&[\mathbf{H}_1,\mathbf{F}_1] = -2\mathbf{F}_1, && [\mathbf{H}_1,\mathbf{F}_2] = \mathbf{F}_2,\\
&[\mathbf{H}_1,\mathbf{F}_3] = -\mathbf{F}_3, && [\mathbf{H}_1,\mathbf{F}_4] = 0,\\
&[\mathbf{H}_2,\mathbf{F}_1] = 2\mathbf{F}_1, && [\mathbf{H}_2,\mathbf{F}_2] = -2\mathbf{F}_2,\\
&[\mathbf{H}_2,\mathbf{F}_3] = 0, && [\mathbf{H}_2,\mathbf{F}_4] = -2\mathbf{F}_4,\\
&[\mathbf{E}_1,\mathbf{E}_2] = \mathbf{E}_3, && [\mathbf{E}_1,\mathbf{E}_3] = 0,\\
&[\mathbf{E}_1,\mathbf{E}_4] = 0, && [\mathbf{E}_2,\mathbf{E}_3] = 2\mathbf{E}_4,\\
&[\mathbf{E}_2,\mathbf{E}_4] = 0, && [\mathbf{E}_3,\mathbf{E}_4] = 0,\\
&[\mathbf{F}_1,\mathbf{F}_2] = -\mathbf{F}_3, && [\mathbf{F}_1,\mathbf{F}_3] = 0,\\
&[\mathbf{F}_1,\mathbf{F}_4] = 0, && [\mathbf{F}_2,\mathbf{F}_3] = -2\mathbf{F}_4,\\
&[\mathbf{F}_2,\mathbf{F}_4] = 0, && [\mathbf{F}_3,\mathbf{F}_4] = 0,\\
&[\mathbf{E}_1,\mathbf{F}_1] = \mathbf{H}_1, && [\mathbf{E}_1,\mathbf{F}_2] = 0,\\
&[\mathbf{E}_1,\mathbf{F}_3] = -\mathbf{F}_2, && [\mathbf{E}_1,\mathbf{F}_4] = 0,\\
&[\mathbf{E}_2,\mathbf{F}_1] = 0, && [\mathbf{E}_2,\mathbf{F}_2] = \mathbf{H}_2,\\
&[\mathbf{E}_2,\mathbf{F}_3] = 2\mathbf{F}_1, && [\mathbf{E}_2,\mathbf{F}_4] = -\mathbf{F}_3,\\
&[\mathbf{E}_3,\mathbf{F}_1] = -\mathbf{E}_2, && [\mathbf{E}_3,\mathbf{F}_2] = 2\mathbf{E}_1,\\
&[\mathbf{E}_3,\mathbf{F}_3] = 2\mathbf{H}_1 + \mathbf{H}_2, && [\mathbf{E}_3,\mathbf{F}_4] = \mathbf{F}_2,\\
&[\mathbf{E}_4,\mathbf{F}_1] = 0, && [\mathbf{E}_4,\mathbf{F}_2] = -\mathbf{E}_3,\\
&[\mathbf{E}_4,\mathbf{F}_3] = \mathbf{E}_2, && [\mathbf{E}_4,\mathbf{F}_4] = \mathbf{H}_1 + \mathbf{H}_2.
\end{aligned}
$$

We can take as $\mathfrak{h}$ the Cartan subalgebra with the bases $\mathbf{H}_1$ and $\mathbf{H}_2$.

We will denote $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{h}^*$, for which we have

$$\lambda(\mathbf{H}_1) = \lambda_1, \qquad \lambda(\mathbf{H}_2) = \lambda_2.$$

The root systems $\mathfrak{g} = B_2$ with respect to these bases $\mathbf{H}_1$ and $\mathbf{H}_2$ are $R = \{\pm\boldsymbol{\alpha}_k\,;\ k = 1,2,3,4\}$, where

$$\boldsymbol{\alpha}_1 = (2,-2), \qquad \boldsymbol{\alpha}_2 = (-1,2),$$
$$\boldsymbol{\alpha}_3 = \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 = (1,0), \quad \boldsymbol{\alpha}_4 = \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_2 = (0,2).$$

If we choose positive roots $R_+ = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4\}$, the basis in root system R is $B = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$.

If we define $\mathbf{H}_3 = 2\mathbf{H}_1 + \mathbf{H}_2$ and $\mathbf{H}_4 = \mathbf{H}_1 + \mathbf{H}_2$, the following relations

$$[\mathbf{H}_k, \mathbf{E}_k] = 2\mathbf{E}_k, \quad [\mathbf{H}_k, \mathbf{F}_k] = -2\mathbf{F}_k, \quad [\mathbf{E}_k, \mathbf{F}_k] = \mathbf{H}_k$$

are valid for any $k = 1, \dots, 4$.

3. The extremal vectors for Verma type representation

We denote by $\mathfrak{n}_+$, and $\mathfrak{n}_-$ the Lie subalgebras generated by elements $\mathbf{E}_k$, and $\mathbf{F}_k$, respectively, where $k = 1, \dots, 4$, and $\mathfrak{b}_+ = \mathfrak{h} + \mathbf{n}_+$. Let us further consider $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{h}^*$ the one-dimensional representation $\tau_{\boldsymbol{\lambda}}$ for the Lie algebra $\mathfrak{b}_+$ such that for any $\mathbf{H} \in \mathfrak{h}$ and $\mathbf{E} \in \mathfrak{n}_+$

$$\tau_{\boldsymbol{\lambda}}(\mathbf{H} + \mathbf{E})|0\rangle = \boldsymbol{\lambda}(\mathbf{H})|0\rangle.$$

The element $|0\rangle$ will be called the lowest-weight vector. Let further be

$$W(\boldsymbol{\lambda}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}|0\rangle,$$

where $\mathfrak{b}_+$-module $\mathbb{C}|0\rangle$ is defined by $\tau_{\boldsymbol{\lambda}}$.

It is clear that $W(\boldsymbol{\lambda}) \sim U(\mathfrak{n}_-)|0\rangle$ and it is the $U(\mathfrak{g})$-module for the left regular representation, which will be called the Verma module. [1]

It is a well-known fact that every $U(\mathfrak{g})$-submodule of the module $W(\boldsymbol{\lambda})$ is isomorphic to module $W(\boldsymbol{\mu})$, where

$$\boldsymbol{\mu} = \boldsymbol{\lambda} - n_1\boldsymbol{\alpha}_1 - n_2\boldsymbol{\alpha}_2,$$

for $n_1, n_2 \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$. For the lowest-weight vector of the representation $W(\boldsymbol{\mu}) \subset W(\boldsymbol{\lambda})$, $|0\rangle_{\boldsymbol{\mu}}$, is fulfilled

$$\mathbf{H}|0\rangle_{\boldsymbol{\mu}} = \boldsymbol{\mu}(\mathbf{H})|0\rangle_{\boldsymbol{\mu}}, \quad \mathbf{H} \in \mathfrak{h}, \quad \mathbf{E}|0\rangle_{\boldsymbol{\mu}} = 0, \quad \mathbf{E} \in \mathfrak{n}_+.$$

Such vectors $|0\rangle_{\boldsymbol{\mu}}$ will be called extremal vectors $W(\boldsymbol{\lambda})$.

From the well-known result for the Verma modules we know that the Verma module $W(\boldsymbol{\lambda})$ is irreducible iff

$$\lambda_1 \notin \mathbb{N}_0, \qquad \lambda_2 \notin \mathbb{N}_0,$$
$$\lambda_1 + \lambda_2 + 1 \notin \mathbb{N}_0, \qquad 2\lambda_1 + \lambda_2 + 2 \notin \mathbb{N}_0.$$

If $\lambda_1 \in \mathbb{N}_0$, resp. $\lambda_2 \in \mathbb{N}_0$, then the extremal vectors are

$$\mathbf{F}_1^{\lambda_1+1}|0\rangle = |0\rangle_{\boldsymbol{\mu}_1}, \quad \text{resp.} \quad \mathbf{F}_2^{\lambda_2+1}|0\rangle = |0\rangle_{\boldsymbol{\mu}_2},$$

where

$$\boldsymbol{\mu}_1 = \boldsymbol{\lambda} - (\lambda_1 + 1)\boldsymbol{\alpha}_1 = (-\lambda_1 - 2, 2\lambda_1 + \lambda_2 + 2),$$
$$\boldsymbol{\mu}_2 = \boldsymbol{\lambda} - (\lambda_2 + 1)\boldsymbol{\alpha}_2 = (\lambda_1 + \lambda_2 + 1, -\lambda_2 - 2). \quad (1)$$

If $W(\boldsymbol{\mu})$ is a submodule $W(\boldsymbol{\lambda})$, we will define the $U(\mathfrak{g})$-factor-module

$$W(\boldsymbol{\lambda}|\boldsymbol{\mu}) = W(\boldsymbol{\lambda})/W(\boldsymbol{\mu}).$$

Now we can study the reducibility of a representation like that.

Again, the extremal vector is called any nonzero vector $\mathbf{v} \in W(\boldsymbol{\lambda}|\boldsymbol{\mu})$ for which there exists $\boldsymbol{\nu} \in \mathfrak{h}^*$ such that

$$\mathbf{H}_k\mathbf{v} = \nu_k\mathbf{v}, \quad \mathbf{E}_k\mathbf{v} = 0, \quad k = 1,2. \quad (2)$$

It is clear that $\mathbf{E}_k\mathbf{v} = 0$ for $k = 1,2,3,4$.

In this paper, we find all such extremal vectors in the space $W(\boldsymbol{\lambda}|\boldsymbol{\mu}_2)$, where $\lambda_2 \in \mathbb{N}_0$ and $\boldsymbol{\mu}_2$ is given by (1).

4. Differential equations for extremal vectors

Let $\lambda_2 \in \mathbb{N}_0$ and $\boldsymbol{\mu}_2$ be given by equation (1). It is easy to see that the basis in the space $W(\boldsymbol{\lambda}|\boldsymbol{\mu}_2)$ is given by the vectors

$$|\mathbf{n}\rangle = |n_1, n_3, n_4, n_2\rangle = (\lambda_2 - n_2)!\,\mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle,$$

where $n_1, n_3, n_4 \in \mathbb{N}_0$ and $n_2 = 0, 1, \dots, \lambda_2$.[2]

Now by direct calculation we obtain

$$\begin{aligned}
\mathbf{H}_1|\mathbf{n}\rangle &= (\lambda_1 - 2n_1 + n_2 - n_3)|\mathbf{n}\rangle,\\
\mathbf{H}_2|\mathbf{n}\rangle &= (\lambda_2 + 2n_1 - 2n_2 - 2n_4)|\mathbf{n}\rangle,\\
\mathbf{E}_1|\mathbf{n}\rangle &= n_1(\lambda_1 - n_1 + n_2 - n_3 + 1)|n_1 - 1, n_3, n_4, n_2\rangle\\
&\quad - (\lambda_2 - n_2)n_3|n_1, n_3 - 1, n_4, n_2 + 1\rangle\\
&\quad + n_3(n_3 - 1)|n_1, n_3 - 2, n_4 + 1, n_2\rangle,\\
\mathbf{E}_2|\mathbf{n}\rangle &= n_2|n_1, n_3, n_4, n_2 - 1\rangle\\
&\quad + 2n_3|n_1 + 1, n_3 - 1, n_4, n_2\rangle\\
&\quad - n_4|n_1, n_3 + 1, n_4 - 1, n_2\rangle. \quad (3)
\end{aligned}$$

It is possible to rewrite the action by the second order differential operators (see [10, 11]) on the polynomial functions z_1, z_2, z_3 a z_4, which are in variable z_2 up to the level λ_2. If we put

$$|n_1, n_3, n_4, n_2\rangle = (\lambda_2 - n_2)!\,\mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle \leftrightarrow z_1^{n_1}z_2^{n_2}z_3^{n_3}z_4^{n_4},$$

we obtain from equations (3) for the action on polynomials $f = f(z_1, z_2, z_3, z_4)$

$$\begin{aligned}
\mathbf{H}_1 f &= \lambda_1 f - 2z_1 f_1 + z_2 f_2 - z_3 f_3,\\
\mathbf{H}_2 f &= \lambda_2 f + 2z_1 f_1 - 2z_2 f_2 - 2z_4 f_4,\\
\mathbf{E}_1 f &= \lambda_1 f_1 - z_1 f_{11} + z_2 f_{12} - z_3 f_{13}\\
&\quad - \lambda_2 z_2 f_3 + z_2^2 f_{23} + z_4 f_{33},\\
\mathbf{E}_2 f &= f_2 + 2z_1 f_3 - z_3 f_4, \quad (4)
\end{aligned}$$

[1] In Dixmier's book the Verma module $M(\boldsymbol{\lambda})$ is defined with respect to $\tau_{\boldsymbol{\lambda}-\boldsymbol{\delta}}$, where $\boldsymbol{\delta} = \frac{1}{2}\sum_{k=1}^{4}\boldsymbol{\alpha}_k = (1,1)$. So we have $W(\boldsymbol{\lambda}) = M(\boldsymbol{\lambda} + \boldsymbol{\delta})$.

[2] If $\lambda_2 \notin \mathbb{Z}$ we can use a similar construction with basis $|\mathbf{n}\rangle = \Gamma(\lambda_2 - n_2 + 1)\mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle$, where $n_1, n_2, n_3, n_4 \in \mathbb{N}_0$.

where $f_k = \dfrac{\partial f}{\partial z_k}$.

The conditions for extremal vectors (2) are now

$$\begin{aligned}\lambda_1 f - 2z_1 f_1 + z_2 f_2 - z_3 f_3 &= \nu_1 f,\\ \lambda_2 f + 2z_1 f_1 - 2z_2 f_2 - 2z_4 f_4 &= \nu_2 f,\\ \lambda_1 f_1 - z_1 f_{11} + z_2 f_{12} &\\ - z_3 f_{13} - \lambda_2 z_2 f_3 + z_2^2 f_{23} + z_4 f_{33} &= 0,\\ f_2 + 2z_1 f_3 - z_3 f_4 &= 0, \qquad (5)\end{aligned}$$

where ν_1 and ν_2 are complex numbers.

The condition on the degree of the polynomial $f(z_1, z_2, z_3, z_4)$ in variable z_2 can be rewritten in the following way

$$\frac{\partial^{\lambda_2+1} f}{\partial z_2^{\lambda_2+1}} = 0.$$

5. The extremal vectors

The extremal vectors are in one-to-one correspondence to polynomial solutions of the systems of equations (5), which are in variable z_2 of maximal degree λ_2. You can find all such solutions in the appendix.

For any λ_1 and λ_2 there exists a constant solution $f(z_1, z_2, z_3, z_4) = 1$. But such a solution gives $\mathbf{v} = |0\rangle$, which is not interesting.

A further solution exists only in the cases $\lambda_1 \in \mathbb{N}_0$, $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$ or $2\lambda_1 + \lambda_2 + 2 \in \mathbb{N}_0$.

For $\lambda_1 \in \mathbb{N}_0$ there is a function $f(z_1, z_2, z_3, z_4) = z_1^{\lambda_1+1}$, and we obtain the extremal vector

$$\mathbf{v} = \mathbf{F}_1^{\lambda_1+1}|0\rangle.$$

For $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$ and $2\lambda_1 + \lambda_2 + 4 \le 0$ we find the solution

$$\begin{aligned}f(z_1, z_2, z_3, z_4) &= \bigl(z_4 + z_2 z_3 - z_1 z_2^2\bigr)^{\lambda_1+\lambda_2+2}\\ &= \sum_{(n_1,n_3)\in\mathcal{D}_\lambda} \frac{(-1)^{n_1}(\lambda_1+\lambda_2+2)!}{n_1!\, n_3!\, (\lambda_1+\lambda_2-n_1-n_3+2)!}\\ &\qquad \times z_1^{n_1} z_2^{2n_1+n_3} z_3^{n_3} z_4^{\lambda_1+\lambda_2-n_1-n_3+2},\end{aligned}$$

where $\mathcal{D}_\lambda = \bigl\{(n_1, n_3) \in \mathbb{N}_0^2\, ;\, n_1 + n_3 \le \lambda_1 + \lambda_2 + 2\bigr\}$. The extremal vector corresponding to this solution is

$$\begin{aligned}\mathbf{v} = \sum_{(n_1,n_3)\in\mathcal{D}_\lambda} &\frac{(-1)^{n_1}(\lambda_2 - 2n_1 - n_3)!}{n_1!\, n_3!\, (\lambda_1+\lambda_2-n_1-n_3+2)!}\\ &\times \mathbf{F}_1^{n_1}\mathbf{F}_3^{n_3}\mathbf{F}_4^{\lambda_1+\lambda_2-n_1-n_3+2}\mathbf{F}_2^{2n_1+n_3}|0\rangle.\end{aligned}$$

If $2\lambda_1 + \lambda_2 + 2 \in \mathbb{N}_0$, we introduce

$$N = 2\lambda_1 + \lambda_2 + 3, \quad \ell_2 = \bigl[\tfrac{1}{2}\lambda_2\bigr], \quad M = \bigl[\tfrac{1}{2}N\bigr].$$

Then we can rewrite the solution from the appendix in the following way:

For λ_1 being a half integer, i.e. $\lambda_1 = \ell_1 - \frac{1}{2}$, where $\ell_1 \in \mathbb{Z}$, we have

$$\begin{aligned}f = \sum_{n_4=0}^{M} \sum_{n_2=0}^{\min(\lambda_2, N-2n_4)} &(-1)^{n_2}\frac{c_{n_2,n_4}}{n_2!\, n_4!}\\ &\times z_1^{n_2+n_4} z_2^{n_2} z_3^{N-n_2-2n_4} z_4^{n_4},\end{aligned}$$

where

$$c_{n_2,n_4} = \begin{cases} \sum_{n=[\frac{1}{2}(n_2+1)]}^{\min(\ell_2, M-n_4)} 2^{2n+n_2+2n_4} \times \frac{\ell_2!\, M!}{(2n-n_2)!\,(\ell_2-n)!\,(M-n-n_4)!}, & \lambda_2 \text{ even},\\ \sum_{n=[\frac{1}{2}n_2]}^{\min(\ell_2, M-n_4)} 2^{2n+n_2+2n_4} \times \frac{\ell_2!\, M!}{(2n-n_2+1)!\,(\ell_2-n)!\,(M-n-n_4)!}, & \lambda_2 \text{ odd}.\end{cases}$$

For these solutions we obtain the extremal vectors

$$\begin{aligned}\mathbf{v} = \sum_{n_4=0}^{M} \sum_{n_2=0}^{\min(\lambda_2, N-2n_4)} &(-1)^{n_2}\frac{(\lambda_2-n_2)!}{n_2!\, n_4!} c_{n_2,n_4}\\ &\times \mathbf{F}_1^{n_2+n_4}\mathbf{F}_3^{N-n_2-2n_4}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle.\end{aligned}$$

If λ_1 is an integer we have $\lambda_1 \le -2$. The solution of the differential equations in this case is

$$\begin{aligned}f = \sum_{n_4=0}^{M} \sum_{n_2=0}^{N-2n_4} &(-1)^{n_2}\frac{d_{n_2,n_4}}{n_2!\, n_4!}\\ &\times z_1^{n_2+n_4} z_2^{n_2} z_3^{N-n_2-2n_4} z_4^{n_4},\end{aligned}$$

where

$$d_{n_2,n_4} = \begin{cases} \sum_{n=[\frac{1}{2}(n_2+1)]}^{M-n_4} 2^{n+n_2+2n_4}\frac{(2\ell_2-1)!!}{(2\ell_2-2n-1)!!} \times \frac{M!}{(2n-n_2)!\,(M-n-n_4)!}, & \lambda_2 \text{ even},\\ \sum_{n=[\frac{1}{2}n_2]}^{M-n_4} 2^{n+n_2+2n_4}\frac{(2\ell_2-1)!!}{(2\ell_2-2n-1)!!} \times \frac{M!}{(2n-n_2+1)!\,(M-n-n_4)!}, & \lambda_2 \text{ odd},\end{cases}$$

and the extremal vectors are

$$\begin{aligned}\mathbf{v} = \sum_{n_4=0}^{M} \sum_{n_2=0}^{N-2n_4} &(-1)^{n_2}\frac{(\lambda_2-n_2)!}{n_2!\, n_4!} d_{n_2,n_4}\\ &\times \mathbf{F}_1^{n_2+n_4}\mathbf{F}_3^{N-n_2-2n_4}\mathbf{F}_4^{n_4}\mathbf{F}_2^{n_2}|0\rangle.\end{aligned}$$

6. Appendix: Polynomial solutions of differential equations

To obtain extremal vectors we need to find the polynomial solutions

$$f(z_1, z_2, z_3, z_4) = \sum_{n_1,n_2,n_3,n_4\ge 0} c_{n_1,n_2,n_3,n_4} z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4}$$

of the system of equations (5), which are of less degree than $(\lambda_2 + 1)$ in the variable z_2.

To simplify the solution of the first equations, we put

$$\begin{aligned}f(z_1&, z_2, z_3, z_4)\\ &= z_1^{-\rho_2}(4z_1z_4 + z_3^2)^{\rho_2+\rho_1/2} g(t, x_1, x_2, x_3),\end{aligned}$$

where $\rho_1 = \lambda_1 - \nu_1$, $\rho_2 = \frac{1}{2}(\lambda_2 - \nu_2)$, $x_2 = z_1$, $x_3 = z_2$ and

$$t = \frac{(2z_1z_2 - z_3)^2}{4z_1z_4 + z_3^2}, \quad x_1 = \frac{2z_1z_2 - z_3}{z_3},$$

or $z_1 = x_2$, $z_2 = x_3$ and

$$z_3 = \frac{2x_2x_3}{1+x_1}, \quad z_4 = \frac{x_2x_3^2(x_1^2-t)}{t(1+x_1)^2}.$$

The first order equations are equivalent to the conditions

$$g_{x_1} = g_{x_2} = g_{x_3} = 0,$$

and so $g(t,x_1,x_2,x_3) = g(t)$.

The equations of the second order give the system of three equations

$$\begin{aligned}&(2\lambda_1\rho_1 + 2\lambda_1\rho_2 + \lambda_2\rho_1 + 2\lambda_2\rho_2\\ &\qquad - \rho_1^2 - 2\rho_1\rho_2 - 2\rho_2^2 + 3\rho_1 + 4\rho_2)g = 0,\\ &(2\lambda_1 + \lambda_2 - \rho_1 - 2\rho_2 + 3)(1-t)g'\\ &\qquad + \rho_2(\lambda_1 - \rho_1 - \rho_2 + 1)g = 0,\\ &4t(1-t)g'' + 2\big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g'\\ &\qquad + (2\lambda_1\rho_1 - \rho_1^2 + 3\rho_1 + 2\rho_2)g = 0. \qquad (6)\end{aligned}$$

As we want to obtain polynomial solutions $f(z_1,z_2,z_3,z_4)$, which are in variable z_2 of less or equal degree $\lambda_2 \in \mathbb{N}_0$, there must be solution $g(t)$ of the system (6), which is the polynomial in $\sqrt{t}$ of less or equal degree λ_2.

If we exclude derivatives of g from the second and the third equations,we find that nonzero solutions can exist only in the following six cases:

(1.) $\rho_1 = 0$, $\rho_2 = 0$;

(2.) $\rho_1 = 2\lambda_1 + 2$, $\rho_2 = -\lambda_1 - 1$;

(3.) $\rho_1 = 0$, $\rho_2 = \lambda_1 + \lambda_2 + 2$;

(4.) $\rho_1 = 2\lambda_1 + 2$, $\rho_2 = \lambda_2 + 1$;

(5.) $\rho_1 = 2\lambda_1 + \lambda_2 + 3$, $\rho_2 = 0$;

(6.) $\rho_1 = -\lambda_2 - 1$, $\rho_2 = \lambda_1 + \lambda_2 + 2$.

Case 1 ($\rho_1 = \rho_2 = 0$)**.** A function that corresponds to the extremal vector is $f(z_1,z_2,z_3,z_4) = g(t)$, where $g(t)$ is the solution of the system

$$\begin{aligned}(2\lambda_1 + \lambda_2 + 3)(1-t)g' &= 0,\\ 2t(1-t)g'' + \big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g' &= 0. \qquad (7)\end{aligned}$$

For each λ_1 and λ_2 this system has the solution $g(t) = 1$ which corresponds to the extremal vector

$$f(z_1,z_2,z_3,z_4) = 1.$$

But for $2\lambda_1 + \lambda_2 + 3 = 0$ we obtained for $g(t)$ the equation

$$2t(1-t)g'' + \big(1 + (\lambda_2 - 2)t\big)g' = 0,$$

which also has a non-constant solution

$$g(t) = G(\sqrt{t}), \quad \text{where} \quad G(x) = \int \big(1-x^2\big)^{(\lambda_2-1)/2}\mathrm{d}x.$$

However this solution does not give a polynomial function $f(z_1,z_2,z_3,z_4)$ for any λ_2.

Case 2 ($\rho_1 = 2\lambda_1 + 2$, $\rho_2 = -\lambda_1 - 1$)**.** The function that corresponds to the extremal vector is in this case

$$f(z_1,z_2,z_3,z_4) = z_1^{\lambda_1+1}g(t),$$

where $g(t)$ is the solution of system (7). As in event 1 we find that the non-constant polynomial solutions

$$f(z_1,z_2,z_3,z_4) = z_1^{\lambda_1+1}$$

get only $\lambda_1 \in \mathbb{N}_0$.

Case 3 ($\rho_1 = 0$, $\rho_2 = \lambda_1 + \lambda_2 + 2$.)**.** The function for the extremal vectors is

$$f(z_1,z_2,z_3,z_4) = \left(\frac{4z_1z_4 + z_3^2}{z_1}\right)^{\lambda_1+\lambda_2+2} g(t),$$

where $g(t)$ is the solution of the system

$$\begin{aligned}(\lambda_2+1)\big((1-t)g' + (\lambda_1+\lambda_2+2)g\big) &= 0,\\ 2t(1-t)g'' + \big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g'&\\ +(\lambda_1+\lambda_2+2)g &= 0. \quad (8)\end{aligned}$$

As we assume that $\lambda_2 \in \mathbb{N}_0$,for each λ_1, λ_2 this system has the solution

$$g(t) = (1-t)^{\lambda_1+\lambda_2+2}.$$

This solution corresponds to the function

$$f(z_1,z_2,z_3,z_4) = \big(z_4 + z_2z_3 - z_1z_2^2\big)^{\lambda_1+\lambda_2+2}, \quad (9)$$

which is a non-constant polynomial for $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$.

This function is a polynomial in the variable z_2 of degree $2\lambda_1 + 2\lambda_2 + 2$. It gives sought solutions for $2\lambda_1 + \lambda_2 + 4 \leq 0$.

Thus, function (9) provides a permissible solution for the $\lambda_2 \in \mathbb{N}_0$ only if $\lambda_1 \in \mathbb{Z}$, $-\lambda_2 - 1 \leq \lambda_1 \leq -\frac{1}{2}\lambda_2 - 2$, from which follows $\lambda_2 \geq 2$.

Case 4 ($\rho_1 = 2\lambda_1 + 2$, $\rho_2 = \lambda_2 + 1$.)**.** In this case, the function that can match the extremal vector is

$$f(z_1,z_2,z_3,z_4) = z_1^{-\lambda_2-1}(4z_1z_4 + z_3^2)^{\lambda_1+\lambda_2+2}g(t),$$

where $g(t)$ is the solution of system (8). So

$$f(z_1,z_2,z_3,z_4) = z_1^{\lambda_1+1}\big(z_4 + z_2z_3 - z_1z_2^2\big)^{\lambda_1+\lambda_2+2}.$$

To give a polynomial solution, which we have found, to this function there must be $\lambda_1 \in \mathbb{N}_0$ and $\lambda_1 + \lambda_2 + 1 \in \mathbb{N}_0$. But in this case, the degree of polynomial f in the variable z_2 is greater than λ_2 and, therefore, is not a permissible solution.

Case 5 ($\rho_1 = 2\lambda_1 + \lambda_2 + 3$, $\rho_2 = 0$.)**.** The function corresponding to the possible extremal vectors is

$$f(z_1, z_2, z_3, z_4) = (4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} g(t),$$

where function $g(t)$ meets the equation

$$4t(1-t)g'' + 2\big(1 + (2\lambda_1 + 2\lambda_2 + 1)t\big)g' - \lambda_2(2\lambda_1 + \lambda_2 + 3)g = 0. \quad (10)$$

This equation has two linearly independent solutions

$$g_1(t) = F\big(-\tfrac{1}{2}\lambda_2, -\lambda_1 - \tfrac{1}{2}\lambda_2 - \tfrac{3}{2}; \tfrac{1}{2}; t\big),$$
$$g_2(t) = \sqrt{t}\, F\big(\tfrac{1}{2} - \tfrac{1}{2}\lambda_2, -\lambda_1 - \tfrac{1}{2}\lambda_2 - 1; \tfrac{3}{2}; t\big),$$

where $F(\alpha, \beta; \gamma; t)$ is the hypergeometric function

$$F(\alpha, \beta; \gamma; t) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!\,(\gamma)_n} t^n,$$

where

$$(\alpha)_n = \frac{\Gamma(\alpha + n)}{\Gamma(\alpha)} = \alpha(\alpha+1)\dots(\alpha+n-1).$$

These solutions correspond to the functions

$$f_1 = \sum_{n=0}^{\infty} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$

$$f_2 = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}.$$

For at least one of these functions to be a nonconstant polynomial, must be $2\lambda_1 + \lambda_2 + 3 \in \mathbb{N}$, i.e. $2\lambda_1 + \lambda_2 + 2 \in \mathbb{N}_0$.

If $2\lambda_1 + \lambda_2 + 3$ is even, we get the solution

$$f_1 = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$

and for $2\lambda_1 + \lambda_2 + 3$ odd, we have the solution

$$f_2 = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + 1} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n+1}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}.$$

If $2\lambda_1 + \lambda_2 + 3$ is even and λ_2 is even, then λ_1 is a half integer, i.e. $\lambda_1 = \ell_1 - \frac{1}{2}$, where $\ell_1 \in \mathbb{Z}$, counts in f_1 only to $n \leq \frac{1}{2}\lambda_2$, i.e.

$$f = \sum_{n=0}^{\min(\frac{1}{2}\lambda_2, \lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2})} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$

and, therefore, f is in the variable z_2 of a polynomial of degree not exceeding λ_2.

If $2\lambda_1 + \lambda_2 + 3$ is even and λ_2 is odd, i.e. λ_1 is an integer, the function

$$f = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} \frac{(-\frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - \frac{3}{2})_n}{n!\,(\frac{1}{2})_n} \times (2z_1z_2 - z_3)^{2n}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + \frac{3}{2}},$$

in the variable z_2 is a polynomial of degree $2\lambda_1 + \lambda_2 + 3$. Thus admissible solutions get only $\lambda_1 \leq -2$.

If $2\lambda_1 + \lambda_2 + 3$ is odd, then solution f_2 comes into play. If $\frac{1}{2}(\lambda_2 - 1) \in \mathbb{N}_0$, i.e. for odd λ_2 and half integer λ_1 sum in f_2 only $n \leq \frac{1}{2}(\lambda_2 - 1)$, then the solutions are

$$f = \sum_{n=0}^{\min(\frac{1}{2}\lambda_2 - \frac{1}{2}, \lambda_1 + \frac{1}{2}\lambda_2 + 1)} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n+1}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}$$

in the z_2 polynomial of degree not exceeding λ_2.

But for $2\lambda_1 + \lambda_2 + 3$ odd and λ_2 even, i.e. $\lambda_1 \in \mathbb{Z}$, the solution is

$$f = \sum_{n=0}^{\lambda_1 + \frac{1}{2}\lambda_2 + 1} \frac{(\frac{1}{2} - \frac{1}{2}\lambda_2)_n(-\lambda_1 - \frac{1}{2}\lambda_2 - 1)_n}{n!\,(\frac{3}{2})_n} \times (2z_1z_2 - z_3)^{2n+1}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 - n + 1}.$$

In the variable z_2 it is a polynomial of degree $2\lambda_1 + \lambda_2 + 3$. Therefore we get a permissible solution for $2\lambda_1 + \lambda_2 + 3 \leq \lambda_2$, i.e. $\lambda_1 \leq -2$.

Case 6 ($\rho_1 = -\lambda_2 - 1$, $\rho_2 = \lambda_1 + \lambda_2 + 2$.)**.** In this case,

$$f(z_1, z_2, z_3, z_4) = z_1^{-\lambda_1 - \lambda_2 - 2}(4z_1z_4 + z_3^2)^{\lambda_1 + \frac{1}{2}\lambda_2 + \frac{3}{2}} g(t),$$

where function $g(t)$ is the solution of equation (10).

For this function f to be polynomial, must be $2\lambda_1 + \lambda_2 + 3 \in \mathbb{N}_0$ and $-\lambda_1 - \lambda_2 - 2 \in \mathbb{N}_0$. But these conditions are not fulfilled for any $\lambda_2 \in \mathbb{N}_0$.

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