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A perturbative solution of the power-law viscoelastic constitutive equation for lithospheric rocks

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Abstract
A power-law, viscoelastic constitutive equation for lithospheric rocks, is considered. The equation is a nonlinear generalization of the Maxwell constitutive equation, in which the viscous deformation depends on the n-th power of deviatoric stress, and describes a medium which is elastic with respect to normal stress, but relaxes deviatoric stress. Power-law exponents equal to 2 and 3, which are most often found in laboratory experiments, are considered. The equation is solved by a perturbative method for a viscoelastic layer subjected to a constant, extensional or compressional, strain rate and yields stress as a function of time, temperature and rock composition. The solution is applied to an ideal extensional boundary zone and shows that the base of the crustal seismogenic layer may be deeper than predicted by a linear rheology.

Key words *lithosphere rheology – viscoelasticity – plate boundaries*

1. Introduction

Rheology has a chief role in determining the state of stress at plate boundary zones. Stress and rheology are also responsible for seismic activity (Meissner and Strehlau, 1982; Sibson, 1982; Dragoni *et al.*, 1986; Handy, 1989). Laboratory experiments show that, at sufficiently high temperatures, brittle behaviour in crystalline materials gives way to ductile behaviour (Jaeger and Cook, 1976; Paterson, 1978). As a consequence, it is inferred that brittle faulting at shallow depth gives way to ductile deformation at greater depth (Scholz, 1990). The transition takes place over a depth range depending largely on lithology and thermal conditions and limits the depth to which earthquakes normally occur.

A number of microscopic processes may produce ductility. If diffusion of crystal impurities prevails, the strain rate is proportional to stress. The motion of crystal dislocations produces instead a power-law creep, with a strain rate exponent (n) usually ranging from 2 to 3 (Kirby and Kronenberg, 1987). At shallow depth, the mechanism of pressure solution, based on the change in solubility of the solid at the sites of high normal stress, again entails a linear relation-sheep between strain rate and applied stress (Rutter, 1976). The local conditions of pressure, temperature and strain rate control which process is going to dominate (see Dragoni, 1993 for a brief review).

The state of stress at a plate boundary can be evaluated by assuming that the boundary zone is subject to a strain rate which is con-

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stant both in space and time. This assumption is equivalent to considering a steady-state motion of rigid plates. The value of this strain rate can be calculated from the relative plate velocity and the horizontal width of the boundary zone. The link between strain and stress is provided by the constitutive equation of the boundary zone material, which depends on temperature, pressure and rock composition. Solving the constitutive equation then yields the stress as a function of time and depth.

Parmentier *et al.* (1976) showed that a nonlinear rheology can be approximated by a linear rheology with appropriate parameters. In fact, the main features of the mechanics of plate boundary zones can be explained by a linear constitutive equation. Analytical solutions for a linear viscoelastic (Maxwell) rheology were found both for transcurrent (Dragoni *et al.*, 1986) and compressional or extensional (Dragoni *et al.*, 1993) plate boundaries.

For a nonlinear, power-law viscoelastic rheology, analytical solutions were found only in the case of a transcurrent plate boundary, for power-law exponents equal to 2, 3 and 4 (Dragoni, 1988). The case of a transcurrent boundary is particularly simple, since only one stress component is relevant to the problem and the constitutive equation is greatly simplified. The effects of nonlinear crustal rheology were recently studied by Reches *et al.* (1994) for the earthquake cycles on the San Andreas fault.

In the present paper we consider a nonlinear, power-law viscoelastic rheology for plate boundary and solve the constitutive equation by a perturbative method. Power-law exponents equal to 2 and 3, which are most often found in laboratory experiments, are considered. We show stress profiles for an ideal, extensional boundary zone, and make a comparison between linear and nonlinear rheologies.

2. Solution of the constitutive equation

The rheological behaviour of any material is described by a constitutive equation relating strain, stress and their time derivatives. Crustal and mantle rocks behave elastically for short stress cycles; in fact they propagate seismic shear waves. For long stress cycles, rocks may undergo large, permanent shear strain, while normal strain remains elastic. Different constitutive equations have been proposed to describe such a behaviour (*e.g.*, Peltier *et al.*, 1981). We consider here a nonlinear generalization of the Maxwell constitutive equation, in which the viscous deformation depends on the n-th power of deviatoric stress. This equation describes a medium which is elastic with respect to normal stresses, but relaxes deviatoric stresses:

$$\dot{\sigma}_{ij} + 2\mu A_n e^{-\frac{E}{RT}} \left(\sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}\right)^n = $$
$$= \lambda \dot{\varepsilon}_{kk}\delta_{ij} + 2\mu\dot{\varepsilon}_{ij} \qquad (2.1)$$

where σ_{ij} is the stress tensor, ε_{ij} is the strain tensor, λ and μ are the Lamè parameters, A_n is a rheological parameter, E is the activation energy, R is the gas constant and T is the absolute temperature. Dots indicate differentiation with respect to time. The dependence of rheology on pressure can be neglected at lithospheric depths.

We consider a cartesian coordinate system xyz and a layer $0 \leq z \leq h$, representing the lithosphere, subject to a strain rate

$$\dot{\varepsilon}_{yy} = -\mathcal{R} \qquad (2.2)$$

where $\mathcal{R} > 0$ for compression and $\mathcal{R} < 0$ for extension. The model is two-dimensional and the surface $z = 0$ is traction-free. The surface $z = h$ (base of the lithosphere) is also taken as a free surface since we consider time scales much longer than the typical relaxation times of the asthenosphere. The solution of (2.1) in the linear case ($n = 1$) is (Dragoni *et al.*, 1993)

$$\sigma^0_{xx}(z, t) = -2\mathcal{R}\eta\left[1 + \frac{1}{2}(e^{-\frac{t}{\tau_1}} - 3e^{-\frac{t}{\tau_2}})\right] \qquad (2.3)$$

$$\sigma^0_{yy}(z, t) = -4\mathcal{R}\eta\left[1 - \frac{1}{4}(e^{-\frac{t}{\tau_1}} - 3e^{-\frac{t}{\tau_2}})\right] \qquad (2.4)$$

where τ_1 and τ_2 are characteristic relaxation times:

$$\tau_1 = \frac{\eta}{\mu} \qquad (2.5)$$

$$\tau_2 = \frac{\lambda + 2\mu}{\lambda + \frac{2}{3}\mu}\,\tau_1 \qquad (2.6)$$

and η is viscosity, given by the Arrhenius formula

$$\eta = \frac{1}{2A_1}\exp\frac{E}{RT}. \qquad (2.7)$$

In order to study the nonlinear case, we introduce a nondimensional time

$$\tau = \frac{t}{\tau_1} \qquad (2.8)$$

and a nondimensional stress

$$\theta_{ij} = \frac{\sigma_{ij}}{\lambda}. \qquad (2.9)$$

Accordingly, the linear constitutive equation can be written in a nondimensional form as

$$\frac{d\theta_{ij}}{d\tau} + \left(\theta_{ij} - \frac{1}{3}\,\theta_{kk}\,\delta_{ij}\right) = \frac{d\varepsilon_{kk}}{d\tau}\,\delta_{ij} + 2\gamma\frac{d\varepsilon_{ij}}{d\tau} \qquad (2.10)$$

in which

$$\gamma = \frac{\mu}{\lambda}. \qquad (2.11)$$

Nonlinear solutions can be studied by introducing a real, positive exponent n in the second term of the left side of (2.10):

$$\frac{d\theta_{ij}}{d\tau} + \left(\theta_{ij} - \frac{1}{3}\,\theta_{kk}\,\delta_{ij}\right)^n = \frac{d\varepsilon_{kk}}{d\tau}\,\delta_{ij} + 2\gamma\frac{d\varepsilon_{ij}}{d\tau}. \qquad (2.12)$$

Denoting by θ^0_{ij} the linear solution, we put

$$\theta_{ij} = \theta^0_{ij} + (n-1)\,\delta\theta_{ij} \qquad (2.13)$$

and substitute this expression in (2.12).

Under simple conditions, up to the first order in the binomial expansion, we obtain the two coupled equations (see Appendix for details):

$$(n-1)\frac{d}{d\tau}\,\theta_{xx} +$$

$$+(n-1)\,n\left[\delta\theta_{xx} - \frac{1}{3}(\delta\theta_{xx} + \delta\theta_{yy})\right](\mathcal{R}\gamma\tau_1)^{n-1}B(\tau)^{n-1} =$$

$$= (\mathcal{R}\gamma\tau_1)\,B(\tau) - (\mathcal{R}\gamma\tau_1)^n\,B(\tau)^n \qquad (2.14)$$

$$(n-1)\frac{d}{d\tau}\,\theta_{yy} +$$

$$+(n-1)\,n\left[\delta\theta_{yy} - \frac{1}{3}(\delta\theta_{xx} + \delta\theta_{yy})\right](-\mathcal{R}\gamma\tau_1)^{n-1}\big[2+$$

$$+B(\tau)\big]^{n-1} = (-\mathcal{R}\gamma\tau_1)\left[2 + B(\tau)\right] -$$

$$-(-\mathcal{R}\gamma\tau_1)^n\left[2 + B(\tau)\right]^n \qquad (2.15)$$

where

$$B(\tau) = (3e^{\tau\frac{\tau_1}{\tau_2}} - e^{-\tau}). \qquad (2.16)$$

We consider the cases $n = 2$ and $n = 3$. By separately studying short and long times, we obtain the (dimensional) solutions:

$$\sigma_{xx} = \sigma^0_{xx} + \delta\sigma_{xx} \qquad (2.17)$$

$$\sigma_{yy} = \sigma^0_{yy} + \delta\sigma_{yy}. \qquad (2.18)$$

below specified:

Short times $t \ll \tau_1$

n = **2**

$$\delta\sigma_{xx} = \frac{\lambda}{2}\left[3 + \frac{1}{2}\left(\mathcal{R}\frac{\eta}{\lambda}\right)\right]\left[1 - e^{-\mathcal{R}\frac{\eta}{\lambda}(\sqrt{79}-4)\frac{t}{\tau_1}}\right] \qquad (2.19)$$

$$\delta\sigma_{yy} = \frac{\lambda}{2}\left[3 + \frac{11}{2}\left(\mathcal{R}\frac{\eta}{\lambda}\right)\right]\left[1 - e^{-\mathcal{R}\frac{\eta}{\lambda}(\sqrt{79}-4)\frac{t}{\tau_1}}\right] \qquad (2.20)$$

n = **3**

$$\delta\sigma_{xx} = \frac{\lambda^2}{3\mathcal{R}\eta}\left[\frac{56}{21} - \frac{373}{21}\left(\mathcal{R}\frac{\eta}{\lambda}\right)^2\right]$$

$$\left[1 - e^{-\left(\mathcal{R}\frac{\eta}{\lambda}\right)^2 (\sqrt{2923} - 40)\frac{t}{\tau_1}}\right] \qquad (2.21)$$

$$\delta\sigma_{yy} = \frac{\lambda^2}{3\mathcal{R}\eta}\left[\frac{52}{21} - \frac{357}{21}\left(\mathcal{R}\frac{\eta}{\lambda}\right)^2\right]$$

$$\left[1 - e^{-\left(\mathcal{R}\frac{\eta}{\lambda}\right)^2 (\sqrt{2923} - 40)\frac{t}{\tau_1}}\right]. \qquad (2.22)$$

Long times $t \gg \tau_1$

n = **2**

$$\delta\sigma_{xx} = 2\mathcal{R}\eta - \sqrt{2|\mathcal{R}|\eta\lambda} \quad \mathrm{sgn}\ \mathcal{R} \quad (2.23)$$

$$\delta\sigma_{yy} = 4\mathcal{R}\eta - 2\sqrt{2|\mathcal{R}|\eta\lambda} \quad \mathrm{sgn}\ \mathcal{R} \quad (2.24)$$

n = **3**

$$\delta\sigma_{xx} = 2\mathcal{R}\eta - \sqrt[3]{2\mathcal{R}\eta\lambda^2} \qquad (2.25)$$

$$\delta\sigma_{yy} = 4\mathcal{R}\eta - 2\sqrt[3]{2\mathcal{R}\eta\lambda^2}. \qquad (2.26)$$

To obtain the total stress field in the boundary zone, we add the lithospheric pressure p to σ_{ij}. The differential stress $\Delta\sigma$ is defined as the difference between the maximum and minimum principal stresses. In our model

$$\Delta\sigma = |\sigma_{yy}|. \qquad (2.27)$$

The asymptotic value of $\Delta\sigma$ ($t \to \infty$) is given by

$$\Delta\bar{\sigma} = 2\sqrt{2|\mathcal{R}|\eta\lambda}, \qquad n = 2 \qquad (2.28)$$

and

$$\Delta\bar{\sigma} = 2\sqrt[3]{2|\mathcal{R}|\eta\lambda^2}, \qquad n = 3 \qquad (2.29)$$

while in the linear case

$$\Delta\bar{\sigma} = 4|\mathcal{R}|\eta, \qquad n = 1. \qquad (2.30)$$

The differential stress can be compared with the frictional resistance over a fault in order to estimate the extent of the seismogenic zone.

3. Application to an extensional plate boundary zone

Previous theoretical models have shown that the differences in mechanical behaviour between compressional and extensional boundary zones are due to different structures and geothermal profiles, rather than to different boundary conditions. Compressive zones usually have a thick lithosphere and lower geothermal gradient, while extensional zones have a thin lithosphere and a higher geothermal gradient (Dragoni *et al.*, 1996). We present graphs of the stress evolution in an ideal extensional boundary zone, according to the nonlinear solutions found in section 2.

The boundary zone is composed of three layers, an upper sedimentary layer, a crustal layer and the lithospheric mantle, with the elastic, rheological and thermal parameters shown in table I, where k is thermal conductivity. The geotherm for a recently stretched lithosphere is constructed as in Dragoni *et al.* (1996), assuming a radiogenic heat production in layer 2 and the base of the lithosphere at the isothermal surface T_a = 1330°C (fig. 1):

$$T(z) = \begin{cases} T_0 + \dfrac{Q_0}{k_1} z \\ T(z_1) + \dfrac{Q_0 - DH_0}{k_2}(z - z_1) + \dfrac{D^2 H_0}{k_2}\left(1 - e^{-\frac{z - z_1}{D}}\right) \\ T(z_2) + \dfrac{T_a - T(z_2)}{z_3 - z_2}(z - z_2) \end{cases} \qquad (3.1)$$

in

$$0 \le z \le z_1, \qquad z_1 \le z \le z_2, \qquad z_2 \le z \le z_3$$

respectively, where T_0 is the temperature at the

Table I. Elastic, thermal and rheological structure of the extensional boundary zone considered in the text.

Layer (i)	z_i (km)	λ, μ (GPa)	k_i ($\mathrm{Wm^{-1}\,{}^\circ K^{-1}}$)	E ($\mathrm{kJ\ mol^{-1}}$)	n	An ($\mathrm{MPa^{-n}\ s^{-1}}$)
1	5	20	2.0	155	2.0	2.6×10^{-4}
2	25	30	2.4	155	2.0	2.6×10^{-4}
3	40	60	–	260	3.0	2.0×10^{-4}

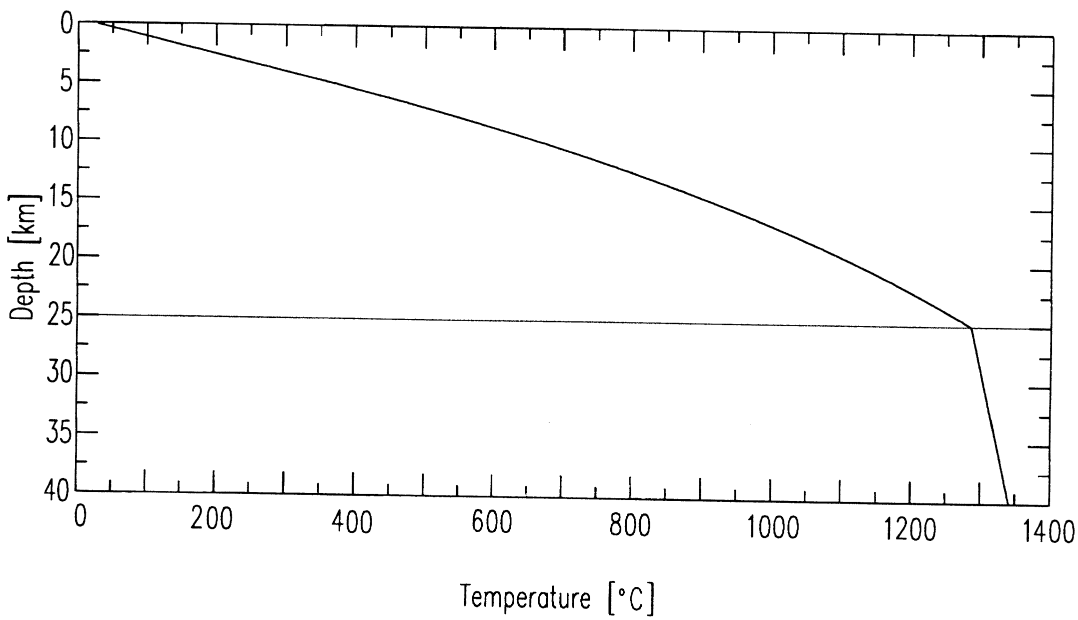

Fig. 1. Geotherm for the extensional boundary zone (after Dragoni *et al.*, 1996).

Earth's surface, Q_0 is the heat flow density at the Earth's surface, H_0 is the radiogenic heat productivity at the top of layer 2 and D is a scale length. We take $T_0 = 20°\mathrm{C}$, $Q_0 = 130\ \mathrm{mW\ m^{-2}}$, $H_0 = 3\ \mu\mathrm{W\ m^{-3}}$, $D = 8$ km.

A rheological discontinuity is assumed at the boundary between crust and mantle, with a change both in the activation energy E and the power-law exponent n: the values are taken from characteristic rock parameters given by Kirby and Kronenberg (1987). We assume $\gamma = 1$, a tipical value for the Earth's crust, and a strain rate $\mathcal{R} = -10^{-15}\ \mathrm{s^{-1}}$.

The stress components σ_{xx} and σ_{yy} are shown in fig. 2a,b, for the case where the deformation of the plate boundary zone start at $t = 0$. Stress profiles show the general features already found in previous works (*e.g.*, Dragoni *et al.*, 1996): a stress concentration in the upper crust, a low viscosity zone in the lower crust, another stress concentration, although of lower amplitude, below the base of the crust.

It is interesting to compare these results with the solution of the linear ($n = 1$) constitutive

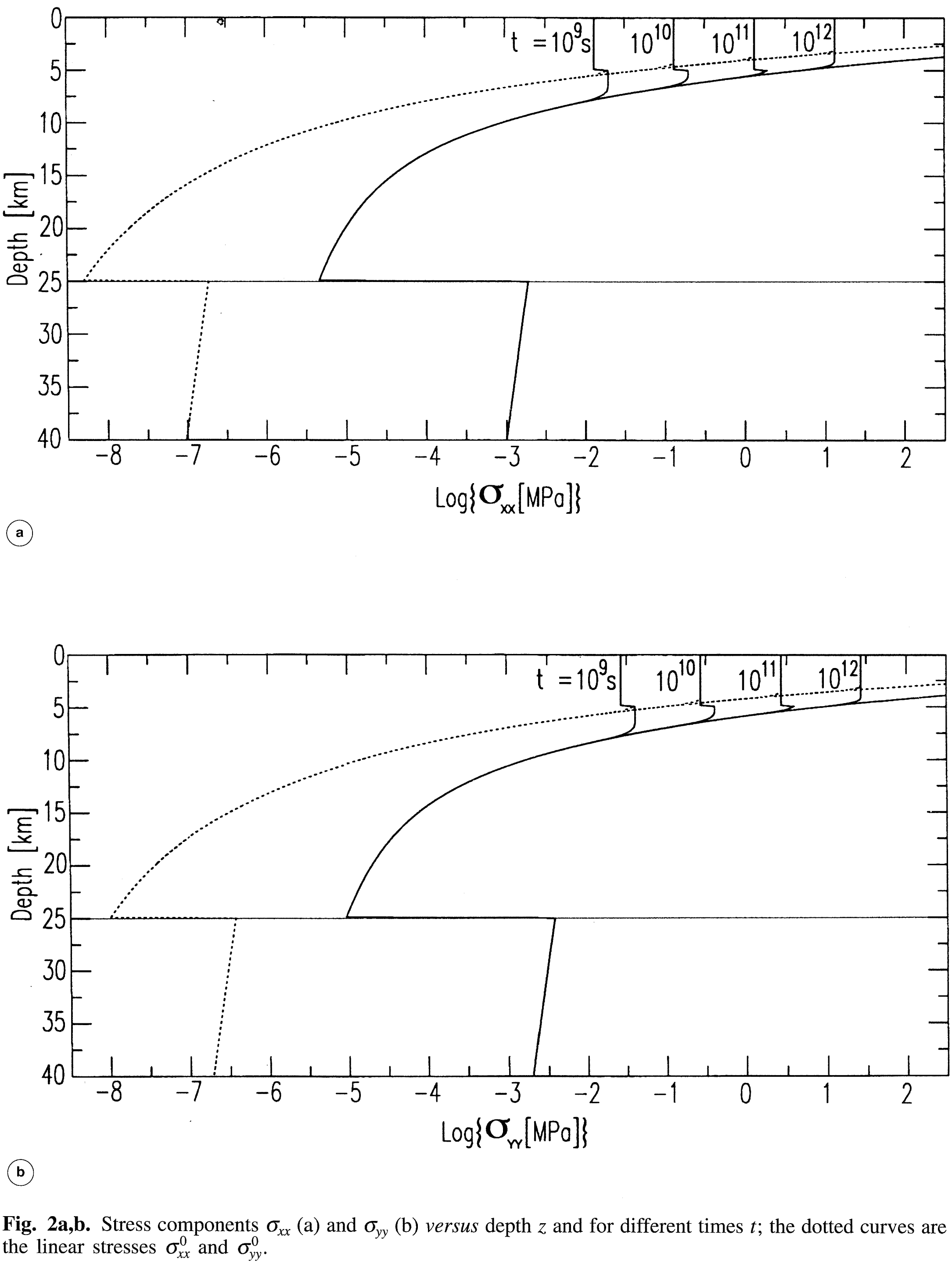

Fig. 2a,b. Stress components σ_{xx} (a) and σ_{yy} (b) *versus* depth z and for different times t; the dotted curves are the linear stresses σ_{xx}^0 and σ_{yy}^0.

equation. Such a comparison is meaningful if we assume that the linear and nonlinear curves in a graph of stress *versus* strain rate intersect at a point representative of laboratory conditions. Accordingly, the linear solution (dotted curves in fig. 2a,b) has been calculated assuming for each layer a value

$$A_1 = A_n \sigma_e^{n-1} \tag{3.2}$$

where σ_e is a typical value of deviatoric stress used in laboratory experiments (we take $\sigma_e = 100$ MPa). The main effect of nonlinear rheology is evident in fig. 2a,b: the asymptotic stress curves are different from the corresponding curves of the linear case, entailing that the stress values at a given depth are higher than in the linear case. Only in the uppermost layer the stress curves are not affected by the choice of the exponent n. This can be understood since the effective viscosity is high enough for the characteristic relaxation times of rocks to be in the order of 10^6 years or more. This entails that, for shorter times, the behaviour is still predominantly elastic and viscous effects are absent, no matter how complex the rheology is.

Figure 3 shows a comparison of the asymptotic differential stress $\Delta\bar{\sigma}$ with a typical frictional resistance σ_f, assumed as linearly increasing with depth with a gradient of 10 MPa km^{-1}. Nonlinear rheology allows rocks to accumulate a greater amount of stress at depths greater than a few kilometres, making seismic activity possible at greater depths than those predicted by linear rheology. This is of particular relevance to regions with high heat flow, where the assumption of a linear rheology would confine seismicity to an extremely shallow layer. For the same reason, subcrustal seismicity may be also favoured by nonlinear rheology, but only in the case of lower geothermal gradients than those considered here.

4. Conclusions

A nonlinear, viscoelastic constitutive equation has been solved for a model lithosphere subject to a compressional or an extensional

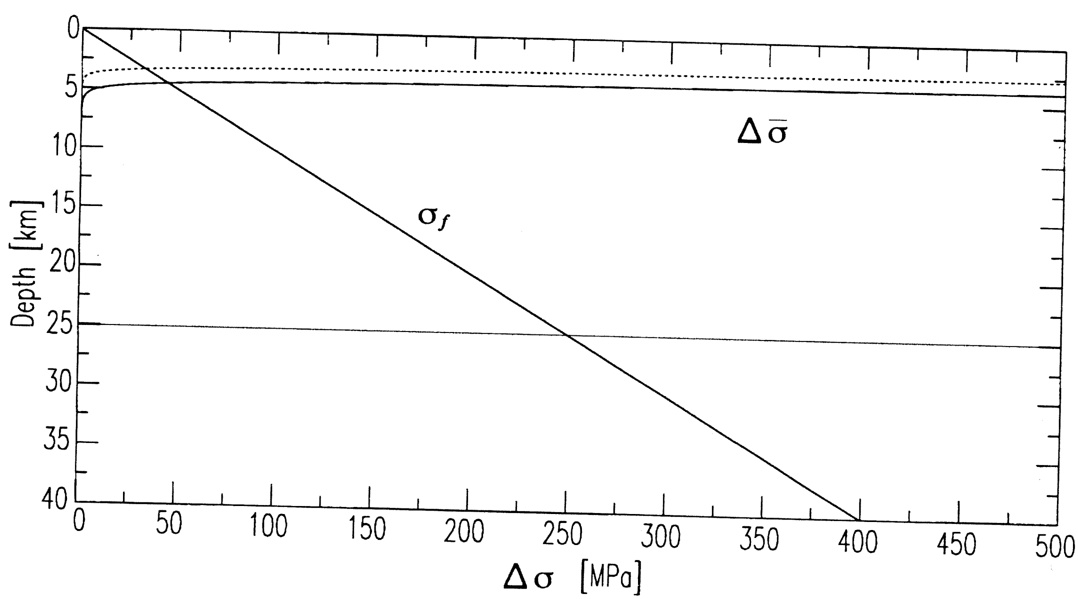

Fig. 3. Asymptotic differential stress $\Delta\bar{\sigma}$ as a function of depth (the dotted curve corresponds to the linear case). The straight line σ_f is a typical frictional resistance with gradient of 10 MPa km^{-1}.

strain rate, which is constant in space and time. The solution is the same in the two cases, apart from the sign of strain rate.

A comparison between linear and nonlinear rheology, in the case of an ideal extensional plate boundary zone, shows that nonlinear rheology allows rocks to accumulate a higher stress at depths greater than a few kilometres, *i.e.* at temperatures greater than 450°C, marking the transition of crustal rocks to ductile behaviour. As a consequence, in the presence of nonlinear rheology, the base of the crustal seismogenic layer may be lowered by a few kilometres or more, depending on the geothermal gradient.

In addition to temperature, the solution is strongly dependent on the rheological parameters. The uncertainty in the values of these parameters remains a weak point of any application of theoretical models to specific boundary zones. Reches *et al.* (1994), in modelling the San Andreas fault, found that only models with viscosity values 10-100 times lower than the experimental viscosity of quartzite fit the field observations, suggesting that the *in situ* viscosity of crustal rocks may be orders of magnitude less than the viscosities determined in laboratory.

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Appendix

In order to use a perturbative approach for nonlinear constitutive equations, it is suitable to transform these equations in nondimensional form: this is the way to overcome any dimensional problem when performing a power series expansion. Introducing θ_{ij} as given by (2.13) into the nonlinear equation (2.12), we obtain the following equations for the unknown functions $\delta\theta_{ij}$:

$$\frac{d}{d\tau}[\theta^0_{xx}+(n-1)\,\delta\theta_{xx}]+\left[\theta^0_{xx}-\frac{1}{3}(\theta^0_{xx}+\theta^0_{yy})\right]^n+\left[1+(n-1)\frac{\delta\theta_{xx}-\frac{1}{3}(\delta\theta_{xx}+\delta\theta_{yy})}{\theta^0_{xx}-\frac{1}{3}(\theta^0_{xx}+\theta^0_{yy})}\right]^n=\frac{d}{d\tau}[\varepsilon_{yy}+\varepsilon_{zz}]$$

$$\frac{d}{d\tau}[\theta^0_{yy}+(n-1)\,\delta\theta_{yy}]+\left[\theta^0_{yy}-\frac{1}{3}(\theta^0_{xx}+\theta^0_{yy})\right]^n+\left[1+(n-1)\frac{\delta\theta_{yy}-\frac{1}{3}(\delta\theta_{xx}+\delta\theta_{yy})}{\theta^0_{yy}-\frac{1}{3}(\theta^0_{xx}+\theta^0_{yy})}\right]^n=\frac{d}{d\tau}[\varepsilon_{yy}+\varepsilon_{zz}]+2\gamma\frac{d\varepsilon_{yy}}{d\tau}$$

(A.1)

Remembering that θ^0_{ij} is the solution for the linear case and using the same strain tensor ε_{ij} as in Dragoni *et al.* (1993), under the hypothesis that

$$(n-1)\,|\,\delta\theta_{xx}-\frac{1}{3}(\delta\theta_{xx}+\delta\theta_{yy})\,|<|\,\theta^0_{xx}-\frac{1}{3}(\theta^0_{xx}+\theta^0_{yy})\,|$$

$$(n-1)\,|\,\delta\theta_{yy}-\frac{1}{3}(\delta\theta_{xx}+\delta\theta_{yy})\,|<|\,\theta^0_{yy}-\frac{1}{3}(\theta^0_{xx}+\theta^0_{yy})\,| \tag{A.2}$$

in order to limit the binomial expansion to the first order, we obtain

$$(n-1)\frac{d}{d\tau}\,\delta\theta_{xx}+(n-1)\,n\left[\delta\theta_{xx}-\frac{1}{3}(\delta\theta_{xx}+\delta\theta_{yy})\right](\mathcal{R}\gamma\tau_1)^{n-1}B(\tau)^{n-1}=(\mathcal{R}\gamma\tau_1)B(\tau)-(\mathcal{R}\gamma\tau_1)^nB(\tau)^n \tag{A.3}$$

$$(n-1)\frac{d}{d\tau}\,\delta\theta_{yy}+(n-1)\,n\left[\delta\theta_{yy}-\frac{1}{3}(\delta\theta_{xx}+\delta\theta_{yy})\right](-\mathcal{R}\gamma\tau_1)^{n-1}\,[2+B(\tau)]^{n-1}=$$
$$=(-\mathcal{R}\gamma\tau_1)\,[2+B(\tau)]-(-\mathcal{R}\gamma\tau_1)]^n\,[2+B(\tau)]^n \tag{A.4}$$

where $B(\tau)$ was given in (2.16). Consider now short times ($\tau \ll 1$, *i.e.*, $t \ll \tau_1$). With a very crude approximation we may write (A.3) as

$$\frac{dX}{d\tau}+n\left(\frac{2}{3}X-\frac{1}{3}Y\right)k_{1n}=k_{2n}$$

$$\frac{dY}{d\tau}+n\left(\frac{2}{3}Y-\frac{1}{3}X\right)k_{3n}=k_{4n} \tag{A.5}$$

where

$$X = (n-1)\,\delta\theta_{xx} \quad ; \quad Y = (n-1)\,\delta\theta_{yy}$$
$$k_{1n} = (\mathcal{R}\gamma\tau_1)^{n-1} B^{n-1}(0) \quad ; \quad k_{2n} = (\mathcal{R}\gamma\tau_1)\,B(0) - (\mathcal{R}\gamma\tau_1)^{n-1} B^{n-1}(0) \tag{A.6}$$
$$k_{3n} = (-\mathcal{R}\gamma\tau_1)^{n-1}(2+B(0))^{n-1} \quad ; \quad k_{4n} = (-\mathcal{R}\gamma\tau_1)(2+B(0)) - (-\mathcal{R}\gamma\tau_1)^{n}(2+B(0))^{n}$$

We look for a solution in the form

$$X(\tau) = C_1 + C_2 e^{\lambda_1 \tau} + C_3 e^{\lambda_2 \tau}$$
$$Y(\tau) = C_4 + C_5 e^{\lambda_1 \tau} + C_6 e^{\lambda_2 \tau} \tag{A.7}$$

where the $C_i (i = 1, \dots 6)$ and λ_j $(j = 1,2)$ are constants. We must require that (A.7) do not diverge throughout the interval $0 \le \tau < 1$ and that in the same interval they satisfy conditions (A.2). With the initial conditions $X(0) = Y(0) = 0$, it is a simple matter to obtain

$$C_1 = \frac{1}{n}\left(\frac{k_{2n}}{k_{1n}} + 2\frac{k_{4n}}{k_{3n}}\right)$$

$$C_4 = \frac{1}{n}\left(2\frac{k_{2n}}{k_{1n}} + \frac{k_{4n}}{k_{3n}}\right)$$

$$C_3 = C_6 = 0 \tag{A.8}$$

$$C_2 = -C_1$$

$$C_5 = -C_4$$

Furthermore λ_1 is the smaller of the two solutions of the equation obtained by the simple requirement that substitution of (A.7) in (A.5), with constants (A.8), leads to a non-zero solution:

$$\begin{vmatrix} \frac{2}{3} n k_{1n} + \lambda & -\frac{n}{3} k_{1n} \\ -\frac{n}{3} k_{3n} & \frac{2}{3} n k_{3n} + \lambda \end{vmatrix} = 0 \tag{A.9}$$

Thus we obtain $(\tau \ll 1)$

$n =$ **2**

$$\delta\theta_{xx} = \frac{1}{2}\left[3 + \frac{1}{2}(\mathcal{R}\gamma\tau_1)\right]\left[1 - e^{-\frac{\mathcal{R}\gamma\tau_1}{3}(\sqrt{79}-4)\,\tau}\right]$$
$$\delta\theta_{yy} = \frac{1}{2}\left[3 + \frac{11}{2}(\mathcal{R}\gamma\tau_1)\right]\left[1 - e^{-\frac{\mathcal{R}\gamma\tau_1}{3}(\sqrt{79}-4)\,\tau}\right] \tag{A.10}$$

$n = 3$

$$2\,\delta\theta_{xx} = \frac{1}{3\mathcal{R}\gamma\tau_1}\left[\frac{56}{21} - \frac{373}{21}(\mathcal{R}\gamma\tau_1)^2\right]\left[1 - e^{-\frac{(\mathcal{R}\gamma\tau_1)^2}{4}(\sqrt{2923}-40)\,\tau}\right]$$

$$2\,\delta\theta_{yy} = \frac{1}{3\mathcal{R}\gamma\tau_1}\left[\frac{52}{21} - \frac{357}{21}(\mathcal{R}\gamma\tau_1)^2\right]\left[1 - e^{-\frac{(\mathcal{R}\gamma\tau_1)^2}{4}(\sqrt{2923}-40)\,\tau}\right] \tag{A.11}$$

It is easy to verify that conditions (A.2) are fulfilled by (A.10) and (A.11) when $\tau \ll 1$, in agreement with the hypothesis of short times.

If we consider now long times ($\tau \gg 1$), it is difficult to prove that the (A.2) are always satisfied by solutions of (A.3). It is better to treat directly the native equations (A.1) in which asymptotic $\delta\theta_{ij}$ are inserted:

$$\theta_{ij} = \theta_{ij}^0 + (n-1)\,\delta\theta_{ij}^{\infty}. \tag{A.12}$$

Imposing that general asymptotic conditions, as expressed in Dragoni *et al.* (1993), are to be satisfied, we obtain

$n = 2$

$$\delta\theta_{xx} = 2\mathcal{R}\gamma\theta_1 - \sqrt{2\,|\mathcal{R}|\,\gamma\theta_1}\ \operatorname{sgn}\ \mathcal{R}$$

$$\delta\theta_{yy} = 4\mathcal{R}\gamma\theta_1 - 2\sqrt{2\,|\mathcal{R}|\,\gamma\theta_1}\ \operatorname{sgn}\ \mathcal{R} \tag{A.13}$$

$n = 3$

$$2\,\delta\theta_{xx} = 2\mathcal{R}\gamma\theta_1 - \sqrt[3]{2\mathcal{R}\gamma\theta_1}$$

$$2\,\delta\theta_{yy} = 4\mathcal{R}\gamma\theta_1 - 2\sqrt[3]{2\mathcal{R}\gamma\theta_1}. \tag{A.14}$$

Multiplication by λ and substitutions (2.4) and (2.6) bring (A.10), (A.11) and (A.13), (A.14) to the dimensional forms presented in the text. For clarity, in our (dimensional) forms, η is the linear (first order), $\lambda\eta$ the second, $\lambda^2\eta$ the third order viscosity and so on, thus determining the value of A_n in (2.1).