Self-adjoint Extensions of Schrödinger Operators with δ-magnetic Fields on Riemannian Manifolds

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Abstract

We consider the magnetic Schrödinger operator on a Riemannian manifold M. We assume the magnetic field is given by the sum of a regular field and the Dirac δ measures supported on a discrete set Γ in M. We give a complete characterization of the self-adjoint extensions of the minimal operator, in terms of the boundary conditions. The result is an extension of the former results by Dabrowski-Šťovíček and Exner-Šťovíček-Vytřas.

Keywords: Spectral theory, functional analysis, self-adjointness, Aharonov-Bohm effect, quantum mechanics, differential geometry, Schrödinger operator.

1 Introduction

Let (M,g) be a two-dimensional, oriented, connected complete C^∞-Riemannian manifold, where g is the Riemannian metric on M. Let $d\mu$ be the measure induced from the Riemannian metric. If we take a local chart (U,φ), $\varphi=(x^1,x^2)$, the measure $d\mu$ is written as $d\mu=\sqrt{G}dx^1dx^2$ in U, where $G=\det(g_{mn})$, $g_{mn}=g(\partial_m,\partial_n)$, and $\partial_m=\partial/\partial x^m$. We denote $L^2(M)=L^2(M;d\mu)$. The set of all 1-forms on M is denoted by $\Lambda^1(M)$. In the coordinate neighborhood U, $A\in\Lambda^1(M)$ is written as

$$A=A_1dx^1+A_2dx^2.$$

In general, the coefficients A_1, A_2 are complex-valued. We say A is real-valued if the coefficients are real-valued. We say A is of the class $C^k\Lambda^1(M)$ if the coefficients are of the class $C^k(U)$ for any local chart (U,φ). We define the class $L^q_{\mathrm{loc}}\Lambda^1(M)$ $(1\le q\le\infty)$[1], etc. similarly. The 2-form dA is called the magnetic field. If $A\in L^1_{\mathrm{loc}}\Lambda^1(M)$, dA can be defined at least in the distribution sense. In U, the magnetic field is given by

$$dA=(\partial_1A_2-\partial_2A_1)dx^1\wedge dx^2.$$

Let $\Gamma=\{\gamma_k\}_{k=1}^K$ be a sequence of mutually distinct points in M. The number K may be infinity, and in this case we assume additionally Γ has no accumulation points in M. Let A be a 1-form on M given by the sum of two 1-forms

(A) $A=A^{(0)}+A^{(1)}$.

The part $A^{(0)}$ corresponds to the δ magnetic fields, that is, we assume the following.

(A0) $A^{(0)}\in C^\infty\Lambda^1(M\setminus\Gamma)\cap L^1_{\mathrm{loc}}\Lambda^1(M)$, real-valued, and

$$dA^{(0)}=\sum_{k=1}^K 2\pi\alpha_k\delta_{\gamma_k}, \qquad (1)$$

where $\alpha_k\in\mathbb{R}$, and δ_γ is the Dirac measure concentrated on the point γ.

More precisely, (1) means

$$-\int_M \mathrm{d}\varphi\wedge A^{(0)}=\sum_{k=1}^K 2\pi\alpha_k\varphi(\gamma_k),$$

for any $\varphi\in C_0^\infty(M)$ (since $A^{(0)}\in L^1_{\mathrm{loc}}\Lambda^1(M)$, the left hand side is well-defined). Notice that this equation is independent of the Riemannian metric g. For the regular part $A^{(1)}$ and the scalar potential V, we assume the following:

(A1) $A^{(1)}\in C^1\Lambda^1(M)$, real-valued.

(V) V is real-valued, $V\in L^2_{\mathrm{loc}}(M)$, and is bounded in some open neighborhood of γ_k for every $k=1,\dots,K$.

Using the local coordinate (x^1,x^2), we define the Schrödinger operator $\mathcal{L}$ in each coordinate neighborhood by

$$\mathcal{L}u = -\frac{1}{\sqrt{G}}\sum_{m,n=1,2}(\partial_m+iA_m)\cdot \left(\sqrt{G}g^{mn}(\partial_n+iA_n)u\right)+Vu,$$

[1] The measure $d\mu$ is omitted, since the class $L^q_{\mathrm{loc}}\Lambda^1(M;d\mu)$ is independent of the choice of $d\mu$. The coefficient A_m is a function on $U\subset M$, however, we denote the pull-back $(\varphi^{-1})^*A_m=A_m\circ\varphi^{-1}$ on $\varphi(U)\subset\mathbb{R}^2$ by the same symbol A_m, for simplicity of notations. This convention is frequently used in this paper.

where (g^{mn}) is the inverse matrix of (g_{mn}). This definition is independent of the choice of local coordinates (see section 2). Define the minimal operator $H_{\min}$ by

$$H_{\min}u = \mathcal{L}u, \quad D(H_{\min}) = \overline{C_0^\infty(M \setminus \Gamma)},$$

where the overline denotes the closure with respect to the graph norm. Define the maximal operator $H_{\max}$ by $H_{\max} = H_{\min}^*$. Then we can show that

$$\begin{aligned} H_{\max}u &= \mathcal{L}u, \\ D(H_{\max}) &= \{u \in L^2(M) \mid \mathcal{L}u \in L^2(M)\}, \end{aligned}$$

where $\mathcal{L}$ is a differential operator on $\mathcal{D}'(M \setminus \Gamma)$. We assume

(SB) The operator $H_{\min}$ is bounded from below.

In the case M is the flat Euclidean plane, it is well-known that the operator $H_{\min}$ is *not* essentially self-adjoint and the structure of the self-adjoint extensions of $H_{\min}$ can be determined via the celebrated Krein-Von Neumann theory of self-adjoint extensions (see e.g. Reed-Simon [13]). In the textbook by Albeverio et al. [3], the case $A^{(0)} = A^{(1)} = 0$ and $V = 0$ (but $\Gamma \neq \emptyset$) is exhaustively studied. Adami-Teta [1] and Dabrowski-Šťovíček [7] study the case $K = 1$, $\alpha_1 \notin \mathbb{Z}$, $A^{(1)} = 0$, and $V = 0$. Exner-Šťovíček-Vytřas [8] study the case $K = 1$, $\alpha_1 \notin \mathbb{Z}$, $dA^{(1)} = Bdx^1 \wedge dx^2$ for some non-zero constant B (the constant magnetic field), and $V = 0$. Moreover, Lisovyy [11] studies the case M is the Poincaré disk, g is the Poincaré metric, $V = 0$ and $dA = B\omega_g + 2\pi\alpha\delta_0$, where B is a non-zero constant and ω_g is the surface form induced from the Poincaré metric g.

In all the results above, they first determine the deficiency subspaces $\mathrm{Ker}(H_{\max} \mp i)$ and apply the Krein-Von Neumann theory. This method cannot be applied in the case $K \geq 2$ and $\alpha_k \notin \mathbb{Z}$, however, this case (and $A^{(1)}$ is the constant field, $V = 0$) on the flat Euclidean plane is studied by the author [12], and the structure of the self-adjoint extensions is determined. Our main purpose in this paper is to generalize the result in [12] on general complete Riemannian manifolds and for more general A and V.

Our first result is about the deficiency indices $n_\pm(H_{\min}) = \dim \mathrm{Ker}(H_{\max} \mp i)$.

Theorem 1.1 *Assume (A), (A0), (A1), (V), and (SB). Then, both deficiency indices $n_\pm(H_{\min})$ are equal to $2K_1 + K_2$, where*

$$K_1 = \#\{\alpha_k \mid \alpha_k \notin \mathbb{Z}\}, \quad K_2 = \#\{\alpha_k \mid \alpha_k \in \mathbb{Z}\}.$$

Note that Bulla-Gesztesy [4] obtain a similar result in the case $A = 0$ and V has singularities, and Iwai-Yabu [9] also obtain a similar result on the two-dimensional torus.

Next, we shall give a complete characterization of the self-adjoint extensions of $H_{\min}$. To this purpose, we introduce some nice coordinates around singularities and some auxiliary functions. For simplicity, we assume $K = \#\Gamma$ is finite for a while.

For $k = 1, \ldots, K$, let (U_k, ϕ_k), $\phi_k = (x^1, x^2)$, be a local chart around γ_k such that U_k is simply connected, $\phi_k(\gamma_k) = 0$, V is bounded in U_k, and $\{U_k\}_{k=1}^K$ are disjoint. Let (r, θ) be the radial coordinate in U_k defined by $x^1 + ix^2 = re^{i\theta}$, $r \geq 0$, $0 \leq \theta < 2\pi$. We assume

$$\begin{aligned} g_{mn}(0,0) &= \delta_{mn}, \\ \partial_j g_{mn}(0,0) &= 0 \ (m,n,j = 1,2), \end{aligned} \tag{2}$$

where δ_{mn} is the Kronecker delta. Condition (2) is satisfied, for example, if we take the normal coordinate[2] as (x^1, x^2).

Let β_k be the fractional part of α_k, that is, $\alpha_k = [\alpha_k] + \beta_k$, $[\alpha_k] \in \mathbb{Z}$ and $0 \leq \beta_k < 1$. Put[3]

$$\begin{aligned} \tilde{A}^{(0)} &= \beta_k r^{-2}(-x^2dx^1 + x^1dx^2), \\ \tilde{A}^{(1)} &= A^{(1)} - A^{(1)}(0). \end{aligned}$$

It is well-known that $d\tilde{A}^{(0)} = 2\pi\beta_k\delta_0$ (see e.g. Aharonov-Bohm [2, 1] or [7]). Define a phase function $\psi_k \in C^\infty(U_k \setminus \{0\})$ by

$$\psi_k(x) = \exp\frac{1}{i}\left(A_1^{(1)}(0)x^1 + A_2^{(1)}(0)x^2 + \int_{x_0}^x \left(A^{(0)} - \tilde{A}^{(0)}\right)\right), \tag{3}$$

where $A^{(1)} = A_1^{(1)}dx^1 + A_2^{(1)}dx^2$, x_0 is some point in $U_k \setminus \{0\}$, and the path of the line integral $\int_{x_0}^x$ lies in $U_k \setminus \{0\}$. Notice that the value of the line integral is independent of the choice of paths modulo $2\pi\mathbb{Z}$, by the Stokes theorem and the assumption $d(A^{(0)} - \tilde{A}^{(0)}) = 2\pi[\alpha_k]\delta_0$ in U_k. Then we have

$$A = \tilde{A} + i\psi_k^{-1}d\psi_k, \quad \tilde{A} = \tilde{A}^{(0)} + \tilde{A}^{(1)} \tag{4}$$

and

$$\mathcal{L} = \psi_k\tilde{\mathcal{L}}\psi_k^{-1} \tag{5}$$

in $U_k \setminus \{0\}$, where $\tilde{\mathcal{L}}$ is the operator $\mathcal{L}$ corresponding to the vector potential $\tilde{A}$ and the scalar potential V.

Let K_1, K_2 be the numbers in Theorem 1.1. In the sequel, we rearrange the index k so that $0 < \beta_k < 1$ for $1 \leq k \leq K_1$. As we prove later, the

[2]The coordinate defined by the local inverse map of the exponential map from the tangent space at γ_k to M.

[3]More precisely, the 1-form $A^{(1)} - A^{(1)}(0)$ is defined as $(A_1^{(1)}(x^1,x^2) - A_1^{(1)}(0,0))dx^1 + (A_2^{(1)}(x^1,x^2) - A_2^{(1)}(0,0))dx^2$, in the coordinate neighborhood U_k.

asymptotics of $u \in D(H_{\max})$ in U_k as $r \to 0$ is given by

$$u = \begin{cases} \psi_k(c_1^k r^{\beta_k - 1} e^{-i\theta} + c_2^k r^{-\beta_k} + \\ c_4^k r^{1-\beta_k} e^{-i\theta} + c_5^k r^{\beta_k}) + \xi \qquad (1 \le k \le K_1), \\ \psi_k(c_3^k \log r + c_6^k) + \xi \qquad (K_1 + 1 \le k \le K), \end{cases}$$

where $c_1^k, \ldots, c_6^k$ are constants and ξ is a regular function in the sense $\xi \in D(H_{\min})$. Define

$$\Phi_j(u) = \begin{cases} {}^t(c_j^1, \ldots, c_j^{K_1}) \in \mathbb{C}^{K_1} & (j = 1, 2, 4, 5), \\ {}^t(c_j^{K_1+1}, \ldots, c_j^K) \in \mathbb{C}^{K_2} & (j = 3, 6), \end{cases}$$

$$\Phi(u) = {}^t({}^t\Phi_1(u) \cdots {}^t\Phi_6(u)) \in \mathbb{C}^{4K_1 + 2K_2}.$$

Define a $(2K_1 + K_2) \times (2K_1 + K_2)$-diagonal matrix D by

$$D = \operatorname{diag}(1 - \beta_1, \ldots, 1 - \beta_{K_1}, \beta_1, \ldots, \beta_{K_1}, -1/2, \ldots, -1/2). \quad (6)$$

Now our theorem is stated as follows.

Theorem 1.2 *Assume (A), (A0), (A1), (V), (SB) and $K < \infty$. Let $\Phi(u)$, D given above.*

(i) Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, where X_1, X_2 are $(2K_1 + K_2) \times (2K_1 + K_2)$ matrices satisfying

$$\operatorname{rank} X = 2K_1 + K_2, \ X_1^* D X_2 = X_2^* D X_1. \quad (7)$$

Then, the operator H_X defined by

$$H_X u = \mathcal{L} u,$$
$$D(H_X) = \{u \in D(H_{\max}) \mid \Phi(u) \in \operatorname{Ran} X\}$$

is a self-adjoint extension of $H_{\min}$.

(ii) For any self-adjoint extension H of $H_{\min}$, there exists some matrix X satisfying (7) and $H = H_X$.

We can consider the case $K = \infty$, but some technical assumptions are necessary. We shall argue this case in section 5.

Thus we can characterize the self-adjoint extensions in terms of the boundary conditions. We can easily prove that the Friedrichs extension corresponds to the case $X_1 = O$, $X_2 = Id$. In the case $M = \mathbb{R}^2$ and $K = 1$, similar results are obtained in [7] and [8], and our theorem is a generalization of their results. As stated in their paper, the choice of matrices X is of course not unique: there are infinitely many matrices X giving same $\operatorname{Ran} X$.

The difficulty in the proof is that we cannot determine the deficiency subspaces explicitly. To overcome this difficulty, we describe the condition of the self-adjointness only using the quotient subspace $D(H_{\max})/D(H_{\min})$. This quotient subspace is essentially the same object as the sum of deficiency subspaces, but much easily tractable than the deficiency subspaces themselves. This idea is also used in [4] or [12].

We note that recently self-adjoint extensions of the Schrödinger operators on $\mathbb{R}^2$ with δ magnetic fields are studied from the viewpoint of the hidden supersymmetric structure; see Correa et al. [5, 6].

The rest of the paper is organized as follows. In section 2, we review basic notations and facts from the differential geometry and the theory of self-adjoint extensions. In section 3, we shall prove the structure of the self-adjoint extensions depends only on the singular part of the vector potentials. In section 4, we shall prove the main theorems. In section 5, we shall consider the case $K = \infty$ and give a complete characterization of the self-adjoint extensions, under some homogeneity conditions.

2 Basic facts

2.1 Formulas in differential geometry

We quote some formulas used in Shubin [14] for the convenience of the readers. Take a local chart (U, φ), $\varphi = (x^1, x^2)$, around $p \in M$. Put $g_{mn} = g(\partial_m, \partial_n)$, and let (g^{mn}) be the inverse matrix of (g_{mn}). For $\alpha, \beta \in \Lambda_p^1(M)$ (the cotangent space at p), we define the scalar product

$$\langle \alpha, \beta \rangle = \sum_{m,n=1,2} g^{mn} \alpha_m \beta_n,$$

where $\alpha = \alpha_1 dx^1 + \alpha_2 dx^2$ and $\beta = \beta_1 dx^1 + \beta_2 dx^2$. Put $|\alpha|^2 = \langle \overline{\alpha}, \alpha \rangle$, where $\overline{\alpha} = \overline{\alpha_1} dx^1 + \overline{\alpha_2} dx^2$. For a 1-form $\omega = \omega_1 dx^2 + \omega_2 dx^2$, we define a function $d^*\omega$ by

$$d^*\omega = -\frac{1}{\sqrt{G}} \sum_{m,n=1,2} \partial_m \left(\sqrt{G} g^{mn} \omega_n \right).$$

This definition is independent of the choice of local coordinates. Actually, operator d^* is characterized by the following relation:

$$\int_M \langle \overline{du}, \omega \rangle \, d\mu = \int_M \overline{u} d^*\omega \, d\mu$$

for any $u \in C_0^\infty(M)$ and $\omega \in C_0^\infty \Lambda^1(M)$.

Let A be a 1-form satisfying our assumptions. For a function f, we define a 1-form $d_A f$ by

$$d_A f = df + i f A,$$

where d is the exterior derivative, and $i = \sqrt{-1}$. For a 1-form ω, we define

$$d_A^* \omega = d^* \omega - i A^* \omega, \quad A^* \omega = \langle A, \omega \rangle.$$

Then we obtain a representation of our Schrödinger operator $\mathcal{L}$ independent of local coordinates:

$$\mathcal{L} = d_A^* d_A + V.$$

For operator d_A^*, the following Leibniz formulas hold: for an appropriate function f and 1-form ω, we have

$$d_A^*(f\omega) = f d^*\omega - \langle df, \omega\rangle - if\langle A, \omega\rangle = f d_A^*\omega - \langle df, \omega\rangle = f d^*\omega - \langle d_A f, \omega\rangle, \quad (8)$$

$$d_A^* d_A(fg) = f d_A^* d_A g - 2\langle df, d_A g\rangle + g d^* df. \quad (9)$$

Proposition 2.1 *Let U, U' be open subsets of $M \setminus \Gamma$ such that $\overline{U}$ is a compact subset of U', and V is bounded in U'. Then, there exists a constant $C > 0$ such that*

$$\int_U |d_A f|^2 \,\mathrm{d}\mu \le C \int_{U'} (|f|^2 + |\mathcal{L}f|^2)\,\mathrm{d}\mu \quad (10)$$

for $f \in D(H_{\max})$.

Proof. According to [14, (5.3)],[4] we have

$$(\mathcal{L}(\phi f), \phi f) = \Re(\phi\mathcal{L}f, \phi f) + \int_M |d\phi|^2 |f|^2 \,\mathrm{d}\mu$$

for $f \in D(H_{\max})$ and $\phi \in C_0^\infty(M \setminus \Gamma)$. Take $\phi \in C_0^\infty(U')$ such that $\phi = 1$ on U. Then the conclusion follows from the above equality,

$$\int_{\mathrm{supp}\,\phi} |d_A(\phi f)|^2 = (\mathcal{L}(\phi f), \phi f) - (V\phi f, \phi f)$$

and assumption V is bounded. □

2.2 Theory of self-adjoint extensions

We quote some notation from the textbook [13]. Let $\mathcal{H}$ be a separable Hilbert space and denote its inner product by $(\cdot, \cdot)$, and norm by $\|\cdot\|$. All the linear operators in this subsection are on the Hilbert space $\mathcal{H}$. For a linear operator X, $D(X)$ denotes the domain of definition of X, $\overline{X}$ the closure of X, X^* the adjoint operator of X. For a linear operator X, the graph inner product of X is defined by

$$(x, y)_X = (Xx, Xy) + (x, y)$$

for $x, y \in D(X)$, and the graph norm by $\|x\|_X = (x, x)_X^{1/2}$.

We introduce some equivalent for the sum of the deficiency subspaces, which is also introduced in [4] or [12]. Let X be a closed, densely defined symmetric operator. Let $\mathcal{D} = D(X^*)/D(X)$, where the right hand side denotes the quotient space. The space $\mathcal{D}$ is a Hilbert space equipped with the norm

$$\|[x]\|_{\mathcal{D}}^2 = \min_{y \in [x]} \|y\|_{X^*}^2 = \|Qx\|_{X^*}^2,$$

where $x \in D(X^*)$, $[x] = x + D(X)$ denotes the equivalence class of x in the quotient space $D(X^*)/D(X)$, and Q denotes the orthogonal projection onto the orthogonal complement of $D(X)$ in $D(X^*)$. For $u, v \in \mathcal{D}$, define

$$[u, v]_{\mathcal{D}} = (X^*x, y) - (x, X^*y), \quad u = [x],\ v = [y],\ x, y \in D(X^*).$$

The value $[u, v]_{\mathcal{D}}$ is independent of the choice of the representatives x, y. Let P be the canonical projection from $D(X^*)$ to $\mathcal{D}$. For a closed subspace V of $\mathcal{D}$, we define a closed linear operator X_V by

$$D(X_V) = \{x \in D(X^*) \mid Px \in V\}, \quad X_V x = X^* x.$$

We also define

$$V^{[\perp]} = \{u \in \mathcal{D} \mid [u, v]_{\mathcal{D}} = 0 \text{ for any } v \in V\}.$$

Then the following proposition immediately follows from the definition of the self-adjointness.

Proposition 2.2

1. *For a closed subspace V of $\mathcal{D}$, the operator X_V is a self-adjoint extension of X if and only if*
$$V^{[\perp]} = V. \quad (11)$$
2. *For any self-adjoint extension $\tilde{X}$ of X, there exists a closed subspace V of $\mathcal{D}$ such that $X_V = \tilde{X}$.*

In terms of the above notations, the Krein-Von Neumann theory can be rephrased as follows.

Proposition 2.3 *Let $\mathcal{N}_\pm = \mathrm{Ker}(X^* \mp i)$ the deficiency subspaces of X, $n_\pm = \dim \mathcal{N}_\pm$ the deficiency indices of X. Then, the following holds.*

(i) The projection operator P gives a Hilbert space isomorphism from the direct sum $\mathcal{N}_+ \oplus \mathcal{N}_-$ to $\mathcal{D}$. In particular, $\dim \mathcal{D} = n_+ + n_-$.

(ii) There exists a one-to-one correspondence between the closed subspaces V of $\mathcal{D}$ satisfying (11) and the unitary operators U from $\mathcal{H}_+$ to $\mathcal{H}_-$, given by

$$V = P(1 + U)\mathcal{H}_+.$$

This proposition says the space $\mathcal{D}$ can play the same role as the sum of deficiency subspaces in the theory of self-adjoint extensions. Particularly when $\mathcal{N}_\pm$ is difficult to determine explicitly (as in our case), the space $\mathcal{D}$ is more tractable, since the element of this space has ambiguity by $D(X)$. Actually, in the next section we shall see that the structure of $\mathcal{D}$ for our Schrödinger operator $H_{\min}$ and the form $[\cdot, \cdot]_{\mathcal{D}}$ is determined only from the singular part $A^{(0)}$ of the vector potential.

[4] Since the function ϕ avoids the singularities, the proof of [14, (5.3)] is also available in our case.

3 Reduction

3.1 Division to the local potential

Let (U_k, ϕ_k), $\phi_k = (x^1, x^2)$, the local coordinate introduced in section 1. Let $\tilde{A}$ be the 1-form given by (4). Take a positive number ϵ_k so small that the closed disc $\{r \le 2\epsilon_k\}$ is contained in U_k. Let $\eta_k \in C_0^\infty(U)$ such that $0 \le \eta_k \le 1$, $\eta_k = 1$ for $r \le \epsilon_k$, $\eta_k = 0$ for $r \ge 2\epsilon_k$. Define functions $\hat{g}_{mn}$, $\hat{A}_m$ and $\hat{V}$ on $\mathbb{R}^2$ by

$$\begin{aligned} \hat{g}_{mn} &= \eta_k g_{mn} + (1-\eta_k)\delta_{mn}, \\ \hat{A}_m &= \tilde{A}_m^{(0)} + \eta_k \tilde{A}_m^{(1)}, \\ \hat{V} &= \eta_k V. \end{aligned}$$

Define a differential operator $\mathcal{L}_k$ on $\mathbb{R}^2$ by

$$\mathcal{L}_k = -\frac{1}{\sqrt{\hat{G}}} \sum_{m,n=1,2} \left(\frac{\partial}{\partial x_m} + i\hat{A}_m \right) \cdot \sqrt{\hat{G}} \hat{g}^{mn} \left(\frac{\partial}{\partial x_n} + i\hat{A}_n \right) + \hat{V},$$

where $\hat{G} = \det(\hat{g}_{mn})$, and $(\hat{g}^{mn})$ is the inverse matrix of $(\hat{g}_{mn})$. Define a linear operator $L_{k,\min}$ on $L^2(\mathbb{R}^2; d\mu_k)$, $d\mu_k = \sqrt{\hat{G}} dx^1 dx^2$, by

$$L_{k,\min} u = \mathcal{L}_k u, \quad D(L_{k,\min}) = \overline{C_0^\infty(\mathbb{R}^2 \setminus \{0\})}.$$

Let $L_{k,\max} = L_{k,\min}^*$. Then

$$\begin{aligned} &L_{k,\max} u = \mathcal{L}_k u, \\ &D(L_{k,\max}) = \{u \in L^2(\mathbb{R}^2; d\mu_k) \mid \mathcal{L}_k u \in L^2(\mathbb{R}^2; d\mu_k)\}, \end{aligned}$$

where $\mathcal{L}_k$ is regarded as a differential operator on $\mathcal{D}'(\mathbb{R}^2 \setminus 0)$. Let $\mathcal{D} = D(H_{\max})/D(H_{\min})$, $\mathcal{D}_k = D(L_{k,\max})/D(L_{k,\min})$. Let $\chi_k \in C_0^\infty(M)$ such that $0 \le \chi_k \le 1$, $\chi_k = 0$ for $r \ge \epsilon_k$ and $\chi_k = 1$ for $r \le \epsilon_k/2$. Define a map T_k from $\mathcal{D}$ to $\mathcal{D}_k$ by

$$T_k[f] = [\psi_k^{-1} \chi_k f],$$

where the function ψ_k is given by (3). Define a map T from $\mathcal{D}$ to the direct sum $\bigoplus_{k=1}^{K} \mathcal{D}_k$ by

$$T[f] = \bigoplus_{k=1}^{k} T_k[f].$$

We also define a map S from $\bigoplus_{k=1}^{K} \mathcal{D}_k$ to $\mathcal{D}$ by[5]

$$S \bigoplus_{k=1}^{K} [f_k] = \left[\sum_{k=1}^{K} \psi_k \chi_k f_k \right].$$

In the sequel, we sometimes write $[f,g]_{\mathcal{D}} = [[f],[g]]_{\mathcal{D}}$ etc. for simplicity of notations.

Lemma 3.1 *1. Assume $K < \infty$. Then, the maps S, T defined above are well-defined and mutually inverse. Moreover, we have*

$$[f,g]_{\mathcal{D}} = \sum_{k=1}^{k} [T_k[f], T_k[g]]_{\mathcal{D}_k} \tag{12}$$

for any $[f],[g] \in \mathcal{D}$.

2. Assume $K = \infty$. Then the map S is well-defined and injective.

Proof. (i) We divide the proof into three steps.
Step 1. The map

$$D(H_{\max}) \ni f \mapsto \psi_k^{-1} \chi_k f \in D(L_{k,\max})$$

is well-defined and continuous.
Proof. Clearly $\psi_k^{-1}\chi_k f \in L^2(\mathbb{R}^2; d\mu_k)$, so it suffices to show that $\mathcal{L}_k(\psi_k^{-1}\chi_k f) \in L^2(\mathbb{R}^2; d\mu_k)$. By (5) and the Leibniz rule (9), we have

$$\begin{aligned} &\mathcal{L}_k(\psi_k^{-1}\chi_k f) = \psi_k^{-1}\mathcal{L}(\chi_k f) = \\ &\psi_k^{-1}\left(\chi_k \mathcal{L} f - 2\langle d\chi_k, d_A f\rangle + (d^* d\chi_k) f\right). \end{aligned}$$

The first term and the third in the parenthesis of the right hand side are in $L^2(\mathbb{R}^2; d\mu_k)$ and continuous with respect to $\|\cdot\|_{H_{\max}}$. Moreover, we can prove the second term is also in L^2 and continuous with respect to $\|\cdot\|_{H_{\max}}$ by using (10). □
Step 2. Let $f \in D(H_{\min})$. Then, we have $\psi_k^{-1}\chi_k f \in D(L_{k,\min})$.
Proof. By definition, there exists a sequence $\{f_n\}_{n=1}^\infty \subset C_0^\infty(M \setminus \Gamma)$ such that $f_n \to f$ in $D(H_{\min})$. Then, $\psi_k^{-1}\chi_k f_n \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and $\psi_k^{-1}\chi_k f_n \to \psi_k^{-1}\chi_k f$ in $D(L_{k,\max})$, by Step 1. Since $D(L_{k,\min})$ is a closed subspace of $D(L_{k,\max})$, we have the conclusion. □

Step 1 and 2 imply the map T is well-defined. We can similarly prove that the map S is also well-defined.
Step 3. The operator ST is the identity map on $\mathcal{D}$.
Proof. By definition, we have

$$(I - ST)[f] = [\psi f], \quad \psi = 1 - \sum_{k=1}^{K} \chi_k^2.$$

So it suffices to prove that $g = \psi f \in D(H_{\min})$.

[5]When $K = \infty$, we define the map S for the elements of $\bigoplus_{k=1}^{\infty} \mathcal{D}_k$ having only finite nonzero components $[f_k]$. So there is no difficulty in the definition of S.

Let (r,θ) be the radial coordinate in U_k and put $B_{k,\epsilon} = \{x \in U_k \mid r < \epsilon\}$. Then we have $\operatorname{supp}\psi \subset M \setminus \bigcup_{k=1}^{K} B_{k,\epsilon_k/2}$. For $c > 0$, let $\xi_c \in C^\infty(M)$ such that $0 \le \xi_c \le 1$, $\xi_c = 1$ in $M \setminus \bigcup_{k=1}^{K} B_{\epsilon_k/c}$, $\xi_c = 0$ in $\bigcup_{k=1}^{K} B_{k,\epsilon_k/(2c)}$. Let $\mathcal{L}_0$, $H_{0,\min}$ and $H_{0,\max}$ be the operators corresponding to the potentials $\xi_4 A$ and $\xi_4 V$. These potentials have no singularities, so we have $H_{0,\min} = H_{0,\max}$ by [14]. Since $\mathcal{L}g = \mathcal{L}_0 g \in L^2$, we have $g \in D(H_{0,\max}) = D(H_{0,\min})$. Thus we can take a sequence $\{g_n\}$ such that $g_n \to g$ in $\|\cdot\|_{H_{0,\min}}$. Then $\xi_2 g_n \in C_0^\infty(M \setminus \Gamma)$ and $\xi_2 g_n \to \xi_2 g = g$ in $\|\cdot\|_{H_{\min}}$. Thus we have $g \in D(H_{\min})$. □

We can prove $TS = I$ similarly. Then (12) follows from (5) and the equality $[f,g]_{\mathcal{D}} = \sum_{k=1}^{K} [\chi_k f, \chi_k g]_{\mathcal{D}}$ (notice that $f - \sum_k \chi_k f \in D(H_{\min})$ can be proved as in Step 3).

(2) Let $K = \infty$. For any positive integer n, we can define $T^{(n)}$ from $\mathcal{D}$ to $\bigoplus_{k=1}^{n} \mathcal{D}_k$, and $S^{(n)}$ from $\bigoplus_{k=1}^{n} \mathcal{D}_k$ to $\mathcal{D}$ similarly, and prove $T^{(n)} S^{(n)} = Id$. This implies the map S is well-defined and injective. □

3.2 Analysis of operators on $\mathbb{R}^2$

We shall analyze the operator $\mathcal{L}_k$ (or $L_{k,\min}$, $L_{k,\max}$) defined in the previous subsection. For simplicity of notation, we omit ˆ and ˜ in the definition of $\mathcal{L}_k$ in the sequel. Then our assumptions are the following:

1. $\mathcal{L}_k = d_A^* d_A + V$ on $\mathbb{R}^2 \setminus \{0\}$, $A = A^{(0)} + A^{(1)}$,
2. $L_{k,\min}$ and $L_{k,\max}$ are operators on $L^2(\mathbb{R}^2; d\mu_k)$, $d\mu_k = \sqrt{G} dx^1 dx^2$,
3. $A^{(0)} = \beta_k r^{-2}(-x^2 dx^1 + x^1 dx^2)$, $0 \le \beta_k < 1$,
4. $A^{(1)} \in C_0^1 \Lambda^1(\{r < 2\epsilon_k\})$, real-valued, $A^{(1)}(0) = 0$,
5. V is bounded, real-valued,
6. $g_{mn}(0) = \delta_{mn}$, $\partial_j g_{mn}(0) = 0$, and $g_{mn} = \delta_{mn}$ for $r \ge 2\epsilon_k$.

We shall show that g_{mn}, $A^{(1)}$ and V have nothing to do with the structure of the self-adjoint extensions. To this purpose, define a differential operator $\mathcal{M}_k$ on $\mathbb{R}^2$ by

$$\mathcal{M}_k = -\sum_{n=1,2} \left(\frac{\partial}{\partial x_n} + i A_n \right)^2 .$$

Define a linear operator $M_{k,\min}$ on $L^2(\mathbb{R}^2; dx^1 dx^2)$ by

$$\begin{aligned} D(M_{k,\min}) &= \overline{C_0^\infty(\mathbb{R}^2 \setminus \{0\})}, \\ M_{k,\min} u &= \mathcal{M}_k u \quad \text{for } u \in D(M_{k,\min}). \end{aligned}$$

Put $M_{k,\max} = M_{k,\min}^*$, and $\mathcal{E}_k = D(M_{k,\max})/D(M_{k,\min})$. We also define $\mathcal{M}_k^{(0)}$, $M_{k,\min}^{(0)}$, $M_{k,\max}^{(0)}$, and $\mathcal{E}_k^{(0)}$, by replacing A_n by $A_n^{(0)}$ in the above definition.

The operator $\mathcal{M}_k^{(0)}$ is already studied in [1] and [7]. Here we quote their results and calculate the form $[\cdot,\cdot]_{\mathcal{E}_k^{(0)}}$.

Proposition 3.2 *Let $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi = 1$ in some neighborhood of 0.*

1. Assume $0 < \beta_k < 1$. Put

$$\begin{aligned} f_k^1 &= \chi e^{-i\theta} r^{\beta_k - 1}, \quad & f_k^2 &= \chi r^{-\beta_k}, \\ f_k^4 &= \chi e^{-i\theta} r^{1-\beta_k}, \quad & f_k^5 &= \chi r^{\beta_k}. \end{aligned}$$

Then, the deficiency indices $n_\pm(M_{k,\min}^{(0)}) = 2$, $\dim \mathcal{E}_k^{(0)} = 4$ and the vectors $\{[f_k^n]\}_{n=1,2,4,5}$ form a basis of $\mathcal{E}_k^{(0)}$. Moreover, for $m,n \in \{1,2,4,5\}$ with $m \le n$,[6] we have

$$[f_k^m, f_k^n]_{\mathcal{E}_k^{(0)}} = \begin{cases} 4\pi(\beta_k - 1) & \text{for } (m,n) = (1,4), \\ -4\pi\beta_k & \text{for } (m,n) = (2,5), \\ 0 & \text{otherwise.} \end{cases}$$

2. Assume $\beta_k = 0$. Put

$$f_k^3 = \chi \log r, \quad f_k^6 = \chi.$$

Then, the deficiency indices $n_\pm(M_{k,\min}^{(0)}) = 1$, $\dim \mathcal{E}_k^{(0)} = 2$, $\{[f_k^j]\}_{j=3,6}$ form a basis of $\mathcal{E}_k^{(0)}$, and

$$[f_k^3, f_k^6]_{\mathcal{E}_k^{(0)}} = 2\pi, \;\; [f_k^3, f_k^3]_{\mathcal{E}_k^{(0)}} = [f_k^6, f_k^6]_{\mathcal{E}_k^{(0)}} = 0.$$

Proof. (i) The first statement follows from the result in [7] or [1]. For the calculation of $[u,v]_{\mathcal{E}_k^{(0)}}$, we use some notation in vector analysis. We use the gradient vector $\nabla = {}^t(\partial_1, \partial_2)$, and identify a 1-form A with the component vector ${}^t(A_1, A_2)$. The dot $\cdot$ denotes the Euclidean inner product. Then we have

$$[u,v]_{\mathcal{E}_k^{(0)}} = \lim_{\epsilon \to 0} \int_{r \ge \epsilon} \Big(-\overline{v (\nabla + iA^{(0)}) \cdot (\nabla + iA^{(0)}) u} + \overline{u} (\nabla + iA^{(0)}) \cdot (\nabla + iA^{(0)}) v \Big) dx^1 dx^2 =$$

[6] Notice that $[f_k^n, f_k^m]_{\mathcal{E}_k^{(0)}} = -\overline{[f_k^m, f_k^n]_{\mathcal{E}_k^{(0)}}}$ by definition.

$$\lim_{\epsilon\to 0}\int_{r=\epsilon}\Big(vn\cdot\overline{(\nabla+iA^{(0)})u}-\overline{u}n\cdot(\nabla+iA^{(0)})v\Big)\,r\,\mathrm{d}\theta = \lim_{\epsilon\to 0}\int_{r=\epsilon}\big(v\overline{\partial_r u}-\overline{u}\partial_r v\big)\,r\,\mathrm{d}\theta, \qquad (13)$$

where $n=(\cos\theta,\sin\theta)$, and the line integral is taken counterclockwise. We used the Green formula and the fact $n\cdot A^{(0)}=0$. Then we can easily prove the second statement by using (13).

(ii) The first part of the statement follows from the results in [3]. The second statement can be justified by using (13). □

Next, we prove that the regular part $A^{(1)}$ does not affect the structure of $\mathcal{E}_k$ and the corresponding form.

Proposition 3.3 *All the statements of Proposition 3.2 hold even if we replace $M^{(0)}_{k,\min}$ by $M_{k,\min}$, and $\mathcal{E}^{(0)}_k$ by $\mathcal{E}_k$.*

Before the proof, we prepare a perturbative lemma, which is an immediate corollary of [10, Theorem IV.5.22].

Lemma 3.4 *Let $\mathcal{H}$ be a separable Hilbert space and $\|\cdot\|$ its norm. Let X, Y be densely defined symmetric operators on $\mathcal{H}$. Assume $D(X)\subset D(Y)$ and there exist positive constants C, δ with $0<\delta<1$ and*

$$\|Yu\|\le\delta\|Xu\|+C\|u\|$$

for every $u\in D(X)$. Then, we have $D(\overline{X+Y})=D(\overline{X})$ and $n_\pm(X+Y)=n_\pm(X)$, where the overline denotes the operator closure.

Proof of Proposition 3.3 We prove only statement (i). Statement (ii) can be proved similarly.

By the Leibniz formula (8), we have for $u\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$

$$(\mathcal{M}_k-\mathcal{M}^{(0)}_k)u = i(d^*A^{(1)})u - 2i\langle A^{(1)},d_{A^{(0)}}u\rangle+|A^{(1)}|^2u. \qquad (14)$$

We denote $\|u\|^2=\int_{\mathbb{R}^2}|u|^2dx^1dx^2$ for a function u, and $\|\omega\|^2=\int_{\mathbb{R}^2}|\omega|^2dx^1dx^2$ for a 1-form ω (notice that $|\omega|^2=\langle\overline{\omega},\omega\rangle$). We denote the essential supremum norm of $|u|$ and $|\omega|$ by $\|u\|_\infty$ and $\|\omega\|_\infty$, respectively. Then we have by the Schwarz inequality

$$\begin{aligned}&\|(\mathcal{M}_k-\mathcal{M}^{(0)}_k)u\|\le\\ &\|d^*A^{(1)}\|_\infty\|u\|+2\|A^{(1)}\|_\infty\|d_{A^{(0)}}u\|+\|A^{(1)}\|^2_\infty\|u\|\le\\ &(\|d^*A^{(1)}\|_\infty+\|A^{(1)}\|^2_\infty)\|u\|+\\ &\|A^{(1)}\|_\infty(\epsilon\|\mathcal{M}^{(0)}_ku\|^2+\epsilon^{-1}\|u\|^2)\end{aligned}$$

for any $\epsilon>0$, where we used the inequality

$$\|d_{A^{(0)}}u\|=(\mathcal{M}^{(0)}_ku,u)^{1/2}\le (\epsilon\|\mathcal{M}^{(0)}_ku\|)^{1/2}(\epsilon^{-1}\|u\|)^{1/2}\le \frac{1}{2}(\epsilon\|\mathcal{M}^{(0)}_ku\|+\epsilon^{-1}\|u\|).$$

Take $\epsilon>0$ sufficiently small and apply Lemma 3.4. Then we have $n_\pm(M_{k,\min})=n_\pm(M^{(0)}_{k,\min})=2$, thus $\dim\mathcal{E}_k=4$ by (i) of Proposition 2.3. Moreover we have $D(M_{k,\min})=D(M^{(0)}_{k,\min})$, so the functions $\{f^j_k\}$ ($j=1,2,4,5$) do not belong to $D(M_{k,\min})$. And we can prove $\mathcal{M}_kf^j_k\in L^2(\mathbb{R}^2)$ by using (14) and the fact $|A^{(1)}|=O(r)$ near the origin. Thus $\{[f^j_k]\}$ form a basis of $\mathcal{E}_k$. For the form $[\cdot,\cdot]_{\mathcal{E}_k}$, we can prove the formula

$$[u,v]_{\mathcal{E}_k}=\lim_{\epsilon\to 0}\int_{r=\epsilon}\Big(v\overline{(\partial_ru)}-\overline{u}(\partial_rv)-2i(n\cdot A^{(1)})\overline{u}v\Big)\,rd\theta$$

in a similar way as in (13). Thus the value $[f^m_k,f^n_k]_{\mathcal{E}_k}$ is not affected by $A^{(1)}$, since $|A^{(1)}|=O(r)$ and $|f^m_kf^n_k|$ is at most $O(r^{-\max(2\beta_k,2(1-\beta_k))})$. □

Next we shall consider the non-flat case. We shall show that metric g also does not affect the structure of $\mathcal{D}_k$ and the corresponding form.

Proposition 3.5 *All the statements of Proposition 3.2 hold even if we replace $M^{(0)}_{k,\min}$ by $L_{k,\min}$ and $\mathcal{E}^{(0)}_k$ by $\mathcal{D}_k$.*

Since V is bounded, we can assume $V=0$. In the sequel, we use the following notation:

$$\mathcal{L}=G^{-1/2}(D+A)\cdot G^{1/2}g^{-1}(D+A),$$

where D is the column vector ${}^t(D_1,D_2)$, $D_j=-i\partial_j$, A is identified with the component vector ${}^t(A_1,A_2)$, and g^{-1} is the inverse matrix of $g=(g_{mn})$.

We shall prepare some elliptic a priori estimate.

Lemma 3.6 *Let $m,n\in\{1,2\}$. Then, there exist $C_m>0$ and $C_{mn}>0$ such that*

$$\begin{aligned}&\|(D_m+A_m)u\|\le C_m(\epsilon\|\mathcal{M}_ku\|+\epsilon^{-1}\|u\|),\\ &\|(D_m+A_m)(D_n+A_n)u\|\le C_{mn}(\|\mathcal{M}_ku\|+\|u\|) \quad (15)\end{aligned}$$

for every $u\in C_0^\infty(\mathbb{R}^2\setminus\{0\})$ and every $\epsilon>0$, where $\|\cdot\|=\|\cdot\|_{L^2(\mathbb{R}^2;dx^1dx^2)}$.

The difficulty is the singularity of our vector potential A at the origin. We can overcome this difficulty by using some commutator technique.

Proof of Lemma 3.6 Put $\Pi_j=D_j+A_j$ ($j=1,2$). Then, since

$$\|\Pi_ju\|^2 = (\epsilon^{1/2}\Pi_j^2u,\epsilon^{-1/2}u)\le \frac{1}{2}\left(\epsilon\|\Pi_j^2u\|^2+\epsilon^{-1}\|u\|^2\right)$$

for $u\in C_0^\infty(\mathbb{R}^2)$, it suffices to prove (15).

Define auxiliary operators

$$\mathcal{A} = i\Pi_1 + \Pi_2, \quad \mathcal{A}^\dagger = -i\Pi_1 + \Pi_2.$$

Let $[X, Y] = XY - YX$ be the commutator of operators X and Y. Then we have

$$[\Pi_1, \Pi_2] = [D_1, A_2] - [D_2, A_1] = -i(b + 2\pi\beta_k\delta_0),$$

where $b = \partial_1 A_2^{(1)} - \partial_2 A_1^{(1)}$ is the magnetic field corresponding to $A^{(1)}$. Thus we have

$$[\mathcal{A}, \mathcal{A}^\dagger] = 2i[\Pi_1, \Pi_2] = 2(b + 2\pi\beta_k\delta_0).$$

Particularly for $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, we have

$$(\mathcal{A}\mathcal{A}^\dagger - \mathcal{A}^\dagger\mathcal{A})u = 2bu.$$

Moreover, we have by definition

$$(\mathcal{A}\mathcal{A}^\dagger + \mathcal{A}^\dagger\mathcal{A})u = 2\mathcal{M}_k u.$$

These equalities imply

$$\mathcal{A}\mathcal{A}^\dagger = \mathcal{M}_k + b, \quad \mathcal{A}^\dagger\mathcal{A} = \mathcal{M}_k - b \tag{16}$$

on $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$.

Since $\Pi_m\Pi_n$ can be written as a finite linear combination of the operators of the form XY, where X, Y are $\mathcal{A}$ or $\mathcal{A}^\dagger$, it suffices to show that there exists some constant $C > 0$ such that

$$\|XYu\| \leq C(\|\mathcal{M}_k u\| + \|u\|) \tag{17}$$

for $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$. For $(X, Y) = (\mathcal{A}, \mathcal{A}^\dagger), (\mathcal{A}^\dagger, \mathcal{A})$, (17) follows from (16), since b is bounded. To estimate $\|\mathcal{A}^2 u\|^2$, we assume $A^{(1)} \in C^\infty$ for a while. Then, we have by (16)

$$\begin{aligned}
&\|\mathcal{A}^2 u\|^2 = (\mathcal{A}^2 u, \mathcal{A}^2 u) = ((\mathcal{A}^\dagger)^2\mathcal{A}^2 u, u) = \\
&(\mathcal{A}^\dagger(\mathcal{A}\mathcal{A}^\dagger - 2b)\mathcal{A}u, u) = \\
&\|\mathcal{A}^\dagger\mathcal{A}u\|^2 - 2(b\mathcal{A}u, \mathcal{A}u) \leq \\
&\|\mathcal{A}^\dagger\mathcal{A}u\|^2 + 2\|b\|_\infty\|\mathcal{A}u\|^2 \leq \\
&\|\mathcal{A}^\dagger\mathcal{A}u\|^2 + 2\|b\|_\infty\|\mathcal{A}^\dagger\mathcal{A}u\|\|u\|.
\end{aligned}$$

When $A^{(1)} \in C^1$, we approximate $A^{(1)}$ by C^∞-potentials w.r.t. C^1-norm on some neighborhood of $\operatorname{supp} u$, then we get the above inequality again. Then, we have (17) by using (16). The case $X = Y = \mathcal{A}^\dagger$ can be treated similarly. □

Proof of Proposition 3.5 First, by assumption (vi), we have

$$\begin{aligned}
g^{-1} &= I + \hat{g}, \quad \max|\hat{g}_{mn}| = O(r^2), \\
G &= 1 + O(r^2), \quad |DG| = O(r),
\end{aligned} \tag{18}$$

as $r \to 0$.

Define a unitary operator U from $L^2(\mathbb{R}^2; \sqrt{G}dx^1dx^2)$ to $L^2(\mathbb{R}^2; dx^1dx^2)$ by

$$Uu = G^{1/4}u.$$

Put $\tilde{\mathcal{L}}_k = U\mathcal{L}_kU^{-1}$, $\tilde{L}_{k,\min} = UL_{k,\min}U^{-1}$, etc. Then we have for $v \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$

$$\tilde{L}_{k,\min}v = G^{-1/4}(D + A) \cdot \sqrt{G}g^{-1}(D + A)G^{-1/4}v.$$

Thus we have

$$\begin{aligned}
\tilde{L}_{k,\min} = &G^{-1/4}(D + A) \cdot G^{1/4}g^{-1}(D + A) + \\
&G^{-1/4}(D + A) \cdot \sqrt{G}g^{-1}(DG^{-1/4}).
\end{aligned} \tag{19}$$

The second term of (19) is written as

$$\begin{aligned}
&G^{-1/4}\left(D \cdot (\sqrt{G}g^{-1}(DG^{-1/4}))\right) + \\
&(DG^{-1/4}) \cdot G^{1/4}g^{-1}(D + A).
\end{aligned} \tag{20}$$

The first term of (20) is bounded, and the second is infinitesimally small w.r.t. $M_{k,\min}$, by Lemma 3.6. The first term of (19) is written as

$$\begin{aligned}
&(D + A) \cdot g^{-1}(D + A) + \\
&G^{-1/4}(DG^{1/4}) \cdot g^{-1}(D + A).
\end{aligned} \tag{21}$$

The second term of (21) is also infinitesimally small w.r.t. $M_{k,\min}$, by Lemma 3.6. The first term of (21) is written as

$$M_{k,\min} + (D + A) \cdot \hat{g}(D + A).$$

The second term of this expression is written as

$$\begin{aligned}
&\sum_{m,n=1,2} (D_m\hat{g}_{mn})(D_n + A_n) + \\
&\sum_{m,n=1,2} \hat{g}_{mn}(D_m + A_m)(D_n + A_n).
\end{aligned} \tag{22}$$

The first sum of (22) is infinitesimally small w.r.t. $M_{k,\min}$. If we take ϵ_k sufficiently small, the second sum is $M_{k,\min}$-bounded with relative bound less than 1, by Lemma 3.6. Now we can apply Lemma 3.4, and conclude that $D(\tilde{L}_{k,\min}) = D(M_{k,\min}) = D(M_{k,\min}^{(0)})$, and $n_\pm(L_{k,\min}) = n_\pm(\tilde{L}_{k,\min}) = n_\pm(M_{k,\min}) = n_\pm(M_{k,\min}^{(0)})$. Moreover, one can show that multiplication by $G^{1/4}$ is a bijective continuous map on $D(M_{k,\min}^{(0)})$. Thus we have

$$\begin{aligned}
&D(L_{k,\min}) = U^{-1}D(\tilde{L}_{k,\min}) = \\
&G^{-1/4}D(M_{k,\min}^{(0)}) = D(M_{k,\min}^{(0)}).
\end{aligned}$$

And then we can prove $\mathcal{L}_k f_k^m \in L^2(\mathbb{R}^2; d\mu_k)$ by the Leibniz formula and (18), and thus $\{[f_k^m]\}_m$ form a basis of $\mathcal{D}_k$.

In a similar way as in (13), we have

$$[u,v]_{\mathcal{D}_k} = \lim_{\epsilon\to 0}\int_{r=\epsilon}\Big(vn\cdot\sqrt{G}g^{-1}\overline{(\nabla+iA)u}- -\overline{u}n\cdot\sqrt{G}g^{-1}(\nabla+iA)v\Big)\, r\,\mathrm{d}\theta.$$

Since $\sqrt{G}g^{-1} = I + O(r^2)$, we can replace $\sqrt{G}g^{-1}$ by I in the calculation of $[f_k^m, f_k^n]_{\mathcal{D}_k}$, and we have $[f_k^m, f_k^n]_{\mathcal{D}_k} = [f_k^m, f_k^n]_{\mathcal{E}_k}$. Thus we have the conclusion. □

4 Proof of main theorems

Proof of Theorem 1.1 Since $H_{\min}$ is semi-bounded, we have $n_+(H_{\min}) = n_-(H_{\min}) = \dim\mathcal{D}/2$. By Lemma 3.1 and Proposition 3.5, we have for $K<\infty$

$$\dim\mathcal{D} = \sum_{k=1}^{K}\dim\mathcal{D}_k = 4K_1 + 2K_2,$$

and for $K=\infty$

$$\dim\mathcal{D} \geq \sum_{k=1}^{\infty}\dim\mathcal{D}_k = \infty.$$

Thus we have the conclusion. □

Proof of Theorem 1.2 By Lemma 3.1 and Proposition 3.5, we have for $u, v \in D(H_{\max})$

$$[u,v]_{\mathcal{D}} = 4\pi\Phi(u)^*\begin{pmatrix} O & -D \\ D & O\end{pmatrix}\Phi(v),$$

where $\Phi(u)^*$ is the row-vector ${}^t\overline{\Phi(u)}$ and D is the matrix given by (6). Let $X = {}^t(X_1, X_2)$ be the matrix satisfying (7). Then we have

$$X^*\begin{pmatrix} O & -D \\ D & O\end{pmatrix}X = O,$$

which implies $V\subset V^{[\perp]}$ for $V = \mathrm{Ran}\,X$. Moreover, if $\mathrm{rank}\,X = 2K_1+K_2$, we have

$$\dim V^{[\perp]} = 4K_1 + 2K_2 - \dim V = 2K_1 + K_2 = \dim V.$$

Thus we have (11), and therefore H_X is self-adjoint. Conversely, for a given self-adjoint extension H of $H_{\min}$, we can construct a $(4K_1+2K_2)\times(2K_1+K_2)$-matrix X by arranging the coefficients of an arbitrary basis of $V = PD(H)$ with respect to the basis $\{[\psi_k f_k^j]\}$. □

5 Infinite singularities

Let us consider the case $K=\infty$, and extend Theorem 1.2. Even in this case, for $u\in D(H_{\max})$ and for each k, we can define the asymptotic coefficients c_j^k at γ_k. However, the sequence $\Phi_j(u)$ is an infinite sequence. We shall find appropriate assumptions which make these infinite sequences square summable.

In the sequel, U_k, β_k, g_{mn} are those introduced in section 1. However, we may replace ψ_k defined by (3) more appropriate one satisfying (4), if such one exists. For simplicity, we assume $V=0$.

(U) (i) There exists $\epsilon_0>0$, independent of k, such that $U_k = \{r<\epsilon_0\}$ for every k.

(ii) There exist β_-, β_+ such that $0<\beta_-\leq\beta_k\leq\beta_+<1$ or $\beta_k = 0$, for every k.

(iii) There exists $C_1>0$ independent of k such that g_{mn} satisfies (2) and

$$|\partial_i\partial_j g_{mn}|\leq C_1$$

in U_k, for every $i, j, m, n = 1, 2$.

(iv) There exists $C_2>0$ independent of k, and phase functions $\psi_k\in C^\infty(U_k\setminus\{0\})$ satisfying $|\psi_k| = 1$, (4) and

$$|\partial_j A_m^{(1)}|\leq C_2$$

in U_k, for $j, m = 1, 2$.

Thus we assume some homogeneity for g, $A^{(0)}$, and $A^{(1)}$. Since the open sets $\{U_k\}_{k=1}^\infty$ are required to be disjoint, assumption (i) says the points of Γ are uniformly separated in some sense. Assumption (ii) seems a little strange, but we need this assumption if we want to make the boundary value $\Phi(u)$ square summable.[7] Assumption (iii) binds the curvature of M, and (iv) the intensity of the magnetic field. In [12], the author considers a similar assumption when M is the flat Euclidean plane and $dA^{(1)}$ is a constant magnetic field.

In the sequel, we use the notation

$$\mathbb{C}^\infty = l^2 = \{(c_j)_{j=1}^\infty \mid \sum_{j=1}^\infty |c_j|^2<\infty\},$$

and define its inner product by usual l^2-inner product. Let

$$\mathcal{H} = \mathbb{C}^{K_1}\oplus\mathbb{C}^{K_1}\oplus\mathbb{C}^{K_2}.$$

Proposition 5.1 *Assume (A), (A0), (A1), (SB), (U), $V=0$, and $K=\infty$. Then, the following linear map*

[7] If we consider another type of characterization, assumption (ii) may be dropped.

$$\mathcal{D} \ni [u] \mapsto \Phi(u) \in \mathcal{H} \oplus \mathcal{H}, \tag{23}$$

is a well-defined homeomorphism. Moreover,

$$[u, v]_{\mathcal{D}} = 4\pi(\Phi(u), \tilde{D}\Phi(v)), \tag{24}$$
$$\tilde{D} = \begin{pmatrix} O & -D \\ D & O \end{pmatrix},$$

where $\mathcal{D}$ is a bounded operator on $\mathcal{H}$ defined by (6).

Once this proposition is established, our theorem can be proved similarly as in the proof of Theorem 1.2. So we omit the proof.

Theorem 5.2 *Assume the same conditions as in Proposition 5.1. Then, the statements of Theorem 1.2 hold with the following changes: X_1, X_2 are bounded operators on $\mathcal{H}$, and condition (7) is replaced by the condition*

$$\operatorname{Ran} X = \operatorname{Ker} X^* \tilde{D},$$

where $\tilde{D}$ is the bounded operator on $\mathcal{H} \oplus \mathcal{H}$ defined in Proposition 5.1.

We conclude this paper by proving Proposition 5.1.

Proof of Proposition 5.1. We divide the proof into two steps.

Step 1. The map

$$\mathcal{D} \ni [f] \mapsto \bigoplus_{k=1}^{\infty} T_k[f] \in \bigoplus_{k=1}^{\infty} \mathcal{D}_k$$

is continuous, bijective and its inverse is also continuous.

Proof. By our assumption (U) and the calculation in section 3, we can prove there exists $C > 0$ independent of k such that

$$\|\psi_k^{-1}\chi_k f\|^2_{L_{k,\max}} \leq C \int_{U_k} (|\mathcal{L}f|^2 + |f|^2) d\mu_k.$$

Summing up these equalities with respect to k, we conclude the map

$$D(H_{\max}) \ni f \mapsto \bigoplus_{k=1}^{\infty} \psi_k^{-1}\chi_k f \in \bigoplus_{k=1}^{\infty} D(L_{k,\max})$$

is continuous. Then the well-definedness of the map (23) can be proved similarly as in section 3. Since $\mathcal{D}$ is identified with the closed subspace $D(H_{\min})^{\perp}$ of $D(H_{\max})$ and the projection from $D(L_{k,\max})$ to $\mathcal{D}_k$ is continuous, we conclude the map (23) is continuous. Moreover, we can prove the inverse map is also well-defined and continuous, so we have the conclusion. □

Step 2. There exists $C > 1$ independent of k such that

$$C^{-1}|c^k| \leq \|[u]\|_{\mathcal{D}_k} \leq C|c^k|$$

for every $[u] \in \mathcal{D}_k$, where $c^k = (c_1^k, c_2^k, c_4^k, c_5^k)$ for $0 < \beta_k < 1$, $c^k = (c_3^k, c_6^k)$ for $\beta_k = 0$, and c_j^k are asymptotic coefficients of u defined in section 1.

Proof. We only consider the case $0 < \beta_k < 1$. Consider the following formula for c_1^k

$$c_1^k = \frac{1}{4\pi(1-\beta_k)}[f_k^4, u]_{\mathcal{D}_k},$$

which can be verified by substituting all the basis functions into u. By choosing the representative $u \in D(L_{k,\min})^{\perp}$ (so $\|u\|_{L_{k,\max}} = \|[u]\|_{\mathcal{D}_k}$) and using the Schwarz inequality, we have

$$|c_1^k| \leq \frac{1}{2\pi(1-\beta_k)}\|f_k^4\|_{L_{k,\max}}\|[u]\|_{\mathcal{D}_k}.$$

The fraction is bounded uniformly w.r.t. k, by our assumption (ii) of (U). Moreover, we can prove $\|f_k^j\|_{L_{k,\max}}$ is also uniformly bounded, by (U) and the calculations in section 3 (first decompose $\mathcal{L}_k$ as in section 3, and estimate all terms). Thus we have

$$|c_j^k| \leq C\|[u]\|_{\mathcal{D}_k}.$$

for $j = 1$. The case $j = 2, 4, 5$ can be treated similarly.

Conversely,

$$\sum_{j=1,2,4,5} \|c_j^k[f_k^j]\|_{\mathcal{D}_k} \leq |c^k|\Big(\sum_{j=1,2,4,5} \|f_k^j\|^2_{L_{k,\max}}\Big)^{1/2},$$

and the sum in the right hand side is uniformly bounded. Thus the conclusion holds. □

By Step 1 and 2, we have proved the map (23) is well-defined and homeomorphism. Equation (24) is confirmed by substituting each f_k^j as u or v. □

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