Tilings Generated by Ito-Sadahiro and Balanced $(-\beta)$-numeration Systems

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Abstract

Let $\beta > 1$ be a cubic Pisot unit. We study forms of Thurston tilings arising from the classical β-numeration system and from the $(-\beta)$-numeration system for both the Ito-Sadahiro and balanced definition of the $(-\beta)$-transformation.

Keywords: beta-expansion, negative base, tiling.

1 Introduction

Representations of real numbers in a positional numeration system with an arbitrary base $\beta > 1$, so-called β-expansions, were introduced by Rényi [10]. During the fifty years since the publication of this seminal paper, β-expansions have been extensively studied from various points of view.

This paper considers tilings generated by β-expansions in the case when β is a Pisot unit. A general method for constructing the tiling of a Euclidean space by a Pisot unit was proposed by Thurston [11], although an example of such a tiling had already appeared in the work of Rauzy [9]. Fundamental properties of these tilings were later studied by Praggastis [8] and Akiyama [1, 2].

In 2009, Ito and Sadahiro introduced a new numeration system [6], using a non-integer negative base $-\beta < -1$. Their approach is very similar to the approach by Rényi. Another definition of a system using a non-integer negative base $-\beta < -1$, obtained as a slight modification of the system by Ito and Sadahiro, was considered by Dombek [4].

The main subject of this paper is to transfer the construction by Thurston into the framework of $(-\beta)$-numeration (both cases) and to provide examples of how tilings (for fixed β) in the positive and negative case can resemble and/or differ from each other. The paper is intended as an entry point into a study of the properties of these tilings.

2 Rényi β-expansions

Let $\beta > 1$ be a real number and let the transformation $T_\beta : [0, 1) \to [0, 1)$ be defined by the prescription $T_\beta(x) := \beta x - \lfloor \beta x \rfloor$. The representation of a number $x \in [0, 1)$ of the form

$$x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots ,$$

where $x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$, is called the *$\beta$-expansion of x*. Since $\beta T(x) \in [0, \beta)$ the coefficients x_i (called *digits*) are elements of the set $\{0, 1, \ldots , \lceil \beta \rceil - 1\}$.

The β-expansion of an arbitrary real number $x \geq 1$ can be naturally defined in the following way: Find an exponent $k \in \mathbb{N}$ such that $\frac{x}{\beta^k} \in [0, 1)$. Using the transformation T_β derive the β-expansion of $\frac{x}{\beta^k}$ of the form

$$\frac{x}{\beta^k} = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \ldots ,$$

so that

$$x = x_1\beta^{k-1} + x_2\beta^{k-2} + \ldots + x_{k-1}\beta + x_k + \frac{x_{k+1}}{\beta} + \ldots$$

The β-expansion of $x \in \mathbb{R}_+$ is denoted by $\mathrm{d}_\beta(x)$, and as usual we write

$$\mathrm{d}_\beta(x) = x_1x_2 \ldots x_k \bullet x_{k+1}x_{k+2} \ldots$$

The digit string $x_1x_2x_3 \cdots$ is said to be *β-admissible* if there exists a number $x \in [0, 1)$ so that $\mathrm{d}_\beta(x) = \bullet x_1x_2x_3 \ldots$ is its β-expansion. The set of admissible digit strings can be described using the Rényi expansion of 1, denoted by $\mathrm{d}_\beta(1) = t_1t_2t_3\ldots$, where $t_1 = \lfloor\beta\rfloor$ and $\mathrm{d}_\beta(\beta - \lfloor\beta\rfloor) = t_2t_3t_4\ldots$. The Rényi expansion of 1 may or may not be finite (i.e., ending in infinitely many 0's which are omitted). The infinite Rényi expansion of 1, denoted by $\mathrm{d}^*_\beta(1)$ is defined by

$$\mathrm{d}^*_\beta(1) = \lim_{\varepsilon\to 0+} \mathrm{d}_\beta(1-\varepsilon)\,,$$

where the limit is taken over the usual product topology on $\{0,1,\ldots,\lceil\beta\rceil - 1\}^{\mathbb{N}}$. It can be shown that

$$\mathrm{d}^*_\beta(1) = \begin{cases} \mathrm{d}_\beta(1) & \text{if } \mathrm{d}_\beta(1) \text{ is infinite,} \\ \big(t_1\cdots t_{m-1}(t_m-1)\big)^\omega & \text{if } \mathrm{d}_\beta(1) = t_1\cdots t_m0^\omega \text{ with } t_m \neq 0. \end{cases}$$

The characterization of admissible stings is given by the following theorem due to Parry.
Theorem 1 ([7]) *A string $x_1x_2x_3\ldots$ over the alphabet $\{0,1,\ldots,\lceil\beta\rceil-1\}$ is β-admissible, if and only if for all $i = 1,2,3,\ldots$,*

$$0^\omega \preceq_{lex} x_ix_{i+1}x_{i+2}\ldots \prec_{lex} \mathrm{d}^*_\beta(1)\,,$$

where $\preceq_{lex}$ is the lexicographical order.

Using β-admissible digit strings, one can define the set of non-negative β-*integers*, denoted $\mathbb{Z}_\beta$,

$$\mathbb{Z}_\beta := \{a_k\beta^k + \ldots a_1\beta + a_0 \mid a_k\cdots a_1a_00^\omega \text{ is a } \beta\text{-admissible digit string}\}\,,$$

and the set $\mathrm{Fin}(\beta)$ of those $x \in \mathbb{R}_+$ whose β-expansions have only finitely many non-zero coefficients to the right from the fractional point

$$\mathrm{Fin}(\beta) := \bigcup_{n\in\mathbb{N}} \frac{1}{\beta^n}\mathbb{Z}_\beta\,.$$

The distances between consecutive β-integers are described in [11]. It is shown that they take values in the set $\{\Delta_i \mid i = 0,1,\ldots\}$, where $\Delta_i = \sum_{j=1}^{\infty}\frac{t_{i+j}}{\beta^j}$ and $\mathrm{d}_\beta(1) = t_1t_2\ldots$. Moreover, the sequence coding the distances in $\mathbb{Z}_\beta$ is known to be invariant under a substitution provided $\mathrm{d}_\beta(1)$ is eventually periodic [5]. The form of this substitution also depends on $\mathrm{d}_\beta(1)$.

If we consider β an algebraic integer, then obviously $\mathrm{Fin}(\beta) \subset \mathbb{Z}[\beta^{-1}]_+$. The converse inclusion, which is very important for the construction of the tiling and also for the arithmetical properties of the system, does not hold in general. An algebraic integer β for which

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_+$$

holds, is said to have Property (F).

3 Ito-Sadahiro $(-\beta)$-expansions

Now consider the real base $-\beta < -1$ and the transformation $T_{-\beta} : \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right) \to \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ defined by the prescription

$$T_{-\beta}(x) = -\beta x - \Big\lfloor -\beta x + \frac{\beta}{\beta+1}\Big\rfloor\,.$$

Every number $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ can be represented in the form

$$x = \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \frac{x_3}{(-\beta)^3} + \cdots\,, \qquad \text{where} \quad x_i = \Big\lfloor -\beta T^{i-1}_{-\beta}(x) + \frac{\beta}{\beta+1}\Big\rfloor\,.$$

The representation of x in such a form is called the $(-\beta)$-*expansion of* x and is denoted

$$\mathrm{d}_{-\beta}(x) = \bullet x_1x_2x_3\ldots$$

By analogy to the case of Rényi β-expansions, we use for the $(-\beta)$-expansion of $x \in \mathbb{R}$ a suitable exponent $l \in \mathbb{N}$ such that $\frac{x}{(-\beta)^l} \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. It is shown easily that the digits x_i of a $(-\beta)$-expansion belong to the set $\{0, 1, \ldots, \lfloor\beta\rfloor\}$.

In order to describe strings that arise as $(-\beta)$-expansions of some $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, so-called $(-\beta)$-*admissible* digit strings, we will use the notation introduced in [6]. We denote $l_\beta = \frac{-\beta}{\beta+1}$ and $r_\beta = \frac{1}{\beta+1}$ the left and right end-point of the definition interval I_β of the transformation $T_{-\beta}$, respectively. That is $I_\beta = [l_\beta, r_\beta)$. We also denote

$$\mathrm{d}_{-\beta}(l_\beta) = d_1 d_2 d_3 \ldots$$

Theorem 2 ([6]) *A string $x_1x_2x_3\cdots$ over the alphabet $\{0, 1, \ldots, \lfloor\beta\rfloor\}$ is $(-\beta)$-admissible, if and only if for all $i = 1, 2, 3, \ldots$,*

$$\mathrm{d}_{-\beta}(l_\beta) \preceq_{alt} x_i x_{i+1} x_{i+2} \prec_{alt} \mathrm{d}^*_{-\beta}(r_\beta)\,,$$

where $\mathrm{d}^*_{-\beta}(r_\beta) = \lim_{\varepsilon\to 0+} \mathrm{d}_{-\beta}(r_\beta - \varepsilon)$ and $\preceq_{alt}$ is the alternate order.

Recall that the alternate order is defined as follows: We say that $x_1x_2x_3\ldots \prec_{\mathrm{alt}} y_1y_2y_3\ldots$, if $(-1)^i(x_i - y_i) > 0$ for the smallest index i satisfying $x_i \neq y_i$. The relation between $\mathrm{d}^*_{-\beta}(r_\beta)$ and $\mathrm{d}_{-\beta}(l_\beta)$ is described in the same paper.

Theorem 3 ([6]) *Let $\mathrm{d}_{-\beta}(l_\beta) = d_1d_2d_3\ldots$ If $\mathrm{d}_{-\beta}(l_\beta)$ is purely periodic with odd period-length, i.e., $\mathrm{d}_{-\beta}(l_\beta) = (d_1d_2\cdots d_{2l+1})^\omega$, then $\mathrm{d}^*_{-\beta}(r_\beta) = (0d_1d_2\cdots d_{2l}(d_{2l+1}-1))^\omega$. Otherwise, $\mathrm{d}^*_{-\beta}(r_\beta) = 0\mathrm{d}_{-\beta}(l_\beta)$.*

Similarly to the Rényi case, one can define the set of $(-\beta)$-*integers*, denoted $\mathbb{Z}_{-\beta}$, using the admissible digit strings.

$$\mathbb{Z}_{-\beta} := \{a_k(-\beta)^k + \cdots a_1(-\beta) + a_0 \mid a_k\cdots a_1a_0 0^\omega \text{ is a } (-\beta)\text{-admissible digit string}\}\,.$$

The set of distances between consecutive $(-\beta)$-integers has been described only for a particular class of β, cf. [3].

4 Balanced $(-\beta)$-numeration system

The last numeration system used in this paper is a slight modification of $(-\beta)$-numeration defined by Ito and Sadahiro. Let $-\beta < -1$ be the base and consider the transformation $S_{-\beta} : \left[-\frac{1}{2}, \frac{1}{2}\right) \to \left[-\frac{1}{2}, \frac{1}{2}\right)$ given by

$$S_{-\beta} = -\beta x - \left\lfloor -\beta x + \frac{1}{2}\right\rfloor\,.$$

The *balanced $(-\beta)$-expansion of a number* $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, denoted $\mathrm{d}_{B,-\beta}(x) = \bullet x_1x_2x_3\ldots$, is

$$x = \frac{x_1}{-\beta} + \frac{x_2}{(-\beta)^2} + \frac{x_3}{(-\beta)^3} + \ldots\,, \qquad \text{where} \quad x_i = \left\lfloor -\beta S^{i-1}_{-\beta}(x) + \frac{1}{2}\right\rfloor\,.$$

Also in this case we use for the $(-\beta)$-expansion of $x \in \mathbb{R}$ a suitable exponent $l \in \mathbb{N}$ such that $\frac{x}{(-\beta)^l} \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. It is shown easily that the digits x_i of a balanced $(-\beta)$-expansion belong to the set $\left\{-\left\lfloor\frac{\beta+1}{2}\right\rfloor, \ldots, \left\lfloor\frac{\beta+1}{2}\right\rfloor\right\}$. Note that sometimes $\overline{d}$ is used instead of $-d$.

A digit string $x_1x_2x_3\ldots$ is called *balanced $(-\beta)$-admissible* if it arises as the balanced $(-\beta)$-expansion of some $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. The two following theorems by Dombek [4] prove that also in this case the admissible strings are characterized by the balanced $(-\beta)$-expansions of the endpoints of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

Theorem 4 **([4])** *A string $x_1x_2x_3\ldots$ over the alphabet $\left\{-\left\lfloor\frac{\beta+1}{2}\right\rfloor,\ldots,\left\lfloor\frac{\beta+1}{2}\right\rfloor\right\}$ is balanced $(-\beta)$-admissible if and only if for all $i=1,2,3,\ldots$*

$$\mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right)\preceq_{alt} x_ix_{i+1}x_{i+2}\ldots\prec_{alt}\mathrm{d}^*_{B,-\beta}\left(\frac{1}{2}\right),$$

where $\mathrm{d}^*_{B,-\beta}\left(\frac{1}{2}\right)=\lim_{\varepsilon\to 0+}\mathrm{d}_{B,-\beta}\left(\frac{1}{2}-\varepsilon\right)$.

Theorem 5 **([4])** *Let* $\mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right)=d_1d_2d_3\ldots$ *Then*

$$\mathrm{d}^*_{B,-\beta}(\frac{1}{2})=\begin{cases}\left(\overline{d_1}\ldots\overline{d_{2l}}\,\overline{(d_{2l+1}-1)}\,d_1\ldots d_{2l}(d_{2l+1}-1)\right)^\omega & \text{if } \mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right)=(d_1\ldots d_{2l+1})^\omega,\\ \overline{d_1}\,\overline{d_2}\,\overline{d_3}\ldots & \text{otherwise.}\end{cases}$$

The set of *balanced $(-\beta)$-integers*, denoted $\mathbb{Z}_{B,-\beta}$, is defined by analogy to the two previous cases.

$$\mathbb{Z}_{B,-\beta}:=\{a_k(-\beta)^k+\ldots a_1(-\beta)+a_0 \mid a_k\ldots a_1a_00^\omega \text{ is a balanced } (-\beta)\text{-admissible string}\}.$$

5 Constructing of the tiling

Recall that a Pisot number is an algebraic integer such that all its algebraic conjugates are in modulus strictly smaller than one.

Let $\beta>1$ be a Pisot number of degree $d=r+2s$. We denote $\beta=\beta^{(1)}$ and we assume that $\beta^{(2)},\ldots,\beta^{(r)}$ are real conjugates of β and $\beta^{(r+1)},\ldots,\beta^{(r+2s)}$ are complex conjugates of β such that $\beta^{(r+j)}=\overline{\beta^{(r+s+j)}}$ for $j=1,\ldots,s$. Denote by $x^{(j)}$, $j=1,\ldots,n$ the corresponding conjugate of $x\in\mathbb{Q}(\beta)$, i.e.,

$$x=q_0+q_1\beta+\ldots+q_{d-1}\beta^{q-1}\quad\mapsto\quad x^{(j)}=q_0+q_1\beta^{(j)}+\ldots+q_{d-1}(\beta^{(j)})^{q-1}.$$

Consider the map $\Phi:\mathbb{Q}(\beta)\to\mathbb{R}^{d-1}$ defined by

$$\Phi(x):=\left(x^{(2)},\ldots,x^{(r)},\Re(x^{(r+1)}),\Im(x^{(r+1)}),\ldots,\Re(x^{(r+s)}),\Im(x^{(r+s)}\right).$$

Proposition 6 **([1])** *Let $\beta>1$ be a Pisot number of degree d. Then $\Phi(\mathbb{Z}[\beta])$ is dense in $\mathbb{R}^{d-1}$.*

The map Φ is used to construct the tiling in the following way. Let $w=w_1\ldots w_l\in\{0,1,\ldots,\lceil\beta\rceil-1\}^*$ be a finite word such that $w0^\omega$ is an admissible digit string. We define the tile T_w as

$$T_w:=\overline{\{\Phi(x)\mid x\in\mathrm{Fin}(\beta)\text{ and }(x)_\beta=a_k\ldots x_1x_0\bullet w_1\cdots w_l\}}.$$

The properties of the tiling of the Euclidean space using tiles T_w were described by Akiyama; the results are summarized in the following theorems.

Theorem 7 **([1])** *Let β be a Pisot unit of degree d with Property (F). Then*

- $\mathbb{R}^{d-1}=\bigcup_{w0^\omega \text{ admissible}} T_w$,
- *for each $x\in\mathbb{Z}_\beta$ we have $\Phi(x)\in\mathrm{Inn}(T_\epsilon)$, where ϵ is the empty word and $\mathrm{Inn}(X)$ denotes the set of inner points of X; especially, the origin 0 is an inner point of the so-called central tile T_ϵ,*
- *for each tile T_w we have $\overline{\mathrm{Inn}(T_w)}=T_w$,*
- *$\partial(T_w)$ is closed and nowhere dense in $\mathbb{R}^{d-1}$, where $\partial(T_w)$ is the set of boundary elements of T_w,*
- *if $\mathrm{d}_\beta(1)=t_1\cdots t_{m-1}1$ then each tile T_w is arc-wise connected.*

Theorem 8 ([2]) *Let β be a Pisot unit of degree d such that $\mathrm{d}_\beta(1) = t_1 \dots t_m(t_{m+1} \cdots t_{m+p})^\omega$ with m, p the smallest possible. Then there are exactly $m + p$ different tiles up to translation.*

Note that $\mathbb{Q}(\beta) = \mathbb{Q}(-\beta)$ and $\mathbb{Z}[\beta] = \mathbb{Z}[-\beta]$. Thus the construction of the tiling associated to $(-\beta)$-numeration follows the same lines, the corresponding mapping Φ_- being defined using isomorphisms of the extension fields $\mathbb{Q}(-\beta)$ and $\mathbb{Q}(-\beta^{(j)})$, and the following variant of Proposition 6 holds; its proof follows the same lines as in the proof of the original proposition.

Proposition 9 *Let $\beta > 1$ be a Pisot number of degree d. Then $\Phi_-(\mathbb{Z}[-\beta])$ is dense in the space $\mathbb{R}^{d-1}$.*

6 Examples of tilings

In the rest of the paper we provide several examples of tilings associated with β cubic Pisot units, i.e., the minimal polynomial of β is of the form $x^3 - ax^2 - bx \pm 1$. Every time all the tiles T_w with w of length $0, 1, 2$ are plotted.

So far no properties of tilings in the negative case similar to those in Theorem 7 and Theorem 8 have been proved. However, the following examples demonstrate that it is reasonable to anticipate that most of the properties remain valid. On the other hand, one can also observe that for a fixed β when we change the β-numeration into the $(-\beta)$-numeration (either Ito-Sadahiro or balanced) the shape and form of the tiles can be either preserved or changed slightly or completely.

6.1 Minimal polynomial $x^3 - x^2 - 1$

The tilings associated to $-\beta$ are trivial in this case. Indeed, $\mathrm{d}_{-\beta}(l_\beta) = 1001^\omega$ and $\mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right) = \big(10(-1)(-1)(-1)(-1)(-1)010(-1)011\big)^\omega$, hence $\mathbb{Z}_{-\beta} = \mathbb{Z}_{B,-\beta} = \{0\}$ (cf. [3, 4]).

6.2 Minimal polynomial $x^3 - 2x^2 - 2x - 1$

This β is an example of a base for which the three considered tilings almost do not change. We have

$$\mathrm{d}_\beta(1) = 211\,, \qquad \mathrm{d}_{-\beta}(l_\beta) = 201^\omega\,, \qquad \mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right) = (1(-1)1)^\omega\,.$$

All three sets $\mathbb{Z}_\beta$, $\mathbb{Z}_{-\beta}$ and $\mathbb{Z}_{B,-\beta}$ have the same set of three possible distances between consecutive elements, namely $\{1, \beta - 2, \beta^2 - 2\beta - 2\}$. The codings of the distances in these sets are generated by substitutions which are pairwise conjugated. Recall that substitutions φ and ψ over an alphabet $\mathcal{A}$ are said to be conjugated if there exists a word $w \in \mathcal{A}^*$ such that $\varphi(a) = w\psi(a)w^{-1}$ for all $a \in \mathcal{A}$. The tilings are composed of the same tiles (up to rotation). See Figure 1.

6.3 Minimal polynomial $x^3 - 3x^2 + x - 1$

In this case

$$\mathrm{d}_\beta(1) = 2201\,, \qquad \mathrm{d}_{-\beta}(l_\beta) = (201)^\omega\,, \qquad \mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right) = (1(-1)00)^\omega\,,$$

and again all three sets of integers have the same possible distances between consecutive elements, $\Delta_i \in \{1, \beta - 2, \beta^2 - 2\beta - 2, \beta^2 - 3\beta + 1\}$. However, in this case the associated substitutions are not conjugated (the condition is not fulfilled on exactly one of four letters) and even though the tilings do look similar, they are composed of different tiles. See Figure 2.

6.4 Minimal polynomial $x^3 - 2x^2 - 1$

This β is an example of a base for which two tilings (and the corresponding properties of the sets of integers) are very similar, but the third tiling differs substantially. We have

$$\mathrm{d}_\beta(1) = 201\,, \qquad \mathrm{d}_{-\beta}(l_\beta) = (2101)^\omega\,, \qquad \mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right) = (101)^\omega\,.$$

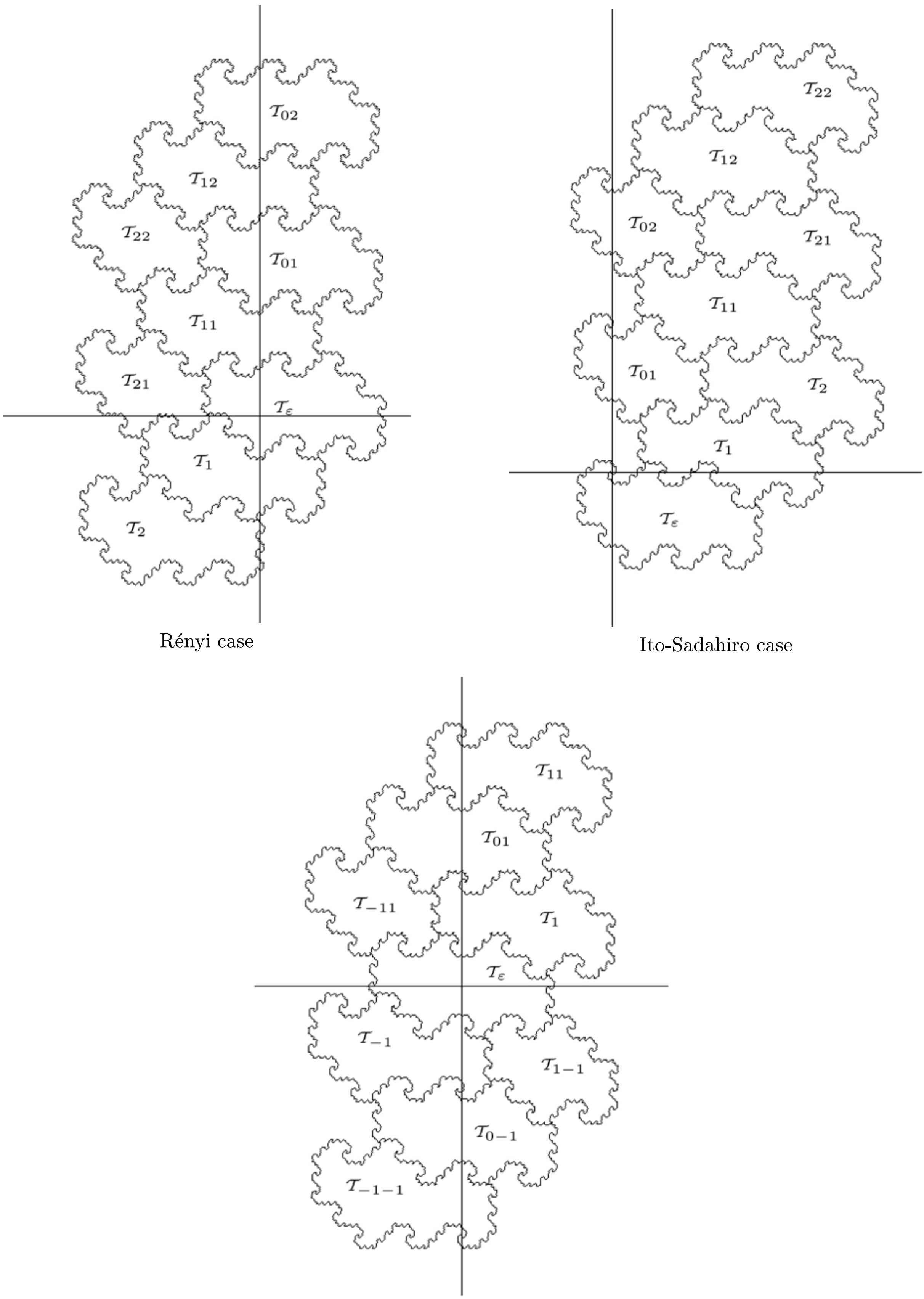
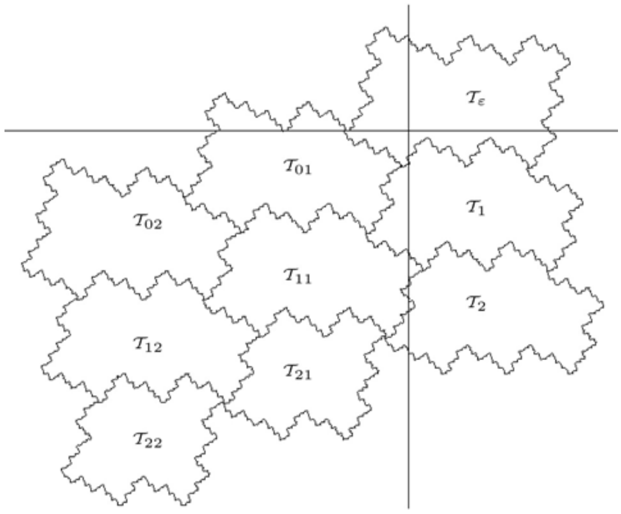
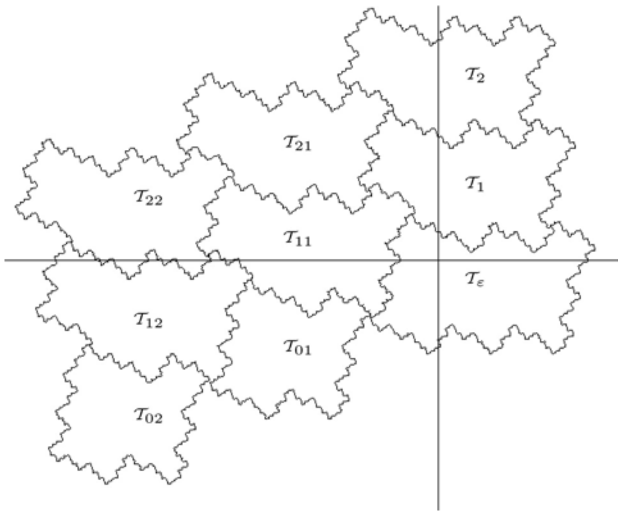


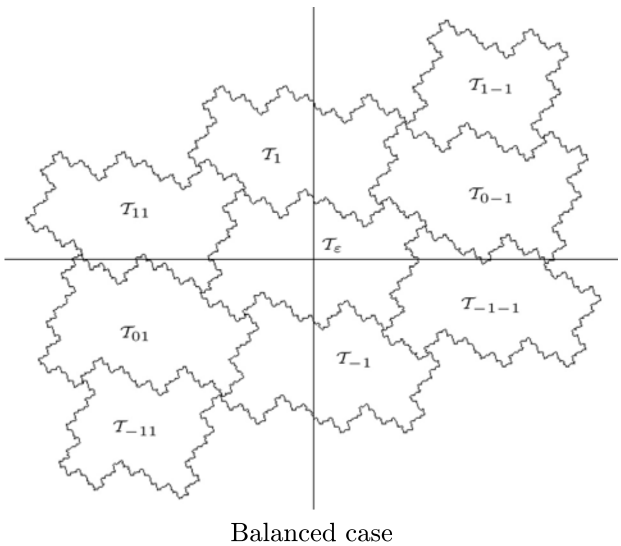
Fig. 1: Minimal polynomial $x^3 - 2x^2 - 2x - 1$



Rényi case

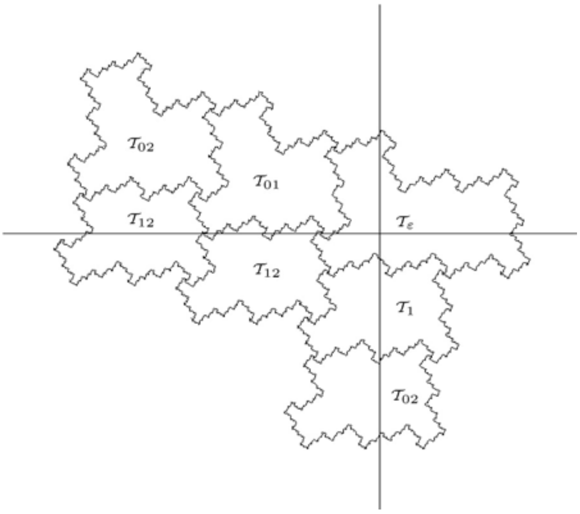


Ito-Sadahiro case

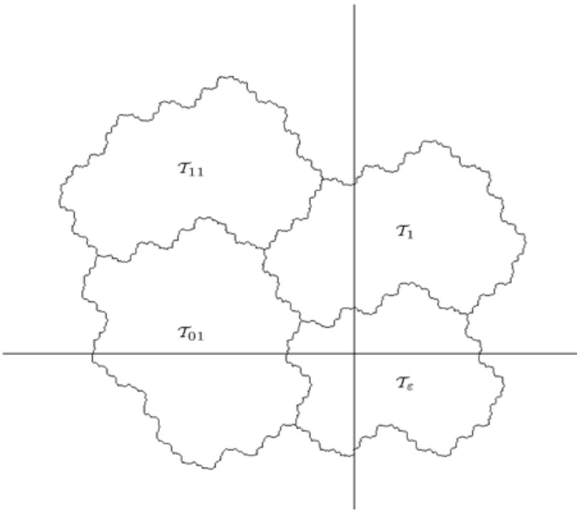


Balanced case

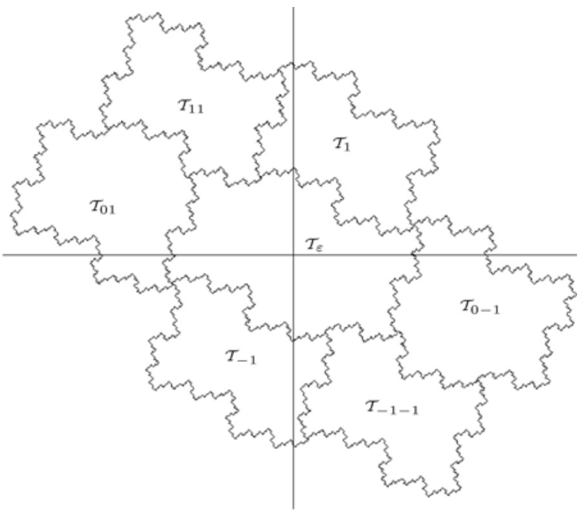
Fig. 2: Minimal polynomial $x^3 - 3x^2 + x - 1$



Rényi case



Ito-Sadahiro case



Balanced case

Fig. 3: Minimal polynomial $x^3 - 2x^2 - 1$

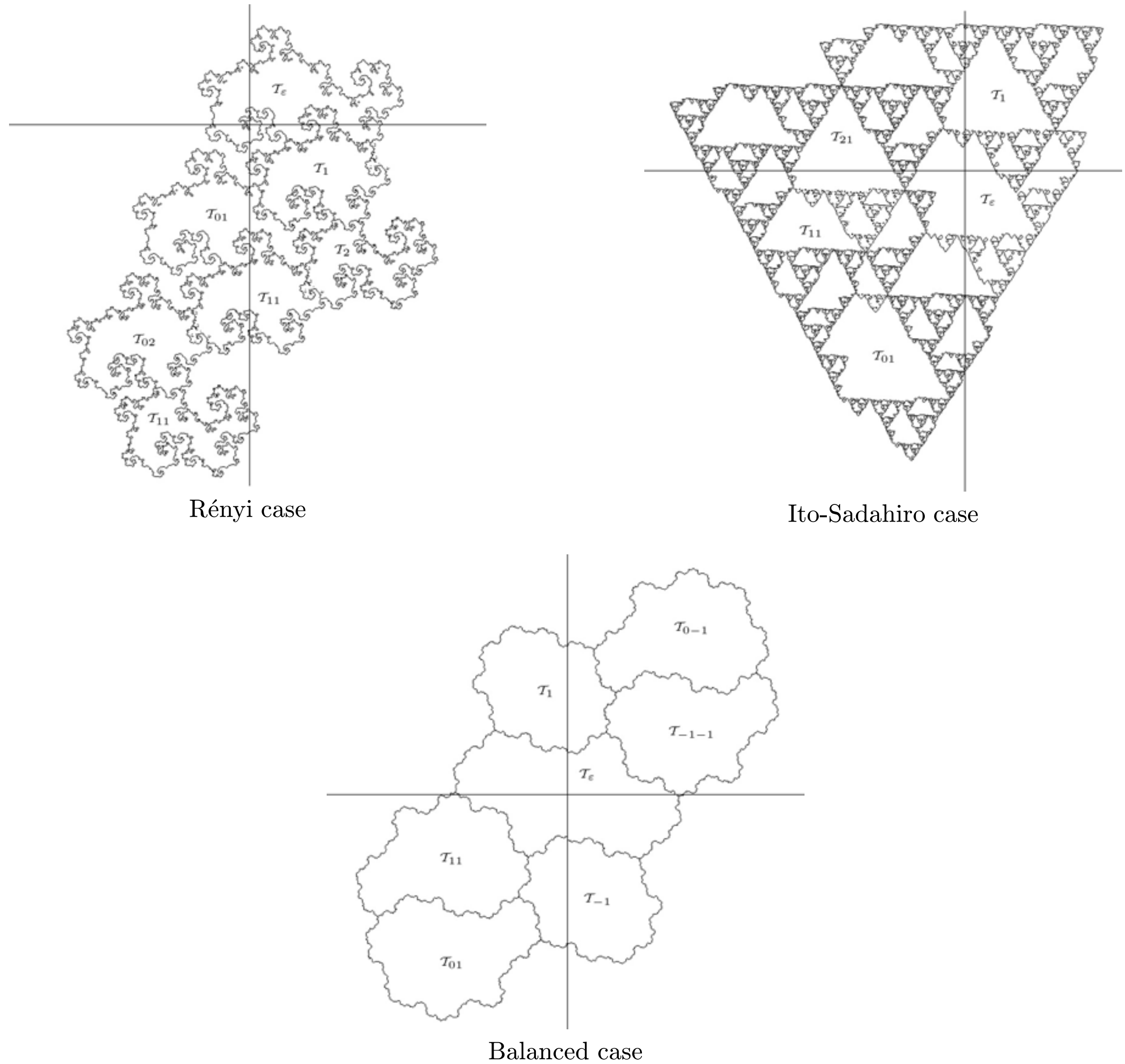


Fig. 4: Minimal polynomial $x^3 - 3x^2 + 2x - 1$

The sets $\mathbb{Z}_\beta$ and $\mathbb{Z}_{B,-\beta}$ have the same set of distances $\{1, \beta-2, \beta^2-2\beta\}$, however the associated substitutions are not conjugated. On the other hand there are five distances between consecutive elements in the set $\mathbb{Z}_{-\beta}$, namely $\{1, \beta^2-\beta-1, \beta-1, \beta, \beta^2-\beta\}$. The forms of the tilings comply: the tiling in the Rényi case and the tiling in the balanced case are somewhat similar, but the tiling in the Ito-Sadahiro case is completely different. See Figure 3.

6.5 Minimal polynomial $x^3 - 3x^2 + 2x - 1$

The last example demonstrates that the tiling can change fundamentally when considering different numeration systems with fixed β. In this case

$$\mathrm{d}_\beta(1) = 201^\omega\,, \quad \mathrm{d}_{-\beta}(l_\beta) = (211)^\omega\,, \quad \mathrm{d}_{B,-\beta}\left(-\frac{1}{2}\right) = (1010(-1)(-1)(-1)(-1)0(-1)0111)^\omega\,,$$

there are three distances between consecutive elements in the set $\mathbb{Z}_\beta$, four in the set $\mathbb{Z}_{-\beta}$ and seven in the set $\mathbb{Z}_{B,-\beta}$. The tilings are completely different. See Figure 4.

7 Conclusion

Due to the similar nature of β-numeration and $(-\beta)$-numeration, the transfer of the construction of the tiling of a space due to Thurston into the framework of $(-\beta)$-numeration is quite straightforward.

In this paper we have provided several examples of these tilings (for both the Ito-Sadahiro definition and the balanced definition of the $-(\beta)$-transformation). Although the shape and form of tiling can change dramatically when one changes (for a fixed β) the β-numeration into the $-(\beta)$-numeration, in general the examples demonstrate that the validity of most of the properties derived by Akiyama and Praggastis in the positive case should be preserved. It remains an open question to provide proofs of such properties.

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