Erlangen Programme at Large 3.2 Ladder Operators in Hypercomplex Mechanics

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Abstract

We revise the construction of creation/annihilation operators in quantum mechanics based on the representation theory of the Heisenberg and symplectic groups. Besides the standard harmonic oscillator (the elliptic case) we similarly treat the repulsive oscillator (hyperbolic case) and the free particle (the parabolic case). The respective hypercomplex numbers turn out to be handy on this occasion. This provides a further illustration to the Similarity and Correspondence Principle.

Keywords: Heisenberg group, Kirillov's method of orbits, geometric quantisation, quantum mechanics, classical mechanics, Planck constant, dual numbers, double numbers, hypercomplex, jet spaces, hyperbolic mechanics, interference, Fock-Segal-Bargmann representation, Schrödinger representation, dynamics equation, harmonic and unharmonic oscillator, contextual probability, symplectic group, metaplectic representation, Shale-Weil representation.

1 Introduction

Harmonic oscillators are treated in most textbooks on quantum mechanics. This is efficiently done through creation/annihilation (ladder) operators [9, 3]. The underlying structure is the representation theory of the Heisenberg and symplectic groups [28, § VI.2], [34, § 8.2], [12, 8]. It is also known that quantum mechanics and field theory can benefit from the introduction of Clifford algebra-valued group representations [20, 5, 4, 10].

The dynamics of a harmonic oscillator generates the symplectic transformation of the phase space of the elliptic type. The respective parabolic and hyperbolic counterparts are also of interest [37, § 3.8], [35]. As we will see, they are naturally connected with the respective hypercomplex numbers.

To make this correspondence explicit we recall that the symplectic group Sp(2) [8, § 1.2] consists of 2×2 matrices with real entries and the unit determinant. It is isomorphic to the group $\mathrm{SL}_2(\mathbb{R})$ [28, 13, 30] and provides linear symplectomorphisms of the two-dimensional phase space. It has three types of non-isomorphic one-dimensional subgroups represented by:

$$K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, \right.$$
$$\left. t \in (-\pi, \pi] \right\}, \tag{1}$$

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, t \in \mathbb{R} \right\}, \tag{2}$$

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, t \in \mathbb{R} \right\}. \tag{3}$$

We will refer to them as elliptic, parabolic and hyperbolic subgroups, respectively.

On the other hand, there are three non-isomorphic types of commutative, associative two-dimensional algebras known as complex, dual and double numbers [38, App. C], [29, § 5]. They are represented by expressions $x + \iota y$, where ι stands for one of the hypercomplex units i, ε or j with the properties:

$$i^2 = -1, \qquad \varepsilon^2 = 0, \qquad j^2 = 1.$$

These units can also be labelled as elliptic, parabolic and hyperbolic.

In an earlier paper [25], we considered representations of the Heisenberg group which are induced by hypercomplex characters of its centre. The elliptic case (complex numbers) describes the traditional framework of quantum mechanics, of course.

Double-valued representations, with the imaginary unit $j^2 = 1$, are a natural source of hyperbolic quantum mechanics developed for a while [14, 15, 17, 16, 18]. The representation acts on a Krein space with an indefinite inner product [2]. This aroused significant recent interest in connection with $\mathcal{PT}$-symmetric quantum mechanics [10]. However, our approach is different from the classical treatment of Krein spaces: we use the hyperbolic unit j and build the hyperbolic analytic function theory on its own basis [21, 27]. In the traditional approach, the indefinite metric is mapped to a definite inner product through auxiliary operators.

The representation with values in dual numbers provides a convenient description of the classical mechanics. To this end we do not take any sort of semiclassical limit, rather the nilpotency of the imaginary unit ($\varepsilon^2 = 0$) performs the task. This removes the vicious necessity to consider the Planck *constant* tending to zero. Mixing this with complex numbers we get a convenient tool for modelling the interaction between quantum and classical systems [22, 24].

Our construction [25] provides three different types of dynamics and also generates the respective rules for addition of probabilities. In this paper we analyse the three types of dynamics produced by transformations (1–3) from the symplectic group Sp(2) by means of ladder operators. As a result we obtain further illustrations to the following:

Principle (Similarity and Correspondence)
[23, Principle 29]
  1. *Subgroups K, N and A play a similar role in the structure of the group* Sp(2) *and its representations.*
  2. *The subgroups shall be swapped simultaneously with the respective replacement of hypercomplex unit* ι.

Here the two parts are interrelated: without a swap of imaginary units there can be no similarity between different subgroups.

In this paper we work with the simplest case of a particle with only one degree of freedom. Higher dimensions and the respective group of symplectomorphisms Sp($2n$) may require consideration of Clifford algebras [32].

2   Heisenberg group and its automorphisms

Let (s, x, y), where s, x, $y \in \mathbb{R}$, be an element of the one-dimensional Heisenberg group $\mathbb{H}^1$ [8, 12]. Consideration of the general case of $\mathbb{H}^n$ will be similar, but is beyond the scope of present paper. The group law on $\mathbb{H}^1$ is given as follows:

$$(s, x, y) \cdot (s', x', y') = \left(s + s' + \frac{1}{2}\omega(x, y; x', y'), x + x', y + y' \right), \qquad (4)$$

where the non-commutativity is due to ω — the *symplectic form* on $\mathbb{R}^{2n}$ [1, § 37]:

$$\omega(x, y; x', y') = xy' - x'y. \qquad (5)$$

The Heisenberg group is a non-commutative Lie group. The left shifts

$$\Lambda(g) : f(g') \mapsto f(g^{-1}g') \qquad (6)$$

act as a representation of $\mathbb{H}^1$ on a certain linear space of functions. For example, an action on $L_2(\mathbb{H}, dg)$ with respect to the Haar measure $dg = ds\, dx\, dy$ is the *left regular* representation, which is unitary.

The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^1$ is spanned by left-(right-)invariant vector fields

$$
\begin{aligned}
S^{l(r)} &= \pm\partial_s, \\
X^{l(r)} &= \pm\partial_x - \frac{1}{2}y\partial_s, \qquad (7) \\
Y^{l(r)} &= \pm\partial_y + \frac{1}{2}x\partial_s
\end{aligned}
$$

on $\mathbb{H}^1$ with the Heisenberg *commutator relation*

$$[X^{l(r)}, Y^{l(r)}] = S^{l(r)} \qquad (8)$$

and all other commutators vanishing. We will sometime omit the superscript l for left-invariant field.

The group of outer automorphisms of $\mathbb{H}^1$, which trivially acts on the centre of $\mathbb{H}^1$, is the symplectic group Sp(2) defined in the previous section. It is the group of symmetries of the symplectic form ω [8, Thm. 1.22], [11, p. 830]. The symplectic group is isomorphic to $SL_2(\mathbb{R})$ [28], [34, Ch. 8]. The explicit action of Sp(2) on the Heisenberg group is:

$$g : h = (s, x, y) \mapsto g(h) = (s, x', y'), \qquad (9)$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}(2),$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The Shale-Weil theorem [8, § 4.2], [11, p. 830] states that any representation $\rho_\hbar$ of the Heisenberg groups generates a unitary *oscillator* (or *metaplectic*) representation $\rho_\hbar^{\text{SW}}$ of $\widetilde{\text{Sp}}(2)$, the two-fold cover of the symplectic group [8, Thm. 4.58].

We can consider the semidirect product $G = \mathbb{H}^1 \rtimes \widetilde{\text{Sp}}(2)$ with the standard group law:

$$(h, g) * (h', g') = (h * g(h'), g * g'),$$

where

$$h, h' \in \mathbb{H}^1, \quad g, g' \in \widetilde{\text{Sp}}(2),$$

and the stars denote the respective group operations while the action $g(h')$ is defined as the composition of the projection map $\widetilde{\text{Sp}}(2) \to \text{Sp}(2)$ and the action (9). This group is sometimes called the Schrödinger group, and it is known as the maximal kinematical invariance group of both the free Schrödinger equation and the quantum harmonic oscillator [31]. This group is of interest not only in quantum mechanics but also in optics [36, 35].

Consider the Lie algebra $\mathfrak{sp}_2$ of the group Sp(2). Pick up the following basis in $\mathfrak{sp}_2$ [34, § 8.1]:

$$A = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The commutation relations between the elements are:

$$[Z, A] = 2B, \quad [Z, B] = -2A, \quad [A, B] = -\frac{1}{2}Z. \quad (10)$$

Vectors Z, $B + Z/2$ and $-A$ are generators of the one-parameter subgroups K, N and A (1–3) respectively.

Furthermore, we can consider the basis $\{S, X, Y, A, B, Z\}$ of the Lie algebra $\mathfrak{g}$ of the Lie group $G = \mathbb{H}^1 \rtimes \widetilde{\mathrm{Sp}}(2)$. All non-zero commutators besides those already listed in (8) and (10) are:

$$[A, X] = \frac{1}{2}X, \quad [B, X] = -\frac{1}{2}Y, \quad [Z, X] = Y; \quad (11)$$

$$[A, Y] = -\frac{1}{2}Y, \quad [B, Y] = -\frac{1}{2}X, \quad [Z, Y] = -X. \quad (12)$$

The Shale-Weil theorem allows us to expand any representation $\rho_\hbar$ of the Heisenberg group to the representation $\tilde{\rho}_\hbar = \rho_\hbar \oplus \rho_\hbar^{\mathrm{SW}}$ of group G.

Example 1 *Let $\rho_\hbar$ be the Schrödinger representation [8, § 1.3] of $\mathbb{H}^1$ in $L_2(\mathbb{R})$, that is [25, (3.5)]:*

$$[\rho_\chi(s, x, y)f](q) = e^{2\pi i\hbar(s - xy/2) + 2\pi ixq} \cdot f(q - \hbar y). \quad (13)$$

Thus the action of the derived representation on the Lie algebra $\mathfrak{h}_1$ is:

$$\rho_\hbar(X) = 2\pi iq, \qquad \rho_\hbar(Y) = -\hbar \frac{\mathrm{d}}{\mathrm{d}q},$$

$$\rho_\hbar(S) = 2\pi i\hbar I. \quad (14)$$

Then the associated Shale-Weil representation of Sp(2) in $L_2(\mathbb{R})$ has the derived action, cf. [35, (2.2)], [8, § 4.3]:

$$\rho_\hbar^{\mathrm{SW}}(A) = -\frac{q}{2}\frac{\mathrm{d}}{\mathrm{d}q} - \frac{1}{4},$$

$$\rho_\hbar^{\mathrm{SW}}(B) = -\frac{\hbar i}{8\pi}\frac{\mathrm{d}^2}{\mathrm{d}q^2} - \frac{\pi iq^2}{2\hbar}, \quad (15)$$

$$\rho_\hbar^{\mathrm{SW}}(Z) = \frac{\hbar i}{4\pi}\frac{\mathrm{d}^2}{\mathrm{d}q^2} - \frac{\pi iq^2}{\hbar}.$$

We can verify commutators (8) and (10–12) for operators (14–15). It is also obvious that in this representation the following algebraic relations hold:

$$\rho_\hbar^{\mathrm{SW}}(A) = \frac{i}{4\pi\hbar}(\rho_\hbar(X)\rho_\hbar(Y) - \frac{1}{2}\rho_\hbar(S)) = \quad (16)$$

$$\frac{i}{8\pi\hbar}(\rho_\hbar(X)\rho_\hbar(Y) + \rho_\hbar(Y)\rho_\hbar(X)),$$

$$\rho_\hbar^{\mathrm{SW}}(B) = \frac{i}{8\pi\hbar}(\rho_\hbar(X)^2 - \rho_\hbar(Y)^2), \quad (17)$$

$$\rho_\hbar^{\mathrm{SW}}(Z) = \frac{i}{4\pi\hbar}(\rho_\hbar(X)^2 + \rho_\hbar(Y)^2). \quad (18)$$

Thus it is common in quantum optics to name $\mathfrak{g}$ as a Lie algebra with quadratic generators, see [9, § 2.2.4].

Note that $\rho_\hbar^{\mathrm{SW}}(Z)$ is the Hamiltonian of the harmonic oscillator (up to a factor). Then we can consider $\rho_\hbar^{\mathrm{SW}}(B)$ as the Hamiltonian of a repulsive (hyperbolic) oscillator. The operator $\rho_\hbar^{\mathrm{SW}}(B - Z/2) = \frac{\hbar i}{4\pi}\frac{\mathrm{d}^2}{\mathrm{d}q^2}$ is the parabolic analog. A graphical representation of all three transformations is given in Figure 1, and a further discussion of these Hamiltonians can be found in [37, § 3.8]. An important observation, which is often missed, is that the three linear symplectic transformations are unitary rotations in the corresponding hypercomplex algebra. This means that the symplectomorphisms generated by operators Z, $B - Z/2$, B within time t coincide with the multiplication of hypercomplex number $q + \iota p$ by $e^{\iota t}$ [23, § 3], which is just another illustration of the Similarity and Correspondence Principle.
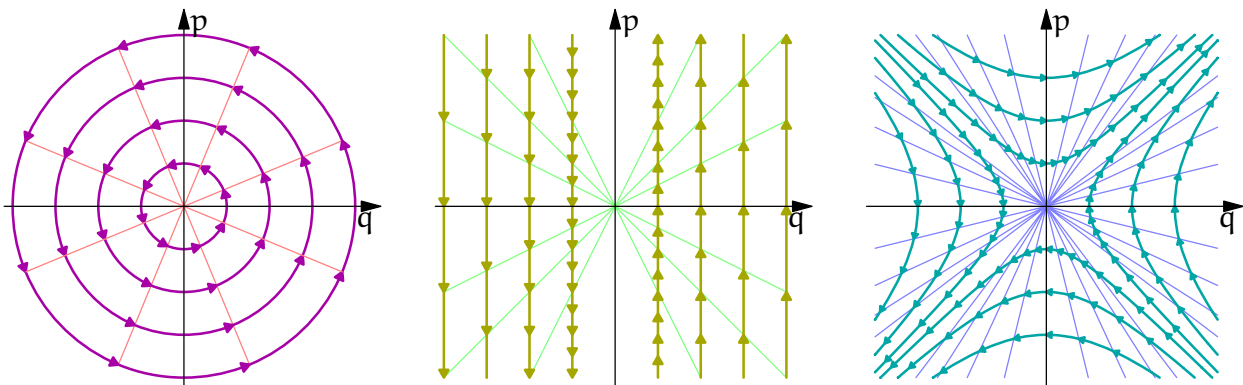

Fig. 1: Three types (elliptic, parabolic and hyperbolic) of linear symplectic transformations on the plane

Example 2 *There are many advantages of considering representations of the Heisenberg group on the phase space [12, § 1.7], [8, § 1.6], [6]. A convenient expression for Fock-Segal-Bargmann (FSB) representation on the phase space is [25, (3.2)]:*

$$[\rho_F(s,x,y)f](q,p) \;=\; e^{-2\pi i(\hbar s + qx + py)} \cdot \tag{19}$$
$$f\left(q - \frac{\hbar}{2}y, p + \frac{\hbar}{2}x\right).$$

Then the derived representation of $\mathfrak{h}_1$ is:

$$\rho_F(X) \;=\; -2\pi i q + \frac{\hbar}{2}\partial_p,$$
$$\rho_F(Y) \;=\; -2\pi i p - \frac{\hbar}{2}\partial_q, \tag{20}$$
$$\rho_F(S) \;=\; -2\pi i \hbar I.$$

This produces the derived form of the Shale-Weil representation:

$$\rho_F^{\text{SW}}(A) \;=\; \frac{1}{2}\left(q\partial_q - p\partial_p\right),$$
$$\rho_F^{\text{SW}}(B) \;=\; -\frac{1}{2}\left(p\partial_q + q\partial_p\right), \tag{21}$$
$$\rho_F^{\text{SW}}(Z) \;=\; p\partial_q - q\partial_p.$$

Note that this representation does not contain the parameter $\hbar$, unlike the equivalent representation (15). Thus the FSB model explicitly shows the equivalence of $\rho_{\hbar_1}^{\text{SW}}$ and $\rho_{\hbar_2}^{\text{SW}}$ if $\hbar_1\hbar_2 > 0$ [8, Thm. 4.57].

As we will also see below, the FSB-type representations in hypercomplex numbers produce almost the same Shale-Weil representations.

3 Ladder operators in quantum mechanics

Let ρ be a representation of the group $G = \mathbb{H}^1 \rtimes \widetilde{\text{Sp}}(2)$ in a space V. Consider the derived representation of the Lie algebra $\mathfrak{g}$ [28, § VI.1] and denote $\tilde{X} = \rho(X)$ for $X \in \mathfrak{g}$. To see the structure of the representation ρ we can decompose the space V into eigenspaces of the operator $\tilde{X}$ for some $X \in \mathfrak{g}$. The canonical example is the Taylor series in complex analysis.

We are going to consider three cases corresponding to three non-isomorphic subgroups (1–3) of Sp(2) starting from the compact case. Let $H = Z$ be a generator of the compact subgroup K. Corresponding symplectomorphisms (9) of the phase space are given by orthogonal rotations with matrices $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$. The Shale-Weil representation (15) coincides with the Hamiltonian of the harmonic oscillator.

Since this is a double cover of a compact group, the corresponding eigenspaces $\tilde{Z}v_k = ikv_k$ are

parametrised by a half-integer $k \in \mathbb{Z}/2$. Explicitly for a half-integer k:

$$v_k(q) = H_{k+\frac{1}{2}}\left(\sqrt{\frac{2\pi}{\hbar}}\,q\right) e^{-\frac{\pi}{\hbar}q^2}, \tag{22}$$

where H_k is the Hermite polynomial [8, § 1.7], [7, 8.2(9)].

From the point of view of quantum mechanics and representation theory (which may be the same), it is beneficial to introduce the ladder operators $L^\pm$, known as *creation/annihilation* in quantum mechanics [8, p. 49] or *raising/lowering* in representation theory [28, § VI.2], [34, § 8.2], [3]. They are defined by the following commutation relations:

$$[\tilde{Z}, L^\pm] = \lambda_\pm L^\pm. \tag{23}$$

In other words, $L^\pm$ are eigenvectors for operators ad Z of the adjoint representation of $\mathfrak{g}$ [28, § VI.2].

Remark 1 The existence of such ladder operators follows from the general properties of Lie algebras if the Hamiltonian belongs to a Cartan subalgebra. This is the case for vectors Z and B, which are the only two non-isomorphic types of Cartan subalgebras in $\mathfrak{sl}_2$. However, the third case considered in this paper, the parabolic vector $B + Z/2$, does not belong to a Cartan subalgebra, yet a sort of ladder operators is still possible with dual number coefficients. Moreover, for the hyperbolic vector B, besides the standard ladder operators an additional pair with double number coefficients will also be described.

From the commutators (23) we deduce that if v_k is an eigenvector of $\tilde{Z}$ then L^+v_k is an eigenvector as well:

$$\tilde{Z}(L^+v_k) \;=\; (L^+\tilde{Z} + \lambda_+L^+)v_k =$$
$$L^+(\tilde{Z}v_k) + \lambda_+L^+v_k =$$
$$ikL^+v_k + \lambda_+L^+v_k =$$
$$(ik + \lambda_+)L^+v_k. \tag{24}$$

Thus the action of ladder operators on the respective eigenspaces V_k can be visualised by the diagram:

$$\cdots \underset{L^-}{\overset{L^+}{\rightleftarrows}} V_{ik-\lambda} \underset{L^-}{\overset{L^+}{\rightleftarrows}} V_{ik} \underset{L^-}{\overset{L^+}{\rightleftarrows}} V_{ik+\lambda} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \cdots \tag{25}$$

There are two ways to search for ladder operators: in (complexified) Lie algebras $\mathfrak{h}_1$ and $\mathfrak{sp}_2$. We will consider them in sequence.

3.1 Ladder operators from the Heisenberg group

Assuming $L^+ = a\tilde{X} + b\tilde{Y}$ we obtain from the relations (11–12) and (23) the linear equations with unknown a and b:

$$a = \lambda_+ b, \qquad -b = \lambda_+ a.$$

The equations have a solution if and only if $\lambda_+^2 + 1 = 0$, and the raising/lowering operators are $L^\pm = \tilde{X} \mp i\tilde{Y}$.

Remark 2 Here we have an interesting asymmetric response: due to the structure of the semidirect product $\mathbb{H}^1 \rtimes \widetilde{\mathrm{Sp}}(2)$ it is the symplectic group which acts on $\mathbb{H}^1$, not vise versa. However, the Heisenberg group has a weak action in the opposite direction: it shifts eigenfunctions of $\mathrm{Sp}(2)$.

In the Schrödinger representation (14) the ladder operators are

$$\rho_\hbar(L^\pm) = 2\pi i q \pm i\hbar \frac{\mathrm{d}}{\mathrm{d}q}. \tag{26}$$

The standard treatment of the harmonic oscillator in quantum mechanics, which can be found in many textbooks, e.g. [8, § 1.7], [9, § 2.2.3], is as follows. The vector $v_{-1/2}(q) = e^{-\pi q^2/\hbar}$ is an eigenvector of $\tilde{Z}$ with the eigenvalue $-\dfrac{i}{2}$. In addition $v_{-1/2}$ is annihilated by L^+. Thus the chain (25) terminates to the right and the complete set of eigenvectors of the harmonic oscillator Hamiltonian is presented by $(L^-)^k v_{-1/2}$ with $k = 0, 1, 2, \ldots$

We can make a wavelet transform generated by the Heisenberg group with the mother wavelet $v_{-1/2}$, and the image will be the Fock-Segal-Bargmann (FSB) space [12], [8, § 1.6]. Since $v_{-1/2}$ is the null solution of $L^+ = \tilde{X} - i\tilde{Y}$, then by the general result [26, Cor. 24] the image of the wavelet transform will be null-solutions of the corresponding linear combination of the Lie derivatives (7):

$$D = \overline{X^r - iY^r} = (\partial_x + i\partial_y) - \pi\hbar(x - iy), \tag{27}$$

which turns out to be the Cauchy-Riemann equation on a weighted FSB-type space.

3.2 Symplectic ladder operators

We can also look for ladder operators within the Lie algebra $\mathfrak{sp}_2$, see [23, § 8]. Assuming $L_2^+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ from the relations (10) and defining condition (23) we obtain the linear equations with unknown a, b and c:

$$c = 0, \qquad 2a = \lambda_+ b, \qquad -2b = \lambda_+ a.$$

The equations have a solution if and only if $\lambda_+^2 + 4 = 0$, and the raising/lowering operators are $L_2^\pm = \pm i\tilde{A} + \tilde{B}$. In the Shale-Weil representation (15) they turn out to be:

$$L_2^\pm = \pm i\left(\frac{q}{2}\frac{d}{dq} + \frac{1}{4}\right) - \frac{\hbar i}{8\pi}\frac{d^2}{dq^2} - \frac{\pi i q^2}{2\hbar} =$$

$$-\frac{i}{8\pi\hbar}\left(\mp 2\pi q + \hbar\frac{d}{dq}\right)^2. \tag{28}$$

Since this time $\lambda_+ = 2i$ the ladder operators $L_2^\pm$ produce a shift on the diagram (25) twice bigger than the operators $L^\pm$ from the Heisenberg group. After all, this is not surprising since from the explicit representations (26) and (28) we get:

$$L_2^\pm = -\frac{i}{8\pi\hbar}(L^\pm)^2.$$

4 Ladder operators for the hyperbolic subgroup

Consider the case of the Hamiltonian $H = 2B$, which is a repulsive (hyperbolic) harmonic oscillator [37, § 3.8]. The corresponding one-dimensional subgroup of symplectomorphisms produces hyperbolic rotations of the phase space. The eigenvectors v_μ of the operator

$$\rho_\hbar^{\mathrm{SW}}(2B)v_\nu = -i\left(\frac{\hbar}{4\pi}\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \frac{\pi q^2}{\hbar}\right)v_\nu = i\nu v_\nu,$$

are Weber-Hermite (or parabolic cylinder) functions $v_\nu = D_{\nu-\frac{1}{2}}\left(\pm 2e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{\hbar}}q\right)$, see [7, § 8.2], [33] for fundamentals of Weber-Hermite functions and [35] for further illustrations and applications in optics.

The corresponding one-parameter group is not compact and the eigenvalues of the operator $2\tilde{B}$ are not restricted by any integrality condition, but the raising/lowering operators are still important [13, § II.1], [30, § 1.1]. We again seek solutions in two subalgebras $\mathfrak{h}_1$ and $\mathfrak{sp}_2$ separately. However, the additional options will be provided by a choice of the number system: either complex or double.

4.1 Complex ladder operators

Assuming $L_h^+ = a\tilde{X} + b\tilde{Y}$ from the commutators (11–12), we obtain the linear equations:

$$-a = \lambda_+ b, \qquad -b = \lambda_+ a. \tag{29}$$

The equations have a solution if and only if $\lambda_+^2 - 1 = 0$. Taking the real roots $\lambda = \pm 1$ we obtain that the raising/lowering operators are $L_h^\pm = \tilde{X} \mp \tilde{Y}$. In the Schrödinger representation (14) the ladder operators are

$$L_h^\pm = 2\pi i q \pm \hbar\frac{d}{dq}. \tag{30}$$

The null solutions $v_{\pm\frac{1}{2}}(q) = e^{\pm\frac{\pi i}{\hbar}q^2}$ to operators $\rho_\hbar(L^\pm)$ are also eigenvectors of the Hamiltonian

$\rho_\hbar^{\text{SW}}(2B)$ with the eigenvalue $\pm\frac{1}{2}$. However the important distinction from the elliptic case is that they are not square-integrable on the real line anymore.

We can also look for ladder operators within the $\mathfrak{sp}_2$, that is in the form $L_{2h}^+ = a\tilde{A} + b\tilde{B} + c\tilde{Z}$ for the commutator $[2\tilde{B}, L_h^+] = \lambda L_h^+$. We will get the system:

$$4c = \lambda a, \qquad b = 0, \qquad a = \lambda c.$$

A solution again exists if and only if $\lambda^2 = 4$. Within complex numbers we get only the values $\lambda = \pm 2$ with the ladder operators $L_{2h}^\pm = \pm 2\tilde{A} + \tilde{Z}/2$, see [13, § II.1], [30, § 1.1]. Each indecomposable $\mathfrak{h}_1$- or $\mathfrak{sp}_2$-module is formed by a one-dimensional chain of eigenvalues with a transitive action of ladder operators $L_h^\pm$ or $L_{2h}^\pm$ respectively. And we again have a quadratic relation between the ladder operators:

$$L_{2h}^\pm = \frac{\text{i}}{4\pi\hbar}(L_h^\pm)^2.$$

4.2 Double ladder operators

There are extra possibilities in the context of hyperbolic quantum mechanics [17,16,18]. Here we use the representation of $\mathbb{H}^1$ induced by a hyperbolic character $e^{\text{j}ht} = \cosh(ht) + \text{j}\sinh(ht)$, see [25, (4.5)], and obtain the hyperbolic representation of $\mathbb{H}^1$, cf. (13):

$$[\rho_h^\text{j}(s', x', y')\hat{f}](q) = e^{\text{j}h(s'-x'y'/2)+\text{j}x'q} \cdot$$
$$\hat{f}(q - hy'). \qquad (31)$$

The corresponding derived representation is

$$\begin{aligned}
\rho_h^\text{j}(X) &= \text{j}q, \\
\rho_h^\text{j}(Y) &= -h\frac{\text{d}}{\text{d}q}, \qquad (32) \\
\rho_h^\text{j}(S) &= \text{j}hI.
\end{aligned}$$

Then the associated Shale–Weil derived representation of $\mathfrak{sp}_2$ in the Schwartz space $S(\mathbb{R})$ is, cf. (15):

$$\begin{aligned}
\rho_h^{\text{SW}}(A) &= -\frac{q}{2}\frac{\text{d}}{\text{d}q} - \frac{1}{4}, \\
\rho_h^{\text{SW}}(B) &= \frac{\text{j}h}{4}\frac{\text{d}^2}{\text{d}q^2} - \frac{\text{j}q^2}{4h}, \qquad (33) \\
\rho_h^{\text{SW}}(Z) &= -\frac{\text{j}h}{2}\frac{\text{d}^2}{\text{d}q^2} - \frac{\text{j}q^2}{2h}.
\end{aligned}$$

Note that $\rho_h^{\text{SW}}(B)$ now generates a usual harmonic oscillator, not the repulsive one like $\rho_\hbar^{\text{SW}}(B)$ in (15). However, the expressions in the quadratic algebra are still the same (up to a factor), cf. (16–18):

$$\rho_h^{\text{SW}}(A) = -\frac{\text{j}}{2h}(\rho_h^\text{j}(X)\rho_h^\text{j}(Y) - \tfrac{1}{2}\rho_h^\text{j}(S)) = \quad (34)$$

$$-\frac{\text{j}}{4h}(\rho_h^\text{j}(X)\rho_h^\text{j}(Y) + \rho_h^\text{j}(Y)\rho_h^\text{j}(X)),$$

$$\rho_h^{\text{SW}}(B) = \frac{\text{j}}{4h}(\rho_h^\text{j}(X)^2 - \rho_h^\text{j}(Y)^2), \qquad (35)$$

$$\rho_h^{\text{SW}}(Z) = -\frac{\text{j}}{2h}(\rho_h^\text{j}(X)^2 + \rho_h^\text{j}(Y)^2). \qquad (36)$$

This is due to the Principle of similarity and correspondence: we can swap operators Z and B with simultaneous replacement of hypercomplex units i and j.

The eigenspace of the operator $2\rho_h^{\text{SW}}(B)$ with an eigenvalue $\text{j}\nu$ are spanned by the Weber-Hermite functions $D_{-\nu-\frac{1}{2}}\left(\pm\sqrt{\frac{2}{h}}x\right)$, see [7, § 8.2]. Functions D_ν are generalisations of the Hermit functions (22).

The compatibility condition for a ladder operator within the Lie algebra $\mathfrak{h}_1$ will be (29) as before, since it depends only on the commutators (11–12). Thus we still have the set of ladder operators corresponding to values $\lambda = \pm 1$:

$$L_h^\pm = \tilde{X} \mp \tilde{Y} = \text{j}q \pm h\frac{\text{d}}{\text{d}q}.$$

Admitting double numbers, we have an extra way to satisfy $\lambda^2 = 1$ in (29) with values $\lambda = \pm\text{j}$. Then there is an additional pair of hyperbolic ladder operators, which are identical (up to factors) to (26):

$$L_\text{j}^\pm = \tilde{X} \mp \text{j}\tilde{Y} = \text{j}q \pm \text{j}h\frac{\text{d}}{\text{d}q}.$$

Pairs $L_h^\pm$ and $L_\text{j}^\pm$ shift eigenvectors in the "orthogonal" directions changing their eigenvalues by ± 1 and $\pm\text{j}$. Therefore an indecomposable $\mathfrak{sp}_2$-module can be parametrised by a two-dimensional lattice of eigenvalues in double numbers, see Table 1.
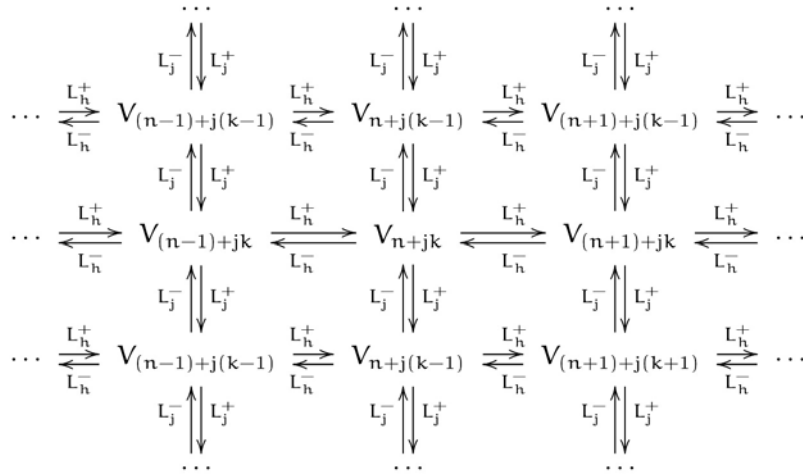
The following functions

$$\begin{aligned}
v_{\frac{1}{2}}^{\pm h}(q) &= e^{\mp\text{j}q^2/(2h)} = \cosh\frac{q^2}{2h} \mp \text{j}\sinh\frac{q^2}{2h}, \\
v_{\frac{1}{2}}^{\pm\text{j}}(q) &= e^{\mp q^2/(2h)}
\end{aligned}$$

are null solutions to the operators $L_h^\pm$ and $L_\text{j}^\pm$, respectively. They are also eigenvectors of $2\rho_h^{\text{SW}}(B)$ with eigenvalues $\mp\frac{\text{j}}{2}$ and $\mp\frac{1}{2}$, respectively. If these functions are used as mother wavelets for the wavelet transforms generated by the Heisenberg group, then the image space will consist of the null-solutions of the following differential operators, see [26, Cor. 24]:

$$\begin{aligned}
D_h &= \overline{X^r - Y^r} = (\partial_x - \partial_y) + \frac{h}{2}(x + y), \\
D_\text{j} &= \overline{X^r - \text{j}Y^r} = (\partial_x + \text{j}\partial_y) - \frac{h}{2}(x - \text{j}y),
\end{aligned}$$

Table 1: The action of hyperbolic ladder operators on a 2D lattice of eigenspaces. Operators $L_h^\pm$ move the eigenvalues by 1, making shifts in the horizontal direction. Operators $L_j^\pm$ change the eigenvalues by j, shown as vertical shifts

for $v_{\frac{1}{2}}^{\pm h}$ and $v_{\frac{1}{2}}^{\pm j}$, respectively. This is again in line with the classical result (27). However annihilation of the eigenvector by a ladder operator does not mean that the part of the 2D-lattice becomes void, since it can be reached via alternative routes. Instead of multiplication by a zero, as happens in the elliptic case, a half-plane of eigenvalues will be multiplied by the divisors of zero $1 \pm j$.

We can also search ladder operators within the algebra $\mathfrak{sp}_2$ and admitting double numbers we will again find two sets of them [23, § 3]:

$$
\begin{aligned}
L_{2h}^\pm &= \pm\tilde{A} + \tilde{Z}/2 = \\
&\mp \frac{q}{2}\frac{d}{dq} \mp \frac{1}{4} - \frac{jh}{4}\frac{d^2}{dq^2} - \frac{jq^2}{4h} = -\frac{j}{4h}(L_h^\pm)^2, \\
L_{2j}^\pm &= \pm j\tilde{A} + \tilde{Z}/2 = \\
&\mp \frac{jq}{2}\frac{d}{dq} \mp \frac{j}{4} - \frac{jh}{4}\frac{d^2}{dq^2} - \frac{jq^2}{4h} = -\frac{j}{4h}(L_j^\pm)^2.
\end{aligned}
$$

Again these operators $L_{2h}^\pm$ and $L_{2h}^\pm$ produce double shifts in the orthogonal directions on the same two-dimensional lattice in Tabular 1.

5 Ladder operator for the nilpotent subgroup

Finally, we look for ladder operators for the Hamiltonian $\tilde{B} + \tilde{Z}/2$ or, equivalently, $-\tilde{B} + \tilde{Z}/2$. It can be identified with a free particle [37, § 3.8].

We can look for ladder operators in the representation (14–15) within the Lie algebra $\mathfrak{h}_1$ in the form $L_\varepsilon^\pm = a\tilde{X} + b\tilde{Y}$. This is possible if and only if

$$-b = \lambda a, \quad 0 = \lambda b. \tag{37}$$

The compatibility condition $\lambda^2 = 0$ implies $\lambda = 0$ within complex numbers. However, such a "ladder"

operator produces only the zero shift on the eigenvectors, cf. (24).

Another possibility appears if we consider the representation of the Heisenberg group induced by dual-valued characters. On the configurational space such a representation is [25, (4.11)]:

$$
[\rho_\chi^\varepsilon(s,x,y)f](q) = e^{2\pi ixq}\left(\left(1 - \varepsilon h\left(s - \frac{1}{2}xy\right)\right)\cdot\right.
$$
$$
\left. f(q) + \frac{\varepsilon hy}{2\pi i}f'(q)\right). \tag{38}
$$

The corresponding derived representation of $\mathfrak{h}_1$ is

$$
\begin{aligned}
\rho_h^p(X) &= 2\pi iq, \\
\rho_h^p(Y) &= \frac{\varepsilon h}{2\pi i}\frac{d}{dq}, \\
\rho_h^p(S) &= -\varepsilon hI.
\end{aligned} \tag{39}
$$

However the Shale-Weil extension generated by this representation is inconvenient. It is better to consider the FSB-type parabolic representation [25, (4.9)] on the phase space induced by the same dual-valued character, cf. (19):

$$
[\rho_h^\varepsilon(s,x,y)f](q,p) = e^{-2\pi i(xq+yp)}\cdot \tag{40}
$$
$$
(f(q,p) + \varepsilon h(sf(q,p) + \frac{y}{4\pi i}f_q'(q,p) - \frac{x}{4\pi i}f_p'(q,p))).
$$

Then the derived representation of $\mathfrak{h}_1$ is:

$$
\begin{aligned}
\rho_h^p(X) &= -2\pi iq - \frac{\varepsilon h}{4\pi i}\partial_p, \\
\rho_h^p(Y) &= -2\pi ip + \frac{\varepsilon h}{4\pi i}\partial_q, \\
\rho_h^p(S) &= \varepsilon hI.
\end{aligned} \tag{41}
$$

An advantage of the FSB representation is that the derived form of the parabolic Shale–Weil representation coincides with the elliptic one (21).

Eigenfunctions with the eigenvalue μ of the parabolic Hamiltonian $\tilde{B} + \tilde{Z}/2 = q\partial_p$ have the form

$$v_\mu(q, p) = e^{\mu p/q} f(q), \qquad (42)$$

with an arbitrary function $f(q)$.

The linear equations defining the corresponding ladder operator $L_\varepsilon^\pm = a\tilde{X} + b\tilde{Y}$ in the algebra $\mathfrak{h}_1$ are (37). The compatibility condition $\lambda^2 = 0$ implies $\lambda = 0$ within complex numbers again. Admitting dual numbers, we have additional values $\lambda = \pm\varepsilon\lambda_1$ with $\lambda_1 \in \mathbb{C}$ with the corresponding ladder operators

$$L_\varepsilon^\pm = \tilde{X} \mp \varepsilon\lambda_1\tilde{Y} = -2\pi\mathrm{i}q - \frac{\varepsilon h}{4\pi\mathrm{i}}\partial_p \pm 2\pi\varepsilon\lambda_1\mathrm{i}p = \\ -2\pi\mathrm{i}q + \varepsilon\mathrm{i}\left(\pm 2\pi\lambda_1 p + \frac{h}{4\pi}\partial_p\right).$$

For the eigenvalue $\mu = \mu_0 + \varepsilon\mu_1$ with $\mu_0, \mu_1 \in \mathbb{C}$ the eigenfunction (42) can be rewritten as:

$$v_\mu(q, p) = e^{\mu p/q} f(q) = e^{\mu_0 p/q}\left(1 + \varepsilon\mu_1\frac{p}{q}\right) f(q) \quad (43)$$

due to the nilpotency of ε. Then the ladder action of $L_\varepsilon^\pm$ is $\mu_0 + \varepsilon\mu_1 \mapsto \mu_0 + \varepsilon(\mu_1 \pm \lambda_1)$. Therefore, these operators are suitable for building $\mathfrak{sp}_2$-modules with a one-dimensional chain of eigenvalues.

Finally, consider the ladder operator for the same element $B + Z/2$ within the Lie algebra $\mathfrak{sp}_2$. According to the above procedure we get the equations:

$$-b + 2c = \lambda a, \qquad a = \lambda b, \qquad \frac{a}{2} = \lambda c,$$

which can again be resolved if and only if $\lambda^2 = 0$. There is the only complex root $\lambda = 0$ with the corresponding operators $L_p^\pm = \tilde{B} + \tilde{Z}/2$, which does not affect the eigenvalues. However the dual number roots $\lambda = \pm\varepsilon\lambda_2$ with $\lambda_2 \in \mathbb{C}$ lead to the operators

$$L_\varepsilon^\pm = \pm\varepsilon\lambda_2\tilde{A} + \tilde{B} + \tilde{Z}/2 = \pm\frac{\varepsilon\lambda_2}{2}(q\partial_q - p\partial_p) + q\partial_p.$$

6 Conclusions: similarity and correspondence

We wish to summarise our findings. Firstly, the appearance of hypercomplex numbers in ladder operators for $\mathfrak{h}_1$ follows exactly the same pattern as was already noted for $\mathfrak{sp}_2$ [23, Rem. 32]:

- the introduction of complex numbers is a necessity for the *existence* of ladder operators in the elliptic case;
- in the parabolic case, we need dual numbers to make ladder operators *useful*;

- in the hyperbolic case, double numbers are not required neither for the existence or for the usability of ladder operators, but they do provide an enhancement.

In the spirit of the Similarity and Correspondence Principle we have the following extension of Prop. 33 from [23]:

Proposition *Let a vector $H \in \mathfrak{sp}_2$ generate the subgroup K, N' or A', that is $H = Z$, $B + Z/2$, or $2B$, respectively. Let ι be the respective hypercomplex unit. Then the ladder operators $L^\pm$ satisfying the commutation relation:*

$$[H, L_2^\pm] = \pm\iota L^\pm$$

are given by:
1. *Within the Lie algebra $\mathfrak{h}_1$: $L^\pm = \tilde{X} \mp \iota\tilde{Y}$.*
2. *Within the Lie algebra $\mathfrak{sp}_2$: $L_2^\pm = \pm\iota\tilde{A} + \tilde{E}$. Here $E \in \mathfrak{sp}_2$ is a linear combination of B and Z with the properties:*
   - *$E = [A, H]$.*
   - *$H = [A, E]$.*
   - *Killings form $K(H, E)$ [19, § 6.2] vanishes. Any of the above properties defines the vector $E \in \mathrm{span}\{B, Z\}$ up to a real constant factor.*

It is worth continuing this investigation and describing in detail hyperbolic and parabolic versions of FSB spaces.

Acknowledgement

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