Topics on n-ary Algebraic Structures

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Abstract

We review the basic definitions and properties of two types of n-ary structures, the Generalized Lie Algebras (GLA) and the Filippov ($\equiv$ n-Lie) algebras (FA), as well as those of their Poisson counterparts, the Generalized Poisson (GPS) and Nambu-Poisson (N-P) structures. We describe the Filippov algebra cohomology complexes relevant for the central extensions and infinitesimal deformations of FAs. It is seen that semisimple FAs do not admit central extensions and, moreover, that they are rigid. This extends Whitehead's lemma to all $n \geq 2$, $n = 2$ being the original Lie algebra case. Some comments on n-Leibniz algebras are also made.

1 Introduction

The Jacobi identity (JI) for Lie algebras $\mathfrak{g}$, $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$, may be looked at in two ways. First, one may see it as a consequence of the associativity of the composition of generators in the Lie bracket. Secondly, it may be viewed as the statement that the adjoint map is a derivation of the Lie algebra, $ad_X\,[Y,Z] = [ad_X\,Y, Z] + [Y, ad_X\,Z]$.

A natural problem is to consider n-ary generalizations, *i.e.* to look for the possible characteristic identities that a n-ary bracket,

$$(X_1,\ldots,X_n) \in \mathcal{G}\times\ldots\times\mathcal{G} \mapsto [X_1,\ldots,X_n] \in \mathcal{G}, \quad (1.1)$$

antisymmetric in its arguments (this may be relaxed; see last section), may satisfy. When $n > 2$ two generalizations of the JI suggest themselves. These are:

(a) Higher order Lie algebras or generalized Lie algebras (GLA) $\mathcal{G}$, proposed independently in [1, 2, 3] and [4, 5, 6, 7]. Their bracket is defined by the full antisymmetrization

$$[X_{i_1},\ldots,X_{i_n}] := \sum_{\sigma\in S_n} (-1)^{\pi(\sigma)} X_{i_{\sigma(1)}} \ldots X_{i_{\sigma(n)}}\,. \quad (1.2)$$

For n even, this definition implies the generalized Jacobi identity (GJI)

$$\sum_{\sigma\in S_{2n-1}} (-1)^{\pi(\sigma)} \left[[X_{i_{\sigma(1)}},\ldots,X_{i_{\sigma(n)}}], X_{i_{\sigma(n+1)}}\ldots,X_{i_{\sigma(2n-1)}}\right] = 0 \quad (1.3)$$

which follows from the associcitivity of the products in (1.2) (for n odd, the $r \cdot h \cdot s$ is $n!(n-1)![X_{i_1},\ldots,X_{i_{2n-1}}]$ rather than zero). Chosen a basis of $\mathcal{G}$, the bracket may be written as $[X_{i_1},\ldots,X_{i_{2p}}] = \Omega_{i_1\ldots i_{2p}}{}^j X_j$, where the $\Omega_{i_1\ldots i_{2p}}{}^j$ are the structure constants of the GLA.

(b) n-Lie or Filippov algebras (FA) $\mathfrak{G}$. The characteristic identity that generalizes the $n = 2$ JI is the Filippov identity (FI) [8]

$$[X_1,\ldots,X_{n-1},[Y_1,\ldots Y_n]] = \sum_{a=1}^{n}[Y_1,\ldots Y_{a-1},[X_1,\ldots,X_{n-1},Y_a],Y_{a+1},\ldots Y_n]\,. \quad (1.4)$$

If we introduce *fundamental objects* $\mathscr{X} = (X_1,\ldots,X_{n-1})$ antisymmetric in their $(n-1)$ entries and acting on $\mathfrak{G}$ as

$$\mathscr{X}\cdot Z \equiv ad_{\mathscr{X}} Z := [X_1,\ldots,X_{n-1},Z] \quad \forall Z \in \mathfrak{G}\,, \quad (1.5)$$

then the FI just expresses that $ad_{\mathscr{X}}$ is a derivation of the bracket,

$$ad_{\mathscr{X}}[Y_1,\ldots,Y_n] = \sum_{a=1}^{n}[Y_1,\ldots,ad_{\mathscr{X}}Y_a,\ldots,Y_n]\,. \quad (1.6)$$

Chosen a basis, a FA may be defined through its structure constants,

$$[X_{a_1}\ldots X_{a_n}] = f_{a_1\ldots a_n}{}^d X_d\,, \quad (1.7)$$

and the FI is written as

$$f_{b_1\ldots b_n}{}^l\, f_{a_1\ldots a_{n-1}l}{}^s = \sum_{k=1}^{n} f_{a_1\ldots a_{n-1}b_k}{}^l\, f_{b_1\ldots b_{k-1}lb_{k+1}\ldots b_n}{}^s\,. \quad (1.8)$$

2 Some definitions and properties of FA

The definitions of ideals, solvable ideals and semisimple algebras can be extended to the $n > 2$ case as follows [9]. A subalgebra I of $\mathfrak{G}$ is an ideal of $\mathfrak{G}$ if

$$[X_1,\ldots,X_{n-1},Z] \subset I \quad \forall X_1,\ldots,X_{n-1}\in\mathfrak{G}\,,\ \forall Z\in I\,. \quad (2.9)$$

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An ideal I is (n-)solvable if the series

$$I^{(0)} := I,\ I^{(1)} := [I^{(0)}, \ldots, I^{(0)}], \ldots, \\ I^{(s)} := [I^{(s-1)}, \ldots, I^{(s-1)}], \ldots \qquad (2.10)$$

ends. A FA is then semisimple if it does not have solvable ideals, and simple if $[\mathfrak{G}, \ldots, \mathfrak{G}] \neq \{0\}$ and does not contain non-trivial ideals. There is also a Cartan-like criterion for semisimplicity [10]. Namely, A FA is semisimple if

$$k(\mathscr{X}, \mathscr{Y}) = k(X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1}) := \\ Tr(ad_{\mathscr{X}} ad_{\mathscr{Y}}) \qquad (2.11)$$

is non-degenerate in the sense that

$$k(Z, \mathfrak{G}, \overset{n-2}{\ldots}, \mathfrak{G}, \mathfrak{G}, \overset{n-1}{\ldots}, \mathfrak{G}) = 0 \Rightarrow Z = 0\,. \qquad (2.12)$$

It can also be shown [11] that a semisimple FA is the sum of simple ideals,

$$\mathfrak{G} = \bigoplus_{s=1}^{k} \mathfrak{G}_s = \mathfrak{G}_{(1)} \oplus \ldots \oplus \mathfrak{G}_{(k)} \qquad (2.13)$$

The derivations of a FA $\mathfrak{G}$ generate a Lie algebra. To see it, introduce first the composition of fundamental objects,

$$\mathscr{X} \cdot \mathscr{Y} := \qquad (2.14) \\ \sum_{a=1}^{n-1} (Y_1, \ldots, Y_{a-1}, [X_1, \ldots, X_{n-1}, Y_a], Y_{a+1}, \ldots, Y_{n-1})$$

which reflects that $\mathscr{X}$ acts as a derivation. It is then seen that FI implies that

$$\mathscr{X} \cdot (\mathscr{Y} \cdot \mathscr{Z}) - \mathscr{Y} \cdot (\mathscr{X} \cdot \mathscr{Z}) = (\mathscr{X} \cdot \mathscr{Y}) \cdot \mathscr{Z}, \qquad (2.15) \\ \forall \mathscr{X}, \mathscr{Y}, \mathscr{Z} \in \wedge^{n-1}\mathfrak{G}$$

$$ad_{\mathscr{X}} ad_{\mathscr{Y}} Z - ad_{\mathscr{Y}} ad_{\mathscr{X}} Z = ad_{\mathscr{X} \cdot \mathscr{Y}} Z, \qquad (2.16) \\ \forall \mathscr{X}, \mathscr{Y} \in \wedge^{n-1}\mathfrak{G},\ \forall Z \in \mathfrak{G}\,,$$

which means that $ad_{\mathscr{X}} \in \mathrm{End}\,\mathfrak{G}$ satisfies $ad_{\mathscr{X} \cdot \mathscr{Y}} = -ad_{\mathscr{Y} \cdot \mathscr{X}}$. These two identities show that the inner derivations $ad_{\mathscr{X}}$ associated with the fundamental objects $\mathscr{X}$ generate (the ad map is not necessarily injective) an ordinary Lie algebra, the Lie algebra associated with the FA $\mathfrak{G}$.

An important type of FAs, because of its relevance in physical applications where a scalar product is usually needed (as in the Bagger-Lambert-Gustavsson model in M-theory), is the class of metric Filippov algebras. These are endowed with a metric $\langle , \rangle$ on $\mathfrak{G}$, $\langle Y, Z \rangle = g_{ab} Y^a Z^b$, $\forall\, Y, Z \in \mathfrak{G}$ that is invariant *i.e.*,

$$\begin{aligned} \mathscr{X} \cdot \langle Y, Z \rangle &= \langle \mathscr{X} \cdot Y, Z \rangle + \langle Y, \mathscr{X} \cdot Z \rangle \\ &= \langle [X_1, \ldots, X_{n-1}, Y], Z \rangle + \\ &\quad \langle Y, [X_1, \ldots, X_{n-1}, Z] \rangle = 0\,. \end{aligned} \qquad (2.17)$$

This means that the structure constants with all indices down $f_{a_1 \ldots a_{n-1} bc}$ are completely antisymmetric since the invariance of g above implies $f_{a_1 \ldots a_{n-1} b}{}^{l}\, g_{lc} + f_{a_1 \ldots a_{n-1} c}{}^{l}\, g_{bl} = 0$. The $f_{a_1 \ldots a_{n+1}}$ define a skewsymmetric invariant tensor under the action of $\mathscr{X}$, since the FI implies

$$\sum_{i=1}^{n+1} f_{a_1 \ldots a_{n-1} b_i}{}^{l}\, f_{b_1 \ldots b_{i-1} l b_{i+1} \ldots b_{n+1}} = 0 \\ \text{or} \qquad L_{\mathcal{X}} . f = 0\,. \qquad (2.18)$$

3 Examples of n-ary structures

3.1 Examples of GLAs

Let $n = 2p$. We look for structure constants $\Omega_{i_1 \ldots i_{2p}}{}^{j}$ that satisfy the GJI (1.3) *i.e.*, such that

$$\Omega_{[j_1 \ldots j_{2p}}{}^{l}\, \Omega_{j_{2p+1} \ldots j_{4p-1}]l}{}^{s} = 0\,. \qquad (3.19)$$

It turns out [3, 2] that given a simple compact Lie algebra, the coordinates of the (odd) cocyles for the Lie algebra cohomology satisfy the GJI identity (1.2). These provide the structure constants of an infinity of GLAs, with brackets with $n = 2(m_i - 1)$ entries (where $i = 1, \ldots, l$ and l is the rank of the algebra), according to the table below:

$\mathfrak{g}$	$\dim \mathfrak{g}$	Orders m_i of invariants (and Casimirs)	Orders $2m_i - 1$ of $\mathfrak{g}$-cocycles
A_l	$(l+1)^2 - 1$ $[l \geq 1]$	$2, 3, \ldots, l+1$	$3, 5, \ldots, 2l+1$
B_l	$l(2l+1)$ $[l \geq 2]$	$2, 4, \ldots, 2l$	$3, 7, \ldots, 4l-1$
C_l	$l(2l+1)$ $[l \geq 3]$	$2, 4, \ldots, 2l$	$3, 7, \ldots, 4l-1$
D_l	$l(2l-1)$ $[l \geq 4]$	$2, 4, \ldots, 2l-2, l$	$3, 7, \ldots, 4l-5, 2l-1$
G_2	14	$2, 6$	$3, 11$
F_4	52	$2, 6, 8, 12$	$3, 11, 15, 23$
E_6	78	$2, 5, 6, 8, 9, 12$	$3, 9, 11, 15, 17, 23$
E_7	133	$2, 6, 8, 10, 12, 14, 18$	$3, 11, 15, 19, 23, 27, 35$
E_8	248	$2, 8, 12, 14, 18, 20, 24, 30$	$3, 15, 23, 27, 35, 39, 47, 59$

3.2 Examples of FAs

An important example of finite Filippov algebras is provided by the real euclidean simple n-Lie algebras A_{n+1} defined on an euclidean $(n+1)$-dimensional vector space. Let us fix a basis $\{e_i\}$ $(i = 1, \ldots, n+1)$. The basic commutators are given by

$$[e_1 \ldots \hat{e}_i \ldots e_{n+1}] = (-1)^{n+1} e_i \qquad \text{or} \quad [e_{i_1} \ldots e_{i_n}] = (-1)^n \sum_{i=1}^{n+1} \epsilon_{i_1 \ldots i_n}{}^i e_i \,. \tag{3.20}$$

There are also infinite-dimensional FAs that generalize the ordinary Poisson algebra by means of the bracket of n functions $f_i = f_i\ (x_1, x_2, \ldots, x_n)$ defined by

$$[f_1, f_2, \ldots, f_n] := \epsilon^{i_1 \ldots i_n}_{1 \ldots n}\ \partial_{i_1} f^1 \ldots \partial_{i_n} f^n = \left| \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x^1, x^2, \ldots, x^n)} \right| , \tag{3.21}$$

considered by Nambu [12] specially for $n = 3$. The conmmutators in (3.20) and the above Jacobian n-bracket satisfy the FI, which can be checked by using the Schouten identities technique. All these examples are also metric FAs.

4 n-ary Poisson generalizations

Both GLAs and FAs have n-ary Poisson structure counterparts. These satisfy the associated GJI and FI characteristic identities, to which Leibniz's rule is added.

4.1 Generalized Poisson structures (GPS)

The generalized Poisson structures [2] (GPS, n even) are defined by brackets $\{f_1, \ldots, f_n\}$ where the f_i, $i = 1, \ldots, n$, are functions on a manifold. They are skewsymmtric

$$\{f_1, \ldots, f_i, \ldots, f_j, \ldots, f_n\} = -\{f_1, \ldots, f_j, \ldots, f_i, \ldots, f_n\} , \tag{4.22}$$

satisfy the Leibniz identity,

$$\{f_1, \ldots, f_{n-1}, gh\} = g\{f_1, \ldots, f_{n-1}, h\} + \{f_1, \ldots, f_{n-1}, g\}h , \tag{4.23}$$

and the characteristic identity of the GLAs, the GJI (1.3),

$$\sum_{\sigma \in S_{4s-1}} (-1)^{\pi(\sigma)} \{f_{\sigma(1)}, \ldots, f_{\sigma(2s-1)}, \{f_{\sigma(2s)}, \ldots, f_{\sigma(4s-1)}\}\} = 0 \,. \tag{4.24}$$

As with ordinary Poisson structures, there are *linear* GPS given in terms of coordinates of the odd cocyles of the $\mathfrak{g}$ in the table of Sec. 3.1. They are given by the multivector

$$\Lambda = \frac{1}{(2m-2)!} \Omega_{i_1 \ldots i_{2m-2}.}{}^{\sigma} x_\sigma \partial^{i_1} \wedge \ldots \wedge \partial^{i_{2m-2}} \tag{4.25}$$

since, as it may be checked [2], Λ has zero Schouten-Nijenhuis bracket with itself, $[\Lambda, \Lambda]_{SN} = 0$, which corresponds to the GJI. All GLAs associated with a simple algebra define linear GPS.

4.2 Nambu-Poisson structures (N-P)

These are defined by relations (4.22) and (4.23), but now the characteristic identity is the FI,

$$\begin{aligned} &\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \\ &\{\{f_1, \ldots, f_{n-1}, g_1\}, g_2, \ldots, g_n\} + \\ &\{g_1, \{f_1, \ldots, f_{n-1}, g_2\}, g_3, \ldots, g_n\} + \ldots + \\ &\{g_1 \ldots, g_{n-1}, \{f_1, \ldots, f_{n-1}, g_n\}\} \,. \end{aligned} \tag{4.26}$$

The Filippov identity for the (Nambu) jacobians of n functions was first written by Filippov [8], and by Sahoo and Valsakumar [13] and Takhtajan [14] (who called it fundamental identity) in the context of Nambu mechanics [12]. Physically, the FI is a consistency condition for the time evolution [13, 14], given in terms of $(n-1)$ 'hamiltonian' functions that correspond to the $ad_{\mathscr{X}}$ derivations of a FA. Every even N-P structure is also a GPS, but the converse does not hold.

The question of the quantization of Nambu-Poisson mechanics has been the subject of a vast literature; it is probably fair to say that it remains a problem (for $n > 2$!) aggravated by the fact that there are not so many physical examples of N-P mechanical systems to be quantized. We shall just refer here to [15, 16, 17], from which the earlier literature can be traced.

5 Lie algebra cohomology, extensions and deformations

Given a Lie algebra $\mathfrak{g}$, the p-cochains of the Lie algebra cohomology are p-antisymmetric, V-valued maps (where V is a $\mathfrak{g}$-module),

$$\Omega^p : \mathfrak{g} \times \overset{p}{\cdots} \times \mathfrak{g} \to V \,, \ \Omega^A = \frac{1}{p!} \Omega^A_{i_1 \ldots i_p} \omega^{i_1} \wedge \ldots \wedge \omega^{i_p} , \tag{5.27}$$

where $\{\omega^i\}$ is a basis of the coalgebra $\mathfrak{g}^*$. The coboundary operator (for the left action) $s : \Omega^p \in C^p(\mathfrak{g}, V) \mapsto (s\Omega^p) \in C^{p+1}(\mathfrak{g}, V)$, $s^2 = 0$, is given by

$$(s\Omega^p)^A(X_1,\ldots,X_{p+1}) := \sum_{i=1}^{p+1}(-)^{i+1}\rho(X_i)^A_{.B}\,\Omega^{pB}(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1}) + \sum_{\substack{j,k=1\\ j<k}}^{p+1}(-)^{j+k}\Omega^{pA}([X_j,X_k],X_1,\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_{p+1})\,. \tag{5.28}$$

For the trivial action ($\rho = 0$), this simplifies to

$$(s\Omega^p)(X_1,\ldots,X_{p+1}) = \sum_{1\le j<k}^{p+1}(-1)^j\Omega^p(X_1,\ldots,\hat{X}_j,\ldots,ad_{X_j}X_k,\ldots,X_{p+1})\,. \tag{5.29}$$

The p-cocycles $\Omega^p \in Z^p_\rho(\mathfrak{g},V)$ are p-cochains such that $s\,\Omega^p = 0$; the p-coboundaries $\Omega^p \in B^p_\rho(\mathfrak{g},V)$ are such that $\Omega^p = s\Omega^{p-1}$ for some $(p-1)$-cochain. The p-th cohomology groups are then $H^p_\rho(\mathfrak{g},V) := Z^p_\rho(\mathfrak{g},V)/B^p_\rho(\mathfrak{g},V)$.

For semisimple Lie algebras, Whitehead's lemma states that $H^2_0(\mathfrak{g}) = 0$, $H^2_{ad}(\mathfrak{g},\mathfrak{g}) = 0$. Hence, semisimple Lie algebras do not admit non-trivial central extensions and are moreover rigid (non-deformable) since central extensions and infinitesimal deformations are governed, respectively, by $H^2_0(\mathfrak{g})$ and $H^2_{ad}(\mathfrak{g},\mathfrak{g})$. Let us now turn to the FA case.

6 Central extensions and deformations of FAs

6.1 Central extensions of a FA

Given a Filippov algebra $\mathfrak{G}$ with n-bracket $[\ldots]$, a central extension $\widetilde{\mathfrak{G}}$ has the form

$$\begin{aligned}&[\tilde{X}_{a_1},\ldots,\tilde{X}_{a_n}] := f^b_{a_1\ldots a_n}\,\tilde{X}_b + \alpha^1(X_1,\ldots,X_n)\Xi\,,\\ &[\tilde{X}_1,\ldots,\tilde{X}_{n-1},\Xi] = 0\,,\\ &\tilde{X}\in\widetilde{\mathfrak{G}}\,,\ \alpha^1 \in \wedge^{n-1}\mathfrak{G}^*\wedge\mathfrak{G}^*\,.\end{aligned} \tag{6.30}$$

If α^1 defines a centrally extended FA, it must satisfy the condition that follows from the FI for the above bracket. If we now introduce p-cochains as maps

$$\begin{aligned}&\alpha^p \in \wedge^{n-1}\mathfrak{G}^*\otimes\ldots\otimes\wedge^{n-1}\mathfrak{G}^*\wedge\mathfrak{G}^*\,,\\ &\alpha^p : (\mathscr{X}_1,\ldots,\mathscr{X}_p,Z)\mapsto\alpha^p(\mathscr{X}_1,\ldots,\mathscr{X}_p,Z)\,,\end{aligned} \tag{6.31}$$

the condition imposed by the FI on the one-cochain in (6.30), written in terms of the fundamental objects with $Y_n = Z$, reads

$$\begin{aligned}&\alpha^1(\mathscr{X},\mathscr{Y}\cdot Z) - \alpha^1(\mathscr{X}\cdot\mathscr{Y},Z) - \alpha^1(\mathscr{Y},\mathscr{X}\cdot Z) \equiv\\ &(\delta\alpha^1)(\mathscr{X},\mathscr{Y},Z) = 0\,.\end{aligned} \tag{6.32}$$

Note that α^1 above would become a two-cochain for $n = 2$; we define the order of the p-cochains for FAs ($n\ge 3$) as the number p of fundamental objects among the arguments of the cochain (for a Lie algebra $\mathscr{X} = X$ and p counts the number of algebra elements).

A central extension is actually trivial if it is possible to find new generators $\widetilde{X}' = \tilde{X} - \beta(X)\Xi$ such that

$$[\widetilde{X}'_{a_1},\ldots,\widetilde{X}'_{a_n}] = f^b_{a_1\ldots a_n}\widetilde{X}'_b = f^b_{a_1\ldots a_n}\tilde{X}_b - \beta([X_{a_1},\ldots,X_{a_n}])\Xi\,.$$

But this is equivalent to saying (with $X_{a_n} = Z$) that

$$\alpha^1(X_1,\ldots,X_{n-1},Z) = -\beta([X_1,\ldots,X_{n-1},Z]), \tag{6.33}$$

which may be rewritten in the form

$$\begin{aligned}&\alpha^1(\mathscr{X},Z) = -\beta([X_1,\ldots,X_{n-1},Z]) \equiv\\ &(\delta\beta)(X_1,\ldots,X_{n-1},Z) \equiv (\delta\beta)(\mathscr{X},Z)\,,\end{aligned} \tag{6.34}$$

where β is the zero-cochain $\beta\in\mathfrak{G}^*$ generating the one-cocycle. Therefore, central extensions of FAs are characterized by one-cocycles modulo one-coboundaries.

The above suffices to define the full FA cohomology complex suitable for central extensions. Let $\alpha^p \in \wedge^{n-1}\mathfrak{G}^*\otimes\ldots\otimes\wedge^{n-1}\mathfrak{G}^*\wedge\mathfrak{G}^*$ be a p-cochain. Then $(C^\bullet_0(\mathfrak{G}),\delta)$ is defined by

$$(\delta\alpha)(\mathscr{X}_1,\ldots,\mathscr{X}_{p+1},Z) = \sum_{1\le i<j}^{p+1}(-1)^i\alpha(\mathscr{X}_1,\ldots,\hat{\mathscr{X}}_i,\ldots,\mathscr{X}_i\cdot\mathscr{X}_j,\ldots,\mathscr{X}_{p+1},Z) + \sum_{i=1}^{p+1}(-1)^i\alpha(\mathscr{X}_1,\ldots,\hat{\mathscr{X}}_i,\ldots,\mathscr{X}_{p+1},\mathscr{X}_i\cdot Z)\,. \tag{6.35}$$

Defining p-cocycles and p-coboundaries as usual, the p-th FA cohomology group (for the trivial action) is $H^p_0(\mathfrak{G}) = Z^p_0(\mathfrak{G})/B^p_0(\mathfrak{G})$. Therefore, a FA admits non-trivial central extensions when $H^1_0(\mathfrak{G})\neq 0$.

6.2 Infinitesimal deformations of FAs

An infinitesimal deformation of a FA in Gerstenhaber's [18] sense is obtained by modifying the n-bracket as

$$\begin{aligned}&[X_1,\ldots,X_n]_t =\\ &[X_1,\ldots,X_n] + t\alpha^1(X_1,\ldots,X_n)\,,\end{aligned} \tag{6.36}$$

where α^1 is now $\mathfrak{G}$-valued, so that $\mathfrak{G}$ will now act on it. Again, the FI constrains α^1 by

$$[X_1,\ldots,X_{n-1},[Y_1,\ldots,Y_n]_t]_t = \sum_{a=1}^{n}[Y_1,\ldots,Y_{a-1},[X_1,\ldots,X_{n-1},Y_a]_t,Y_{a+1},\ldots,Y_n]_t \tag{6.37}$$

which, with $Y_n = Z$, may we rewritten as

$$[\mathscr{X},(\mathscr{Y}\cdot Z)_t]_t = [(\mathscr{X}\cdot\mathscr{Y})_t,Z]_t + [\mathscr{Y},(\mathscr{X}\cdot Z)_t]_t\,. \tag{6.38}$$

At first order in t, this gives the following condition on α^1:

$$\begin{aligned}&[X_1,\ldots,X_{n-1},\alpha^1(Y_1,\ldots,Y_n)]+\\&\alpha^1(X_1,\ldots,X_{n-1},[Y_1,\ldots,Y_n])=\\&\sum_{a=1}^{n}[Y_1,\ldots,Y_{a-1},\alpha^1(X_1,\ldots,X_{n-1},Y_a),Y_{a+1},\ldots,Y_n]+\\&\sum_{a=1}^{n}\alpha^1(Y_1,\ldots,Y_{a-1},[X_1,\ldots,X_{n-1},Y_a],Y_{a+1},\ldots,Y_n)\,.\end{aligned}\tag{6.39}$$

In terms of the fundamental objects and with $Y_n = Z$, this may be read as a one-cocycle conditon for α^1,

$$\begin{aligned}&(\delta\alpha)(\mathscr{X},\mathscr{Y},Z)=ad_{\mathscr{X}}\alpha(\mathscr{Y},Z)-\\&ad_{\mathscr{Y}}\alpha(\mathscr{X},Z)-(\alpha(\mathscr{X},\quad)\cdot\mathscr{Y})\cdot Z-\\&\alpha(\mathscr{X}\cdot\mathscr{Y},Z)-\alpha(\mathscr{Y},\mathscr{X}\cdot Z)+\alpha(\mathscr{X},\mathscr{Y}\cdot Z)=0\,,\end{aligned}\tag{6.40}$$

where, for instance for $n = 3$,

$$\begin{aligned}&\alpha^1(\mathscr{X},\quad)\cdot\mathscr{Y}:=\\&(\alpha^1(\mathscr{X},\quad)\cdot Y_1,\,Y_2)+(Y_1,\,\alpha^1(\mathscr{X},\quad)\cdot Y_2)=\\&(\alpha^1(\mathscr{X},Y_1),Y_2)+(Y_1,\alpha^1(\mathscr{X},Y_2))\,.\end{aligned}\tag{6.41}$$

To see whether the $\mathfrak{G}$-valued cocycle α^1 is a coboundary, we look for the possible triviality of the infinitesimal deformation. It will be trivial if new generators can been found in terms of β, $\beta:\mathfrak{G}\to\mathfrak{G}$, $X'_i = X_i - t\beta(X_i)$, such that

$$\begin{aligned}&[X'_1,\ldots,X'_n]_t=[X_1,\ldots,X_n]'\equiv\\&[X_1,\ldots,X_n]-t\beta([X_1,\ldots,X_n])\,.\end{aligned}\tag{6.42}$$

At first order in t this implies

$$\begin{aligned}&[X'_1,\ldots,X'_n]_t=[X_1,\ldots,X_n]_t-\\&t\sum_{a=1}^{n}[X_1,\ldots,X_{a-1},\beta(X_a),X_{a+1},\ldots,X_n]_t=\\&[X_1,\ldots,X_n]+t\alpha^1(X_1,\ldots,X_n)-\\&t\sum_{a=1}^{n}[X_1,\ldots,X_{a-1},\beta(X_a),X_{a+1},\ldots,X_n]\,.\end{aligned}\tag{6.43}$$

Therefore, the deformation is trivial if

$$\begin{aligned}&(\alpha^1)(X_1,\ldots,X_n):=-\beta([X_1,\ldots,X_n])+\\&\sum_{a=1}^{n}[X_1,\ldots,X_{a-1},\beta(X_a),X_{a+1},\ldots,X_n]\equiv\\&(\delta\beta)(\mathscr{X},X_n)\end{aligned}\tag{6.44}$$

i.e., when

$$\begin{aligned}&\alpha^1(\mathscr{X},Z)=(\delta\beta)(\mathscr{X},Z)=\\&-\beta(\mathscr{X}\cdot Z)+(\beta(\quad)\cdot\mathscr{X})\cdot Z+\mathscr{X}\cdot\beta(Z)\,.\end{aligned}\tag{6.45}$$

If all one-cocycles are trivial, the FA is stable or rigid.

The above allows us to write the full complex $(C^\bullet_{ad}(\mathfrak{G},\mathfrak{G}),\,\delta)$ adapted to the deformations of FA problem (see [21] for details). The p-cochains are maps $\alpha^p:\wedge^{(n-1)}\mathfrak{G}\otimes\overset{p}{\cdots}\otimes\wedge^{(n-1)}\mathfrak{G}\wedge\mathfrak{G}\to\mathfrak{G}$ and the action of the coboundary operator δ is now defined by

$$\begin{aligned}&(\delta\alpha^p)(\mathscr{X}_1,\ldots,\mathscr{X}_p,\mathscr{X}_{p+1},Z)=\\&\sum_{1\le j<k}^{p+1}(-1)^j\alpha^p(\mathscr{X}_1,\ldots,\widehat{\mathscr{X}_j},\ldots,\mathscr{X}_{k-1},\mathscr{X}_j\cdot\\&\mathscr{X}_k,\mathscr{X}_{k+1},\ldots,\mathscr{X}_{p+1},Z)+\\&\sum_{j=1}^{p+1}(-1)^j\alpha^p(\mathscr{X}_1,\ldots,\widehat{\mathscr{X}_j},\ldots,\mathscr{X}_{p+1},\mathscr{X}_j\cdot Z)+\\&\sum_{j=1}^{p+1}(-1)^{j+1}\mathscr{X}_j\cdot\alpha^p(\mathscr{X}_1,\ldots,\widehat{\mathscr{X}_j}\ldots,\mathscr{X}_{p+1},Z)+\\&(-1)^p(\alpha^p(\mathscr{X}_1,\ldots,\mathscr{X}_p,\quad)\cdot\mathscr{X}_{p+1})\cdot Z\,,\end{aligned}\tag{6.46}$$

where in the last term

$$\begin{aligned}&\alpha^p(\mathscr{X}_1,\ldots,\mathscr{X}_p,\quad)\cdot\mathscr{Y}=\\&\sum_{i=1}^{n-1}(Y_1,\ldots,\alpha^p(\mathscr{X}_1,\ldots,\mathscr{X}_p,Y_i),\ldots,Y_{n-1})\,.\end{aligned}\tag{6.47}$$

The above cohomology complex [21] is essentially equivalent to that given by Gautheron [19] and Rotkiewicz [20].

7 Whitehead lemma for FAs

It follows from the above discussion that an analogue of the Whitehead lemma for FAs would require $H^1_0(\mathfrak{G})=0$ and $H^1_{ad}(\mathfrak{G},\mathfrak{G})=0$ for $\mathfrak{G}$ semisimple. This may be proven taking advantage of the fact that all simple FAs have the same general structure [11, 8]. Specifically, in Filippov's notation, they have the form

$$\begin{aligned}&[e_1\ldots\hat{e}_i\ldots e_{n+1}]=(-1)^{n+1}\varepsilon_i e_i\\&\text{or}\quad[e_{i_1}\ldots e_{i_n}]=(-1)^n\sum_{i=1}^{n+1}\varepsilon_i\epsilon_{i_1\ldots i_n}{}^i e_i\,,\end{aligned}\tag{7.48}$$

where $\varepsilon_i=\pm1$. In other words, the simple FAs are the euclidean A_{n+1} and the Lorentzian $A_{s,t}$, $(s+t=n+1)$ generalizations of the $n=2$ $so(3)$ and $so(1,2)$ Lie algebras, $[e_i,e_j]=\sum_k\varepsilon_k\epsilon_{ijk}e_k$, for which Whitehead's lemma does apply.

Define the $Z^1_0(\mathfrak{G})$ and $Z^1_{ad}(\mathfrak{G},\mathfrak{G})$ cocycles by its coordinates,

$$\begin{aligned}&\alpha^1_{i_1\ldots i_n}=\alpha^1(e_{i_1},\ldots,e_{i_n})\,,\\&\alpha^1_{i_1\ldots i_n}{}^j=\alpha^1(e_{i_1},\ldots,e_{i_n})^j\,,\quad i,j=1,\ldots,(n+1)\,.\end{aligned}\tag{7.49}$$

Using the explicit form of the simple FAs, it is possible to show [21] that the above cocycles are necessarily one-coboundaries respectively generated by the zero-cochains β_k, β^r_k *i.e.*, that

$$\begin{aligned}
&\alpha^1_{i_1\dots i_n} = \beta([e_{i_1}\dots e_{i_n}]) = \varepsilon_k \epsilon_{i_1\dots i_n}{}^k \beta_k \Rightarrow \\
&\beta_k = \frac{\varepsilon_k}{n!} \sum_{i_1,\dots,i_n=1}^{n+1} \epsilon^{i_1\dots i_n}{}_k\, \alpha^1_{i_1\dots i_n}\,; \\
&\alpha^1_{i_1\dots i_n}{}^r = -(-1)^n \sum_{s=1}^{n+1} \varepsilon_s \epsilon_{i_1\dots i_n}{}^s \beta^r{}_s + \qquad (7.50) \\
&(-1)^n \sum_{a=1}^{n} \sum_{r=1}^{n+1} \varepsilon_s \epsilon_{i_1\dots i_{a-1} a i_{a+1}\dots i_n}{}^s \beta^r{}_{i_a} \Rightarrow \\
&\beta^{rs} = -\frac{(-1)^n}{2}\left[\varepsilon_s(\alpha^1)^{rs} - \frac{1}{n-1}\sum_{t=1}^{n+1} \varepsilon_t (\alpha^1)^t{}_t \delta^{rs}\right].
\end{aligned}$$

The $(\alpha^1)^{rs}$ above is the Poincaré dual (with $\epsilon^{1_1\dots i_n r}$) of $\alpha^1_{i_1\dots i_n}{}^s$; it may be seen to be (rs)-symmetric because of the cocycle condition. Therefore, $H^1_0(\mathfrak{G}) = 0$, $H^1_{ad}(\mathfrak{G},\mathfrak{G}) = 0$ for a simple FA. Using now that a semisimple FA is the sum (2.13) of simple ideals the following result is obtained [21]:

Lemma (Whitehead lemma for $n \geq 2$)

Semisimple Filippov (n-Lie) algebras, $n \geq 2$, do not admit non-trivial central extensions and are, moreover, rigid.

8 A comment on FA and Leibniz algebra cohomology

Leibniz algebras [22] $\mathscr{L}$ are a non-commutative version of Lie algebras: their bracket does not need being anticommutative ($[X,Y] \neq -[Y,X]$) but still satisfies the (left, say) 'Leibniz' identity

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[[X,Z]]\,. \qquad (8.51)$$

Lie algebras are Leibniz algebras the bracket of which is anticommutative.

Similarly, one may define n-Leibniz algebras $\mathfrak{L}$ [23, 24] by dropping the anticommutatitivity of the FA n-bracket while keeping the (left, say) FI. Introducing also here fundamental objects for $\mathfrak{L}$, the identity reads

$$\begin{aligned}
&\mathscr{X}\cdot(\mathscr{Y}\cdot\mathscr{Z}) = (\mathscr{X}\cdot\mathscr{Y})\cdot\mathscr{Z} + \mathscr{Y}\cdot(\mathscr{X}\cdot\mathscr{Z}) \\
&\forall \mathscr{X},\mathscr{Y},\mathscr{Z} \in \otimes^{n-1}\mathfrak{L}\,.
\end{aligned} \qquad (8.52)$$

Note that now $\mathscr{X} \in \otimes^{n-1}\mathfrak{L}$ since, in contrast with FAs, the anticommutativity of the $(n-1)$ arguments of $\mathscr{X}$ is not assumed. The above is still the (left) FI (1.4) previously defining FAs; n-Lie algebras are n-Leibniz algebras the bracket of which is fully anticommutative. As a result, the characteristic FI

$$\begin{aligned}
&\mathscr{X}\cdot(\mathscr{Y}\cdot\mathscr{Z}) - \mathscr{Y}\cdot(\mathscr{X}\cdot\mathscr{Z}) = (\mathscr{X}\cdot\mathscr{Y})\cdot\mathscr{Z} \\
&\forall\ \mathscr{X},\mathscr{Y},\mathscr{Z} \in \otimes^{n-1}\mathfrak{L}\,,
\end{aligned} \qquad (8.53)$$

which determined the nilpotency of the coboundary operator δ and the different FA cohomology complexes (as the JI for Lie algebras), still holds here. Therefore, with a suitable definition of p-cochains, the n-Leibniz and the above FA cohomological complexes have the same structure. In fact, n-Leibniz cohomology underlies n-Lie cohomology. This is why the N-P cohomology may be studied from the point of view of n-Leibniz cohomology [23].

For instance, for $n = 2$ and reverting to the notation that labels the cochains by the number of elements of the algebra it contains, $\alpha^p \in C^p(\mathscr{L},\mathscr{L}) = \mathrm{Hom}(\otimes^p\mathscr{L},\mathscr{L})$, eq. (6.46) for the n-Lie case reduces to

$$\begin{aligned}
&(\delta\alpha^p)(X_1,\dots,X_p,X_{p+1}) = \\
&\sum_{1\leq j<k}^{p+1} (-1)^j \alpha^p(X_1,\dots,\widehat{X_j},\dots,X_{k-1},[X_j,X_k], \\
&\qquad X_{k+1},\dots,X_{p+1}) + \qquad (8.54) \\
&\sum_{j=1}^{p} (-1)^{j+1} X_j\cdot\alpha^p(X_1,\dots,\widehat{X_j}\dots,X_{p+1}) + \\
&(-1)^{p+1}(\alpha^p(X_1,\dots,X_p)\cdot X_{p+1}\,,
\end{aligned}$$

which coincides with the cohomology complex $(C^\bullet(\mathscr{L},\mathscr{L}),\delta)$ for Leibniz algebras $\mathscr{L}$ [25, 24].

Our proof for the Whitehead Lemma for FAs, however, relied on the antisymmetry of the n-commutator, and thus it will not hold when the anticommutativity is relaxed. Thus, one might expect having a richer deformation structure for Leibniz deformations. This has been observed already for the $n = 2$ case [26] by looking at Leibniz deformations of a Lie algebra, and a specific Leibniz deformation of the euclidean 3-Lie algebra has been found [27]. Thus, a natural extension of our work is to look *e.g.* at n-Leibniz deformations of simple n-Lie algebras to see whether this opens more possibilities. Our results [28] for n-Leibniz deformations with brackets that keep the antisymmetry in their first $n-1$ arguments show that rigidity still holds for $n > 3$.

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