ANNALI DI GEOFISICA, VOL. XXXIX, N. 1, January 1996

The fractality of marine measurement networks and of the Earth's sampled magnetic field

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Abstract
We highlight the fractal behaviour of marine measurement networks when determining the Earth's total magnetic field and the spatial trend of the field itself. This approach is a convenient alternative method of assessing the coverage of an area by a set of measurements whenever the environmental situations do not permit a regular distribution of the measurement points. The Earth's magnetic field is sampled in marine areas when the measuring apparatus is moving, even at low speeds, whilst attempts are made to respect the spatial planning which has been pre-determined on the basis of the resolution sought after. However, the real distribution of the measurements presents numerous disturbances which are mainly due to environmental factors. In the case of distributions containing vast areas with no measurement points it is no longer possible to apply Shannon's theorem in 1-D and 2-D. In our paper we apply the fractal theory to certain 1-D and 2-D measurement distributions order to obtain a coverage estimate of the area and the capacity of reconstructing the field. We also examine the trend of the power spectra S of numerous magnetic profiles noting that almost all of them illustrate the dependency with the frequency f in the form $S \approx f^{-\beta}$ which is characteristic (necessary condition) of self-similar or self affine fractals.

Key words *Earth's magnetic measurements – marine surveys – fractals*

1. Non uniform sampling of limited bandwidth signals

Regarding the sampling of a function $f(x)$ with limited bandwidth W and with all the sampling points outside the interval $(-X, X)$ at exactly zero, Shannon's theorem states that $f(x)$ can be specified by $2WX$ sampling points. It must also be noted that the points do not necessarily have to be equidistant and therefore uniformly distributed. Let us now see which conditions are necessary in order to effectively recognise the signal. According to Cauchy, if a signal is a function of a variable x whose set of values is divided into equal intervals in such a way that each division includes a value X – where X is less than half the period corresponding to the most significant frequency present in the signal – and if an instantaneous sample is taken from each interval in any way, then the knowledge of the instantaneous magnitude of each sample as well as the knowledge of the instant (inside each interval in which the sample was taken) contains all the information of the initial signal. Various authors have considered several methods to reconstruct a continuous signal from its non uniform samples. They have proposed a variety of techniques using step-by-step filtering and Sline and Yen's interpolation (Yen, 1956).

Spline's interpolation method, or hill-function type, is a recently designed special development of polynomials. By using simulations it was shown that the Yen method is superior to the others as it is both insensitive to the migration of samples and it has a noise signal ratio (S_{nr}):

$$S_{nr} = \frac{\sum S_i^2}{\sum (S_i - F_i)^2} \qquad (1.1)$$

where S_i equals signal samples and F_i equals the samples taken from the reconstructed signal. Finally, Yao and Thomas (1967) derived a sampling representation for limited bandwidth functions when the sampling points are not necessarily equidistant, but where each one deviates less than $1/\pi * \ln 2 \sim 0.22$ from its corresponding Nyquist instant, as required by the sampling theorem. The above authors called such a representation «semi-uniform» and used Fourier's non harmonic series to derive it. They remarked that a sampling representation is not possible when every sampling instant demonstrates a non-uniform deviation of 1/4 of a unit from the corresponding Nyquist instant, or if an arbitrary but finite number of sampling instants is positioned arbitrarily or if additional points have been added. More recently, other authors (Bucci *et al.*, 1993) proposed a highly efficient algorithm for the interpolation of limited band-width functions beginning with samples which are not uniformly distributed on the plane. The algorithm can be applied wherever 1) the average sample density is higher than the Nyquist average and 2) there is a one-to-one correspondence between the non uniform samples and those of the uniform network associating the nearest uniform sampling to each non uniform sampling point which has shifted by less than 25%. This procedure proves to be more efficient from a computational point of view than optimal linear estimation algorithms. The need to deal with non uniform distribution cases obviously derives from the practical circumstances which hinder samples from being taken from the pre-established points. The problem of interpolation from samples which are not uniformly distributed was well tackled in the case of limited bandwidth functions defined on the real axis. In actual fact, a function which is square integrable, with a limited bandwidth W, can be obtained from its samples in the points where it was picked up (x_i) if the set of exponentials $e^{j\omega x_i}$ is closed within Hilbert's space $L^2(-W_i, W)$ of the functions which can be integrated to the power of four in the interval $(-W_i, W)$ (Bucci *et al.*, 1993). The stability in a non uniform sampling process, *i.e.* the condition that slight sampling errors lead to slight errors in the reconstructed function, was considered thoroughly by Landau (1967) and by Yao and Thomas (1967). In these papers it was shown that a «stable» sampling cannot be made with an average «cadence» which is lower than Nyquist's, irrespective of the position of the sampling points. When the average sampling velocity is actually equal to Nyquist's – in order to guarantee stability – the sequence of sampling points must be sufficiently regular:

1) it must be possible to associate one uniform point to each non uniform point of the distribution;

2) the distance between the non uniform and corresponding uniform sampling points must be linked by a constant C which is strictly lower than $Q/4$, where $1/Q$ represents Nyquist's sampling speed.

When the sampling speed is strictly greater then Nyquist's, the constant C of condition 2 can be chosen arbitrarily, provided that the distance between the samples is greater than zero. Yen (1956) reports some closed form expressions which allow the interpolation of limited bandwidth functions from samples which are not uniformly distributed.

However, what has been said above is only valid in the following particular cases of distribution:

a) a distribution obtained from the migration of a finite number of sampling points to any other uniform distribution;

b) a distribution obtained by translating all the samples of an arbitrary quantity from the same quantity;

c) a recurring non uniformity distribution.

However, in the first case and in the case of highly non uniform distributions the sampling function can have very high values in the intermediate sampling values. The case of 2-dimensional non uniform sampling has been dealt with less in literature and no analogous results are available as for 1-dimensional sampling. Landau (1967) demonstrated that even in the 2-dimensional case stable sampling is not possible at speeds lower than Nyquist's. Recently, (Rahmat-Sami and Cheung, 1987) two basically similar sampling techniques were presented (Sankur and Gerhardt, 1973), starting off from Yen's results – for the interpolation of limited bandwidth functions from irregularly distributed samples. Another important result which was highlighted in a recent paper (*e.g.*, Bucci *et al.*, 1993), demonstrated that in order for the interpolation algorithm to be optimized from the point of view of computational reliability and efficiency – and also to prove stable and accurate – it is preferable to recover the uniform samples from those which are irregularly equidistant rather than using a direct interpolation formula. In further studies it would be worth considering sensitivity to error. Let us consider the interpolation from non ideal samples, *i.e.* those which are affected by random amplitude and phase errors, by simulating a series of real measured data. For this purpose the stability of the algorithm developed can be tested by adding uniformly distributed random errors to the calculated samples of the function in such a way that relative and absolute errors are both simulated. It is taken for granted that there is background noise and that each sample is affected by a percentage error. For example, an error of 1% or 0.1% is taken as the measurement uncertainty. The error maximum will be observed as the number of repetitions in the interpolation calculation and is varied whenever the latter is recursive. A positive result is obtained when the reconstruction error is fairly near the accuracy of the samples.

2. Fractal geometry

Euclidean geometry studies the forms of natural objects by using their likeness to abstract geometric forms of whole topological dimensions. Fractal geometry, on the other hand, admits dimensions which are not whole, *i.e.* between 0 and 1, between 1 and 2 or between 2 and 3 for objects which can be included in a curve, a plane or a cube respectively since the fractionary dimension is intuitively linked to the capacity of the object to fill out the points of the embedding space. So the greater the fractal dimension of any set of points, the greater the coverage of a Euclidean space appears by such a set. Given a Euclidean set in a metric space S one can consider covering this set by means of spheres whose radius is r, so the following definitions of fractal dimensions can be introduced:

1) fractal dimension of contents (of Hausdorff-Besicovitch).

This is the value D for which we have

$$\text{if} \quad d < D \quad \lim_{r \to 0} \inf_{r_m < r} \sum r_m^d = \infty \qquad (2.1)$$

$$\text{and if} \quad d > D \quad \lim_{r \to 0} \inf_{r_m < r} \sum r_m^d = \infty$$

where r equals the radius of the sphere whose centre C is used to cover the Euclidean set and d equals the Euclidean dimension (Mandelbrot, 1987);

2) fractal dimension of covering:

$$\lim_{r \to 0} \inf \frac{\log N(r)}{\log(1/r)} \qquad (2.2)$$

where $N(r)$ equals the lower number of spheres with radius r used for the covering;

3) fractal dimension of concentration for a measurement (Mandelbrot, 1982): usable in spaces where a measurement m is defined as «dense» everywhere:

$$\lim_{r \to 0} \inf \lim_{r \to 0} \inf \frac{\log(N(r,1))}{\log(1/r)} \qquad (2.3)$$

where l equals m, the measurement of the space which is not covered by the spheres of radius r;

4) fractal dimension of correlation Dc which, by indicating as $c(x)$ the function which defines the number of pairs of points belonging to the set whose distance is lower than a length x, is defined as follows:

$$Dc = \frac{\log c(x)}{\log (x)} \tag{2.4}$$

the latter is linked to the fractal dimension with the relation $Dc \leq D$.

A characteristic of fractal sets is that they are similar to themselves in that they are invariant (statistically) compared to scale transformations. Another property of fractal objects is their self-affinity which allows them to present invariance in the trend compared to scale transformations, which are different with respect to the coordinates. Fractal behaviour was also noticed in function signals of one ($v(x)$) or more coordinates ($v(r)$, with $Dr = \sqrt{Dx^2 + Dy^2 + \ldots..}$) as in Brownian motion. If $v(x)$ is a signal with fractal characteristics the properties of self-affinity result in:

$$\Delta v \sim \Delta x^H \; \Delta v \sim \Delta r^H \tag{2.5}$$

(in the case of more than one dimension) where H equals the scale factor.

For self-similar and fractal Brownian motion it results that D = 2-H, D = 3-H and D = 4-H in the case of one, two and three dimensions respectively. If the motion is one-dimensional, where $1 < D < 2$, its intersection with the x-axes can be considered by obtaining a fractal dimension set D-1. From the point of view of the spectrum in frequency f, a fractal Brownian signal shows a power-like spectrum which varies according to $f^{-\beta}$, where β is called the spectral exponent. It is possible to link the spectral exponent to the scale factor H by means of the following equation:

$$H = \frac{\beta - 1}{2}. \tag{2.6}$$

Generally, the above relation is extended to all fractal signals, although it does not always strictly apply.

3. Applications of fractal geometry in geophysics

As described in section 1 it is not always possible, to reconstruct a function, such as the Earth's magnetic field, defined on a continuous space by a reasonable set of its samples which were obtained using a certain law. In general, deterministic type rules may be used which often place limits on the topology of the set of samples in order to obtain the conditions to reconstruct the field function up to certain frequencies. Fractal geometry is an alternative way of dealing with the problem as it identifies useful new statistical parameters in order to establish the order of scale magnitude within which it is possible to reconstruct the function, and it also supplies the criteria to reconstruct it.

In geophysics, the patterns of the function which represent the magnitude measured are generally made to correspond to the values of geophysical signals. This operation is performed using an electronic processor which permits a visible representation of the signal in question by following various procedures. The intrinsic scale, the resolution and the radius of the reconstruction margin of the represented signal are subject to interpretation and assessment according to several applications of the theory of fractals. There can be various topologies of a set of geophysical measurements. Generally, regular lay-outs are preferred (grids, equal intervals) but very often uncontrollable factors intervene which upset the regularity of the set (*e.g.*, figs. 1a and 2a) making it necessary to carry out an interpolation scheme in order to obtain the regularity of the spatial sampling again. The irregular and uncertain component of the distribution character of a real set of measurements would hint at fractal type behaviour by the points of measurement. Korvin *et al.* (1990) and Lovejoy *et al.* (1986) considered the fractal character of sets of measurements by establishing the fractal dimensions of the South Australian network of gravimetric measurements and of the world meteorological network, quantifying them with the values of 1.4 and 1.75 respectively. They also compared the fractal dimension of a network of fixed stations with the covering efficiency of the same.

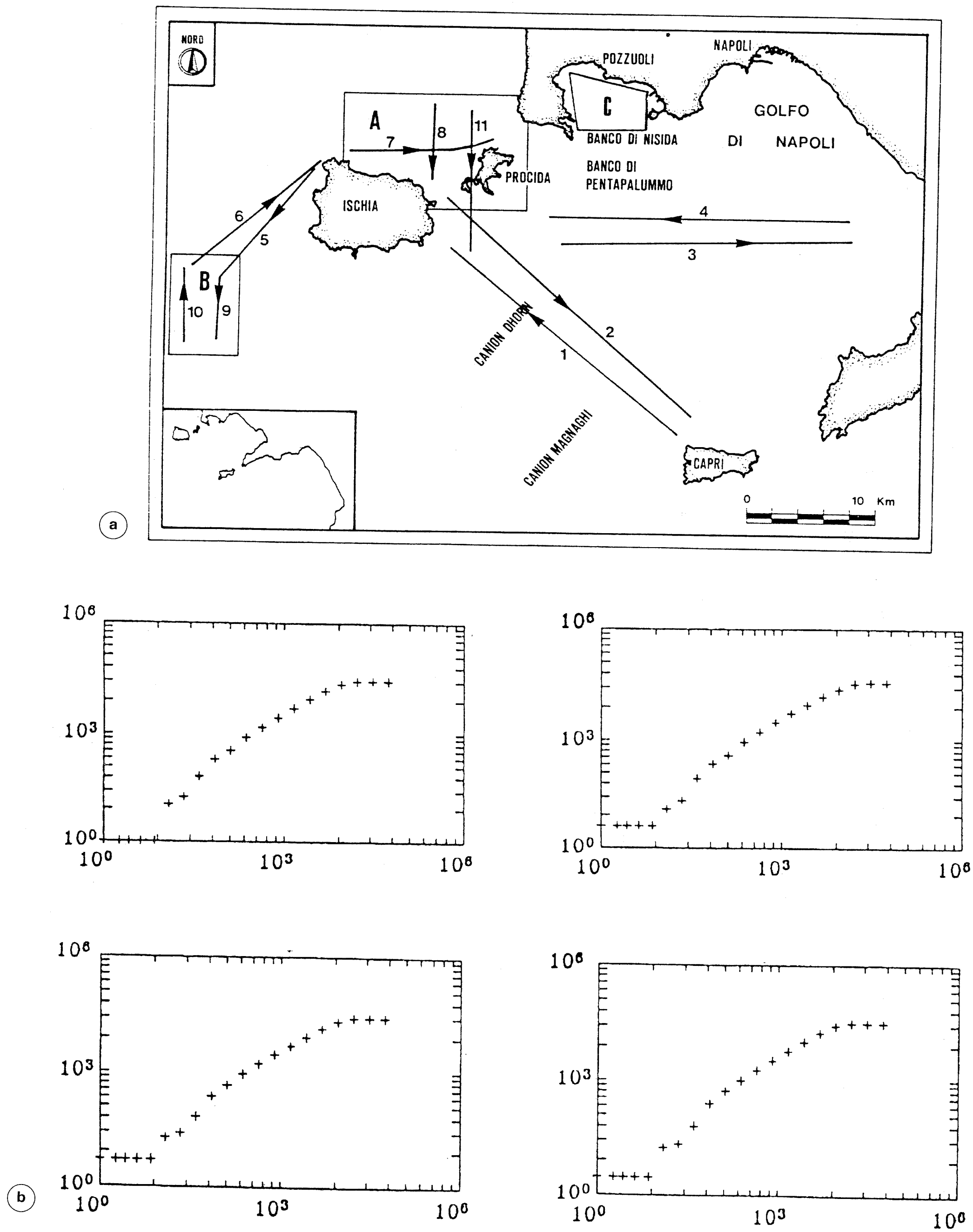

Fig. 1a,b. a) General map of the magnetic survey area; b) correlation dimension function of measurement point distribution along the lines 1, 2, 3 and 4. Horizontal axes show the distances L from the line starting-point; vertical axes show the number of distances less than L.

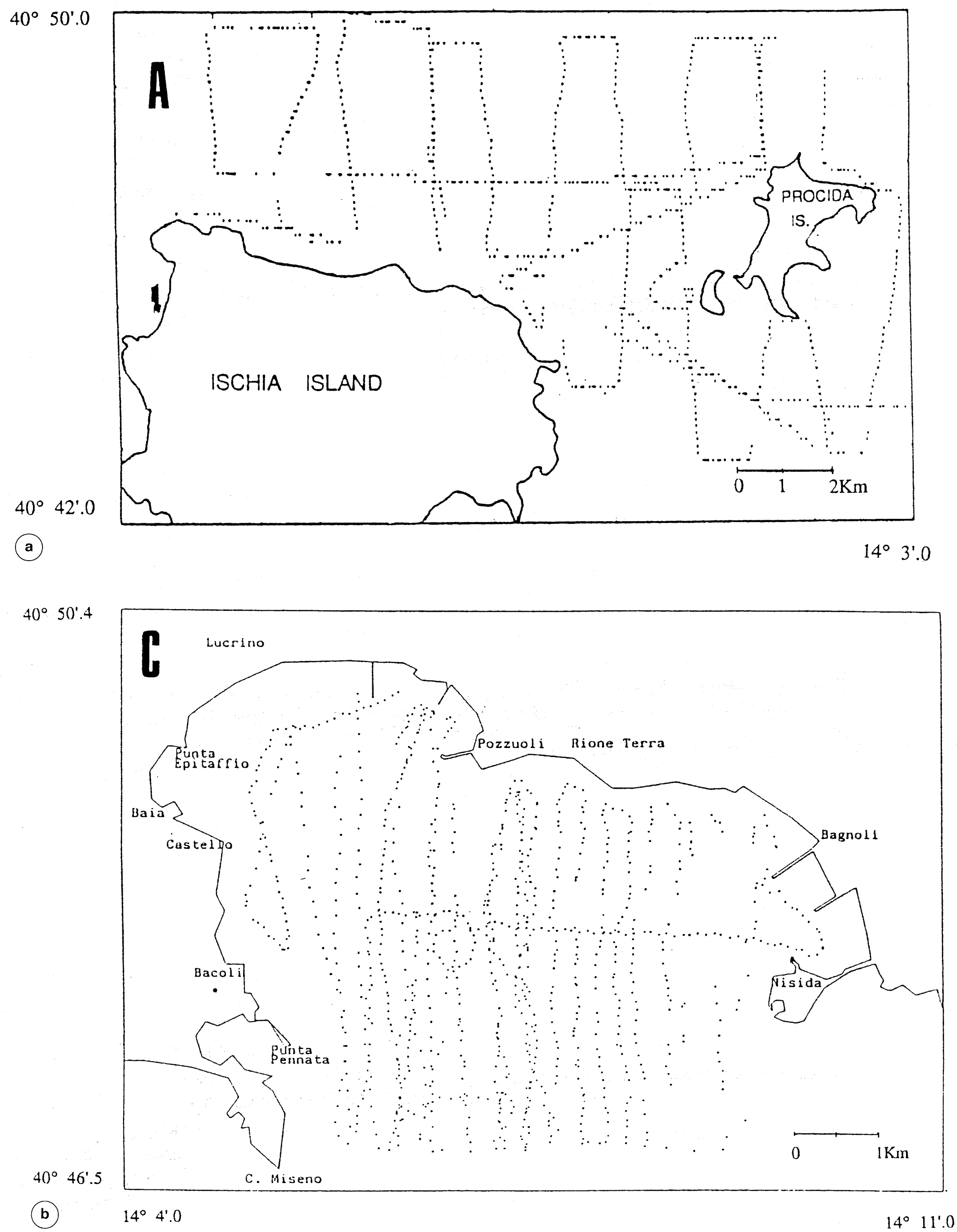

Fig. 2a,b. Spatial distributions of measurements points a) and b) taken in sub-areas «A» and «C» respectively.

It also turns out that many geophysical quantities behave like fractals in both their temporal and spatial trend. Turcotte (1989) pointed out the fractal character of the trend of the Earth's magnetic field over time. Mareshall (1989) used the fractal theory in the topographical reconstruction of sea-floors.

4. Fractal characteristics of sets of measurement points

Over the past few years we have been carrying out small scale measurements of the Earth's Magnetic Field (EMF) on the surface of the sea in one (measurements along profiles) and two (arrays) dimensions. The sea conditions and the inaccessibility of some places due to the presence of ships, islands or rocks have led to the set of measurement points appearing slightly irregular and incomplete in places. In the reconstruction phase of the magnetic field function, the conditions of Shannon's theorem often fail and there is the necessity to use new statistical parameters to define the order of magnitudes within which the actual scale lengths of the measurement points can be defined (Giordano, 1992). The fractal dimension is one possible parameter as it allows a minimum and a maximum scale magnitude to be identified. Figure 1a contains a few sets of measurements, which are one-dimensional: lines 1÷4 and the relative dimension functions, fig. 1b. A scaling regime can be deduced from the latter in an interval L_{min}-L_{max} of approximately 200-20000 m. The lower limit L_{min} identifies the actual spatial resolution of the set of measurements. Table I summarizes the values obtained for several of the sets of measurements.

Table I. Scaling values obtained for the measurement's lines.

Lines	D	L_{min} (m)	L_{max} (m)
1	0.988 ± 0.006	140 ± 35	20000 ± 12500
2	0.977 ± 0.004	140 ± 35	20000 ± 12500
3	0.972 ± 0.003	120 ± 35	20000 ± 12500
4	0.976 ± 0.004	222 ± 45	20000 ± 12500

D is the fractal dimension estimated in the linear part of the log-log plots of fig. 1b.

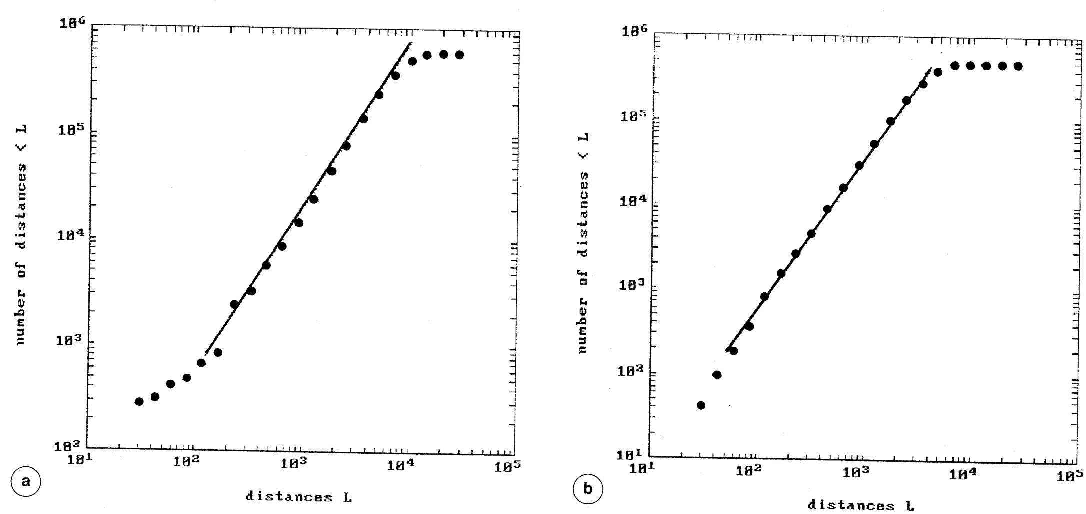

Fig. 3a,b. Dimension correlation fuction: a) for sub-area «A» distribution points and b) for sub-area «C», angular coefficients of the two interpolating lines are respectively $D_c = 1.567 \pm 0.008$ for distribution «A» and $D_c = 1.78 \pm 0.01$ for distribution «C».

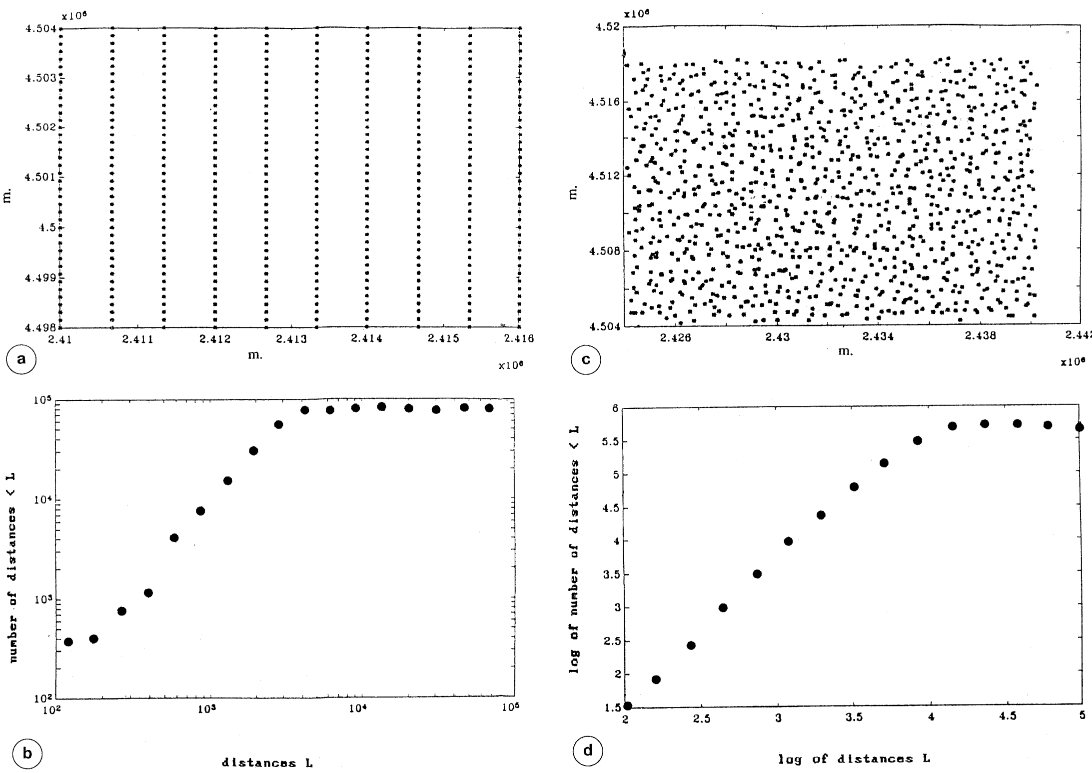

Fig. 4a-d. Two synthetic distributions, with the same number (1000) of measurement points, with their correlation dimension functions: a) an ideal topology of network of marine measurements with a step of 150 m in vertical and 660 m in horizontal respectively; b) the corresponding correlation function with dimension $D_c \cong 1.82$; c) a uniform random distribution of points and d) corresponding correlation function with $D_c \cong 2$.

We also developed the same type of analysis for sets of two-dimensional covering measurements (for the order of the distances at stake the Earth's surface can be considered to be flat) and these were both real (fig. 2a,b) and synthetic (fig. 4a,c).

In order to create the latter we tried to reproduce the ideal topology of a network of marine measurements by presenting different spacings in the two directions with respect to the coordinated axes (fig. 4a). The trend of the correlation function, fig. 4b, presents a much more obvious inclination as the scaling difference increases along the x-axes with respect to the inclination along the ordinates. The length values at the lower knees of both slants correspond to the sampling values along the two directions of the coordinated axes. When the sampling rate along x equals that along y (uniform grid) the fractal dimension is close to the Euclidean embedding dimension (D = 2). If a minor uncertain disturbance is introduced into the sampling module on the points of a regular measuring grid, the fractal dimension decreases as the disturbance module increases. However, this behaviour is maintained as long as the measurement topology prevails over the topology of the uncertain disturbance; if the character of the topology becomes totally random, fig. 4c, the fractal dimension approximates the Euclidean dimension, fig. 4d. Figure 2a,b shows the topologies of two real sets of measurements (sub-areas «A» and «C» of fig. 1) from samplings of the Earth's magnetic field on the the surface of the sea, between the islands of Ischia and Procida (fig. 2a) and in the gulf of Pozzuoli (fig. 2b) and here the fractal dimensions are 1.56 and 1.77 respectively (fig. 3a,b). In these cases the phenomena of dimensions less than 0.44 and 0.23 in the first and second case respectively, cannot be reconstructed (Lovejoy *et al.*, 1986).

5. Spatial trend of the Earth's magnetic field

In general, an $s(x)$ type phenomenon can be said to be fractal if the relation $\Delta s \sim \Delta x^H$ is respected, *i.e.* if there is a parameter H which is the index of the intrinsic scaling of the phenomenon. In a paper in progress we demonstrated how the examined values of the Earth's magnetic field possessed a scaling regime of their own. The spectral power function of the EMF values measured along linear routes presents a behaviour according to $f^{-\beta}$ in the same way as for fractal signals. The fractal dimension of the phenomenon was assessed from the slants of these spectra on the frequency intervals corresponding to the wavelengths identified by the scale regime of the EMF values. Table II shows the β values with a 95% approximation and the calculated values, by eq. (2.5), of the scale factor, H'. The last column of the table shows the H values of the scale factor calculated by linear interpolation on intervals which are congruent to the frequencies being examined on the results obtained by means of a suitable algorithm working on the relation (2.5) to determine the scaling factor, H, straight for measured field values. The forced approximization on the values of β depends on the fact that the signals being dealt with were not filtered. However, the scaling

Table II. Values of β and scale factor calculated with two different methods (see text).

Lines	β	H'	H
1	2.40 ± 0.08	0.70 ± 0.04	0.704 ± 0.004
2	2.1 ± 0.1	0.57 ± 0.06	0.631 ± 0.002
3	1.8 ± 0.2	0.4 ± 0.2	0.721 ± 0.005
4	2.0 ± 0.3	0.5 ± 0.2	0.634 ± 0.004

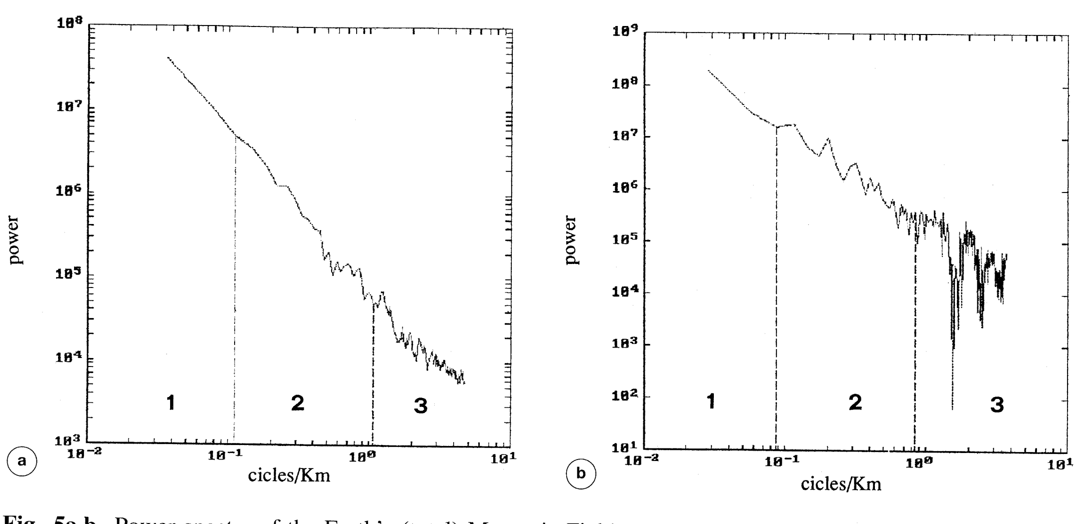

Fig. 5a,b. Power spectra of the Earth's (total) Magnetic Field sampled along lines 1 and 2, respectively. The spectral analysis highlights three regions: the first, «1», concerns the frequencies of the regional field; the second, «2», corresponds to waveforms with fractal regime, the third «3» shows the region where the noise predominates.

values H' calculated from them are comparable – in the context of the approximations adopted – with the H values which were extrapolated in the field of the wavelengths. The spectral analysis, fig. 5a,b, highlights three regions: the first concerns the frequencies of the regional field «1»; the frequencies of the second, «2», correspond to wave forms with a fractal regime; the third, «3», demonstrates signals of disturbance as noise predominates.

6. Conclusions

The considerably irregular topology of a set of EMF marine measurements can be analysed using the fractal method and this has been highlighted by establishing the validity of relation (2.5) in a paper currently being prepared. This allows a minimum scale length, $L_{\min}$, and a maximum scale length, $L_{\max}$, to be identified, and both of these can be correlated to the minimum and maximum wavelengths which can be obtained and used in the interpolation of the EMF values measured. The spatial trend of the EMF shows fractal behaviour in the context of a specific bandwidth of wavelengths and this allows the EMF spectra signals to be analysed by evaluating their slopes to determine the spectral coefficients in the corresponding frequency intervals. The values of the spectral coefficients can be compared with the EMF amplitude scaling coefficient values and allow three types of field components to be identified.

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