Asymptotic Power Series of Field Correlators

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Abstract

We address the problem of ambiguity of a function determined by an asymptotic perturbation expansion. Using a modified form of the Watson lemma recently proved elsewhere, we discuss a large class of functions determined by the same asymptotic power expansion and represented by various forms of integrals of the Laplace-Borel type along a general contour in the Borel complex plane. Some remarks on possible applications in QCD are made.

1 Asymptotic perturbation expansions

Perturbation expansions are known to be divergent both in quantum electrodynamics and in quantum chromodynamics, as well as in many other physically interesting theories and models. In QED, divergence was proved by F. J. Dyson in 1952 (see [1]). His result has been revisited and reformulated by many authors ([2, 3], see also a review in [4]). Dyson proposed to give the divergent series mathematical meaning by interpreting it as an asymptotic series to $F(z)$, the sought function:

$$F(z) \sim \sum_{n=0}^{\infty} F_n z^n, \qquad z \in \mathcal{S}, \quad z \to 0, \tag{1}$$

where $\mathcal{S}$ is a point set having the origin as an accumulation point, z being the perturbation parameter.

To see how dramatically the philosophy of perturbation theory was changed by this step, let us first recall the definition of an asymptotic series:

Definition: Let $\mathcal{S}$ be a region or point set having the origin as an accumulation point. The power series $\sum\limits_{n=0}^{\infty} F_n z^n$ is said to be asymptotic to the function $F(z)$ as $z \to 0$ on $\mathcal{S}$, and we write Eq. (1), if the set of functions $R_N(z)$,

$$R_N(z) = F(z) - \sum_{n=0}^{N} F_n z^n, \tag{2}$$

satisfies the condition

$$R_N(z) = o(z^N) \tag{3}$$

for all $N = 0, 1, 2, \ldots$, $z \to 0$ and $z \in \mathcal{S}$.

Note that the asymptotic series is defined by a *different limiting procedure* than the Taylor one: *taking N fixed*, one observes how $R_N(z)$ behaves for $z \to 0$, $z \in \mathcal{S}$, the procedure being repeated for all $N \geq 0$ integers. Convergence may be provable without knowing $F(z)$, but asymptoticity can be tested only if one knows both the F_n and $F(z)$.

By (1), $F(z)$ is not uniquely determined; there are many different functions having the same asymptotic series, (1) say. The ambiguity of a function given by an asymptotic series is illustrated by the lemma of Watson.

2 Watson lemma

Consider the following integral

$$\Phi_{0,c}(\lambda) = \int_0^c e^{-\lambda x^\alpha}\, x^{\beta-1} f(x)\mathrm{d}x, \tag{4}$$

where $0 < c < \infty$ and $\alpha > 0, \beta > 0$. Let $f(x) \in C^\infty[0,c]$ and $f^{(k)}(0)$ defined as $\lim\limits_{x\to 0+} f^{(k)}(x)$. Let ε be any number from the interval $0 < \varepsilon < \pi/2$.

Lemma 1 (G. N. Watson): *If the above conditions are fulfilled, the asymptotic expansion*

$$\Phi_{0,c}(\lambda) \sim \frac{1}{\alpha}\sum_{k=0}^{\infty} \lambda^{-\frac{k+\beta}{\alpha}}\, \Gamma\left(\frac{k+\beta}{\alpha}\right)\frac{f^{(k)}(0)}{k!} \tag{5}$$

holds for $\lambda \to \infty, \lambda \in S_\varepsilon$, where S_ε is the angle

$$|\arg\lambda| \leq \frac{\pi}{2} - \varepsilon. \tag{6}$$

The expansion (5) can be differentiated with respect to λ any number of times.

For the proof, see for instance [5]. Let us add several remarks:

1) The angle S_ε of validity of (5), (6), is independent of α, β and c.

2) Thanks to the factor $\Gamma\left(\dfrac{k+\beta}{\alpha}\right)$, the expansion coefficients in (5) grow faster with k than those of the Taylor series for $f(x)$.

3) The expansion coefficients in (5) are independent of c. This illustrates the impossibility of a unique determination of a function from its asymptotic expansion.

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In the next section we shall give a modification to the Watson lemma, which shows that under plausible assumptions the straight integration contour can be bent.

3 Modified Watson lemma

The modified Watson lemma we present below (and call Lemma 2') is a special case of Lemma 2, which we publish and prove in Ref. [6]. The special form given here is obtained from that given in [6] by setting $\alpha = \beta = 1$.

Let $G(r)$ be a continuous complex function of the form $G(r) = r\exp(ig(r))$, where $g(r)$ is a real-valued function given on $0 \leq r < c$, with $0 < c \leq \infty$. Assume that the derivative $G'(r)$ is continuous on the interval $0 \leq r < c$ and a constant $r_0 > 0$ exists such that

$$|G'(r)| \leq K_1 r^{\gamma_1}, \qquad r_0 \leq r < c, \tag{7}$$

for a nonnegative K_1 and a real γ_1.

Assume that the parameter $\varepsilon > 0$ exists such that the quantities

$$A = \inf_{r_0 \leq r < c} g(r), \qquad B = \sup_{r_0 \leq r < c} g(r) \tag{8}$$

satisfy the inequality

$$B - A < \pi - 2\varepsilon. \tag{9}$$

Let the function $f(u)$ be defined along the curve $u = G(r)$ and on the disc $|u| < \rho$, where $\rho > r_0$. Let $f(u)$ be holomorphic on the disc and measurable on the curve. Assume that

$$|f(G(r))| \leq K_2 r^{\gamma_2}, \qquad r_0 \leq r < c, \tag{10}$$

hold for a nonnegative K_2 and a real γ_2.

Define the function $\Phi^{(G)}_{b,c}(\lambda)$ for $0 \leq b < c$ by[1]

$$\Phi^{(G)}_{b,c}(\lambda) = \int_{r=b}^{c} e^{-\lambda G(r)} G(r) f(G(r)) dG(r). \tag{11}$$

Lemma 2': *If the above assumptions are fulfilled, then the asymptotic expansion*

$$\Phi^{(G)}_{0,c}(\lambda) \sim \sum_{k=0}^{\infty} \lambda^{-(k+1)}\,\Gamma(k+1)\frac{f^{(k)}(0)}{k!} \tag{12}$$

holds for $\lambda \to \infty, \lambda \in \mathcal{T}_\varepsilon$, *where*

$$\begin{aligned} \mathcal{T}_\varepsilon = \{\lambda : \lambda = |\lambda|\exp(\mathrm{i}\varphi),\\ -\frac{\pi}{2} - A + \varepsilon < \varphi < \frac{\pi}{2} - B - \varepsilon\}. \end{aligned} \tag{13}$$

We refer the reader to Ref. [6] for the proof of Lemma 2 and its discussion. The above simplified version, Lemma 2', is given here to illustrate some special features of the general Lemma 2 and its possible applications.

Let us add several remarks to Lemma 2':

1/ Lemma 2' implies Watson's lemma when the integration contour is chosen to have the special form of a segment of the real positive semiaxis, *i.e.* $g(r) \equiv 0$, and $f(r) \in C^\infty[0, c]$.

2/ Perturbation theory is obtained by setting $\lambda = 1/z$ in (10), (11). Then, the function

$$F^{(G)}_{0,c}(z) = \int_{r=0}^{c} e^{-G(r)/z} f(G(r))\,\mathrm{d}G(r) \tag{14}$$

has the asymptotic expansion

$$F^{(G)}_{0,c}(z) \sim \sum_{k=0}^{\infty} z^{k+1} f^{(k)}(0) \tag{15}$$

for $z \to 0$ and $z \in \mathcal{Z}_\varepsilon$, where

$$\begin{aligned} \mathcal{Z}_\varepsilon = \{z : z = |z|\exp(i\chi),\\ -\frac{\pi}{2} + B + \varepsilon < \chi < \frac{\pi}{2} + A - \varepsilon\}. \end{aligned} \tag{16}$$

3/ The parameter ε in (9) is limited by $0 < \varepsilon < \pi/2 - (B-A)/2$, but is otherwise arbitrary. Note however that the upper limit of ε depends on $B - A$ and may be considerably less than $\pi/2$. This happens, for instance, if the integration contour is bent or meandering.

4/ The parametrization $G(r) = r\exp(ig(r))$ does not include contours that cross a circle centred at $r = 0$, either touching or doubly intersecting it, so that the derivative $G'(r)$ either does not exist or is not bounded. In such cases, the parametrization has to be modified.

5/ Let us remark that the proof of Lemma 2 in Ref. [6] allows us to obtain remarkable correlations between the strength of the bounds on the remainder and the size of the angles within which the asymptotic expansion is valid. It follows from [6] that the bounds are proportional to

$$\frac{1}{(|\lambda| - 1)\sin\varepsilon}\mathrm{e}^{-(|\lambda|-1)r_0\sin\varepsilon} \tag{17}$$

or to

$$C_N(|\lambda|\sin\varepsilon)^{-(N+2)}, \tag{18}$$

where N is the truncation order and the C_N, $N = 0, 1, 2, \ldots$ are λ-independent positive numbers. The bounds decrease with increasing ε, the parameter, which determines the angles $\mathcal{T}_\varepsilon$ and $\mathcal{Z}_\varepsilon$, see (13) and (16) respectively. As a consequence, the larger the angle of validity, the looser the bound, and vice versa.

[1] This integral exists since we assume that $f(u)$ is measurable along the curve $u = G(r)$ and bounded by (10).

4 Some applications to perturbative QCD

To discuss some applications of Lemma 2', we take the Adler function [7],

$$\mathcal{D}(s) = -s\frac{\mathrm{d}\Pi(s)}{\mathrm{d}s} - 1\,. \tag{19}$$

where $\Pi(s)$ is the polarization amplitude defined in terms of the vector current products for light quarks. The Adler function $\mathcal{D}(s)$ is real analytic in the s-plane, except for a cut along the timelike axis produced by unitarity [7, 8]. In perturbative QCD, any finite-order approximant has cuts along the timelike axis, while the renormalization-group improved expansion,

$$\begin{aligned}\mathcal{D}(s) = D_1\,\alpha_s(s)/\pi + D_2\,(\alpha_s(s)/\pi)^2 + \\ D_3\,(\alpha_s(s)/\pi)^3 + \ldots\,,\end{aligned} \tag{20}$$

has, in addition, an unphysical singularity due to the Landau pole in the running coupling $\alpha_s(s)$. (20) is known to be divergent, the D_n growing as $n!$ at large orders [9]–[12].

4.1 On the high ambiguity of perturbative QCD

To discuss the implications of Lemma 2', we first define the Borel transform $B(u)$ by [11],

$$B(u) = \sum_{n\geq 0} b_n\,u^n, \qquad b_n = \frac{D_{n+1}}{\beta_0^n\,n!}\,. \tag{21}$$

It is usually assumed that the series (21) is convergent on a disc of nonvanishing radius (this result was rigorously proved by David et al. [13] for the scalar φ^4 theory in four dimensions). This is what is required in Lemma 2' for the generalized Borel transform $f(G(r))$.

If we assume that the series (20) is asymptotic, Lemma 2' implies a large freedom in recovering the true function from its coefficients. All the functions $\mathcal{D}^G_{0,c}(s)$ of the form

$$\mathcal{D}^G_{0,c}(s) = \frac{1}{\beta_0}\int_{r=0}^{c} e^{-\frac{G(r)}{\beta_0\,a(s)}}\,B(G(r))\,\mathrm{d}G(r)\,, \tag{22}$$

where $a(s) = \alpha_s(s)/\pi$, admit the asymptotic expansion

$$\mathcal{D}^G_{0,c}(s) \sim \sum_{n=1}^{\infty} D_n\,(a(s))^n, \qquad a_s(s)\to 0, \tag{23}$$

in a certain domain of the s-plane, which follows from (13) and the expression of the running coupling $a(s)$ given by the renormalization group. No function of the form $\mathcal{D}^G_{0,c}(s)$, (22), can be a priori preferred when looking for the true Adler function.

Contributing only to the exponentially suppressed remainder, neither the form or length of the contour, nor the values of $B(u)$ outside the convergence disc can affect (23). The remainder to (23) is of the form $h\exp(-d/\beta_0 a(s)) \sim h\left(-\Lambda^2/s\right)^d$. The quantities h and $d>0$ depend on the contour and on $B(u)$ outside the disc, which can be chosen rather freely. As a consequence, (22) contains arbitrary power terms, to be added to (23).

4.2 Analyticity and optimal conformal mapping

In discussing the divergence of (20) and (21), the singularities of $\mathcal{D}(s)$ in the $\alpha_s(s)$ plane and, respectively, those of $B(u)$ in the Borel plane are of importance. As for $B(u)$, some information about the location and nature of the singularities can be obtained from certain classes of Feynman diagrams (which can be summed, see [10]–[12]), and from general arguments based on renormalization theory, [9, 14]. It follows that $B(u)$ has branch points along the rays $u\geq 2$ and $u\leq -1$ (IR and UV renormalons respectively). Other (though nonperturbative) singularities, for $u\geq 4$, are produced by instanton-antiinstanton pairs. (Due to the singularities at $u>0$, the series (20) is not Borel summable.) No other singularities of $B(u)$ in the Borel plane are known, however. It is usually assumed that $B(u)$ is holomorphic elsewhere.

To make full use of the analyticity of $B(u)$ in the whole $\mathcal{B}$, we shall use the method of optimal conformal mapping [15]. Let $\mathcal{K}$ be the disc of convergence of the series (21); clearly, $\mathcal{K}\subset\mathcal{B}$. Then, evidently, the expansion (21) in powers of u can be replaced by that in powers of $w(u)$,

$$B(u) = \sum_{n\geq 0} c_n\,w^n, \tag{24}$$

where the function $w = w(u)$ with the property $w(0) = 0$ represents the conformal mapping of the region of $\mathcal{B}$ onto the disc $|w|<1$, on which (24) converges. It can easily be seen that (24) has better convergence properties than (21) in this case: indeed, as was proved in [15] by using the Schwarz lemma, the larger the region mapped by $w(u)$ onto $|w|<1$, the faster the large-order convergence rate of (24).

If $w(u)$ maps the whole $\mathcal{B}$ onto the unit disc $|w|<1$ in the w plane, the mapping is called optimal. In this case, (24) converges everywhere on $\mathcal{B}$ and the convergence rate is the fastest [15]. The region of convergence of (24) coincides with $\mathcal{B}$, the region of analyticity. In this way, the optimal conformal mapping can express analyticity in terms of convergence.

Inserting (24) into (22) we obtain an alternative asymptotic expansion:

$$\begin{aligned}\mathcal{D}^G_{0,c}(s) = \frac{1}{\beta_0}\int_{r=0}^{c} e^{-\frac{G(r)}{\beta_0\,a(s)}}\,\cdot \\ \sum_{n\geq 0} c_n\,[w(G(r))]^n\,\mathrm{d}G(r)\,.\end{aligned} \tag{25}$$

Containing powers of the optimal conformal mapping $w(u)$ (which has the same location of singularities as the expanded function $B(u)$), this representation implements more information about the singularities of $B(u)$ than the series (21) in powers of u, even at finite orders. Thus, it is to be expected that even the finite-order approximants of (25) will provide a more precise description of the function searched for [16, 17].

4.3 Analyticity may easily get lost

We shall briefly mention an intriguing situation showing that careless manipulation with the integration contour may have a fateful impact on analyticity. In [18], two different integration contours in the u-plane were chosen for the summation of the so-called renormalon chains [10]: for $a(s) > 0$ and $a(s) < 0$, a ray parallel and close to the positive and, respectively, negative semiaxis is chosen. As was expected and later proved [19], analyticity is lost with this choice, the summation being only piecewise analytic in s.

On the other hand, as shown in [20, 21], the Borel summation with the Principal Value (PV) prescription of the same class of diagrams admits an analytic continuation in the s-plane, in agreement with analyticity except for a cut along a segment of the spacelike axis, related to the Landau pole.

5 Conclusion

In this paper we have discussed some special consequences of our general result published in [6], which is based on a modification of the Watson lemma. It follows that a perturbation series, if regarded as asymptotic, implies a huge ambiguity of possible expanded functions having the same asymptotic expansion of the type (1). This mathematical fact is often ignored or overlooked in physical applications. Our contribution consists in the fact that we have specified its special subclass by Lemma 2 of Ref. [6]. Moreover, in the present paper, we have considered a special subclass of Lemma 2 (as defined by Lemma 2' in section 3 of this paper), which we discuss here in more detail due to its direct applicability to perturbative QCD. To find the true solution, additional information inputs are unavoidable.

Applying the result to QCD, we conclude that the contour of the integral representing the QCD correlator can be chosen very freely. The same holds for the Borel transform $B(u)$ outside the convergence circle.

We have kept our discussion on a general level, bearing in mind that little is known, in a rigorous framework, about the analytic properties of the QCD correlators in the Borel plane. If some specific properties are known or assumed, the integral representations will have additional analytic properties. Naturally, the results obtained in [6] may also be useful in other branches of physics where perturbation series are divergent.

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