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Rigorous time domain responses of polarizable media

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Abstract
The scope of this note is to study a model of induced polarization which fits the usually accepted frequency dependent formula of Cole and Cole, but is more general and allows the time domain observations to retrieve the parameters describing the induced polarization phenomena of the medium. By introducing the memory mechanisms, represented by derivatives of fractional order, in the relation between the electric flux density and the electric field and considering the fractional order differential equation which follows, I solve it with mathematically rigorous and closed formulae and compute the responses to a step function, a box, a set of positive boxes and a set of alternating positive and negative boxes. I also introduce a method which retrieves the parameters describing the medium when comparing the theoretical curves with the observed ones. The responses to these signals also allow to estimate the temporary alteration of the medium when repeated positive (negative) signals are input; the response increases (decreases) in amplitude when the signals are all positive (negative), it decreases when the signals are alternatively positive and negative in agreement with the known attitude of the medium to induced polarization.

Key words *constitutive equations – induction – polarization – alterations – Cole-Cole model*

1. Introduction

The polarization of materials was already known to the ancient Greeks but was first quantitatively studied by Faraday. In physics Induced Polarization (IP) was studied by Debye (1928) who recognized in 1912 that all atoms must have displacement polarizability and that this polarization must contribute to the dielectric constant.

The history and development of IP in geoscience is well described by Allaud and Martin (1977); in the nineteenth century IP was first noted by Fox and later by Barus; the first scientific paper is due to Schlumberger (1920), who had previously reported his studies with a patent in 1912, and then discussed it as a current stimulated phenomenon observed as a delayed voltage response in exploration. It is observed mostly where there are pores filled with fluids near deposits of metallic minerals.

Comprehensive reports on the mathematical modelling and mechanisms of IP are those by Wong (1979) and Olhoeft (1985).

The phenomenon has been studied also in the laboratory with results confirming those of the exploration.

The effect of IP is well known. When we inject a continuous current into in the soil, it is believed that a relatively large number of dipoles is formed. These dipoles oppose the passage of the electric current, therefore temporarily increasing the resistivity (charge curve). When the voltage is removed, the dipoles discharge their polarity and generate the observed transient current (decay curve).

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In general the phenomenon of IP is represented mathematically by means of simple laws based on the observation that the charge and decay curves are almost logarithmic.

The Fourier Transforms of these empirical laws are in general of the type proposed by Cole and Cole (1941) who assumed a complex dielectric constant model in the frequency domain and also verified that it satisfies Krönig-Kramer conditions of realizability. (From here the formula of Cole and Cole (1941) or the relative paper or these authors will be indicated with CC). This law is the most useful of all the laws proposed to model IP (Pelton *et al.*, 1983).

The first systematic study of memory formalism in mathematics is due to Liouville (1832) with the introduction of the derivatives of fractional order.

Heaviside (1899), during the last decade of the 19th and the first decade of the 20th century, was the first to study electric phenomena in the frequency domain by means of the operational calculus which corresponds, in the time domain, to the use of fractional order derivatives.

Memory mechanisms to represent dispersive media have been used directly in the time domain by Cisotti (1911), who proved that processes regulated by generic memory mechanisms are irreversible, and by Graffi (1936).

Prior to CC, Gemant (1936) suggested an empirical formula similar to that of CC and used the derivative of order 1/2 in the form of a memory mechanism operating on E (electric field), but he considered it only as a possibility.

In a more recent work Graffi (1962) introduced a generic memory mechanism in the relation between E and D (electric flux density) applying it only to E probably because this allows Maxwell equations to be solved without the use of Laplace Transform (LT).

An extensive study of energy storage in electric networks was been made by Jacquelin (1984, 1988, 1991) using the complex frequency dependent impedence represented by inserting fractional order derivatives in the relation between J (current density of free charges) and E.

Jacquelin's (1984, 1988, 1991) discussion is practically extended to almost all possible circuits; he illustrates his results in the frequency domain at steady state, when the Fresnel terms of the fractional order derivatives are nil, and his technique, based on the observations at many frequencies, relies on convolution for the retrieval of the response in the time domain.

In general we may state that the formalism of this note is now spread to many linear approaches in the studies of dissipative and dispersive properties of many anelastic and dielectric media (*e.g.*, Caputo and Mainardi, 1971; Bagley and Torvik, 1983a, 1983b, 1986; Pelton *et al.*, 1983; Le Mehaute and Crépy, 1983: Jacquelin, 1984: Körnig and Müller, 1989; Caputo, 1989, 1994a,b).

An important work to model the IP was made by Pelton *et al.*, (1983) who gave the interpretation of the CC formula (as well of other formulae) in the time domain by means of derivatives of fractional order. Pelton *et al.* (1983) solved in the time domain the fractional order differential equation in the cases of a delta and of a step input. The time domain solution of Pelton *et al.* (1983) is given by means of series which are practically valid for larger and smaller values of the time measured in units of the relaxation time.

However many aspects of IP modelled by the CC formula, represented in the time domain with constitutive equation for E and D containing fractional order derivatives, have yet to be discussed. It is therefore desirable to obtain rigorous solutions, with rapidly convergent closed form formulae, not only for the step and the boxcar input but also for other types of inputs.

The present approach will be directly in the time domain with special attention to the response to the delta input, which allows solutions to be found also for more complex inputs not considered by Pelton *et al.* (1983) and which may be useful in geophysical exploration.

We will also find the solution to the response to a delta input for a medium governed by a more general constitutive equation containing two fractional derivatives of different order which includes as a particular case that considered by Pelton *et al.* (1983).

This will be implemented by introducing the two fractional order derivatives in the relation between D and E and by solving the fractional order differential equation, representing the constitutive equation, for a delta input, with closed form formulae valid for all positive values of the time and introducing an integral previously used by Caputo (1984) in the solution of anelastic problems.

The case when the two fractional derivatives have the same order will be given special attention and solutions are found when the inputs are the delta, the step, the box, the set of successive positive boxes, the set of alternating positive and negative boxes and the set of saw teeth. The solutions are all expressed by means of closed form formulae.

A method is also given which retrieves from the time domain response to the input of signals of given shape the parameters characterizing the IP phenomena of the medium which was done, with Prony's method, for the case of a box, by Patella *et al.* (1979, 1987).

Besides Pelton *et al.* (1983), many authors have done work in the time domain interpretation of the CC formula. I quoted here only the results essential to the discussion which follows.

2. The *E-D* relation with derivatives of fractional order

Consider that the usual relation $D = \varepsilon E$ is substituted with

$$\alpha \boldsymbol{D} + \gamma \boldsymbol{D}^{(z)} = \sigma \boldsymbol{E} + \varepsilon \boldsymbol{E}^{(w)} \qquad (2.1)$$

where α, γ, ε, σ, w and z are real constants with the appropriate dimensions, with $0 < w \leq z = r/n < 1$, r and n positive integers. The classic Maxwell equations coupled with eq. (2.1) are the equations governing the propagation of electric signals in the medium.

The fractional order derivatives operating on the applied electric field and on induction are here assumed with different order to represent different physical mechanisms.

The derivative of order z of the function $f(t)$ is here defined as the following convolution

$$d^z f(t)/dt^z = (1/\Gamma(1-z)) \int_0^t f'(u)du/(t-u)^z,$$

whose LT is (Caputo, 1969)

$$\mathrm{LT}(d^z f(t)/dt^z) = p^z\, \mathrm{LT}(f(t)) - p^{z-1} f(0).$$

Substituting the LT of (2.1) in the LT of Maxwell equations we find

$$\nabla \times \nabla \times \boldsymbol{e} = -\mu p^2\, [(\sigma + \varepsilon p^w)/(\alpha + \gamma p^z)]\, \boldsymbol{e}. \qquad (2.2)$$

where e is the LT of E.

This equation with $\varepsilon = \gamma = 0$ reproduces the classic case of absence of dispersion.

We have assumed here that $dJ/d\,t = 0$ which gives sufficient generality for our purpose. The results of this note however may be extended to the case when $dJ/d\,t \neq 0$.

A solution to eq. (2.2), obtained by means of separation of variables, is found in Caputo (1993) for the case $z = w$. A solution to eq. (2.2) with $z \neq w$ may be obtained with the same method used for the case when $z = w$.

In some IP works the exponents z and w are equal but the ratio $(\varepsilon p^z + \sigma)/(\gamma p^z + \alpha)$ is elevated to a power inserting again an additional parameter as is done in (2.1) relative to the CC formula. However we prefer formula (2.1) which satisfies the Krönig-Kramer compatibility condition and is easier to handle analytically.

3. The retrieval of the IP parameters and the temporary alterations of the medium

In this section I shall consider a medium whose induction is defined by eq. (2.1), compute the response to the following inputs: a) a step; b) a box: c) a set of positive boxes; d) a set of alternating positive and negative boxes, and discuss how to retrieve the parameters appearing in (2.1).

The LT of eq. (2.1), considering the one-dimensional case, which does not alter the meaning of the results, gives

$$\gamma p^z d + \alpha d = \sigma e + \varepsilon p^w e \qquad (3.1)$$

where d is the LT of D.

Solving for d we find

$$d = e(\varepsilon p^w + \sigma)/\gamma p^z + \alpha). \qquad (3.2)$$

Assuming $p = i\omega$, $\gamma/\alpha = \tau^z$, $\varepsilon/\gamma = \varepsilon_\infty$, $\sigma/\alpha = \varepsilon_0$, and $z = w$ where ε_0 and ε_∞ are the dielectric constants at zero and infinite frequency respectively and τ is a relaxation time, eq. (3.2) coincides with the CC formula. The time domain representation of formula (3.1) was first given by Caputo and Mainardi (1971), in the study of anelastic media, and used by Bagley and Torvik (1983a), by Le Mehaute and Crépy (1983) and by Pelton *et al.* (1983). Bagley and Torvik (1983a,b) already used form (3.2) of the constitutive equation in the frequency domain with two, different exponents for p (Laplace Transform (LT) variable).

To operate in the time domain I rewrite eq. (3.2)

$$d = e\{\sigma/(\alpha + \gamma p^z) + (\varepsilon/\gamma p^{z-w})$$
$$[1 - (\alpha/\gamma)/(\alpha/\gamma + p^z)]\}. \qquad (3.3)$$

Assuming $e = 1$, I find the Green function

$$D_0\{(\sigma/\alpha)\delta(t) - (\varepsilon/\gamma)[t^{z-w-1}/\Gamma(z-w)]\}*$$
$$(\sin \pi z/\pi z)\int_0^\infty (uR)^{1/z} \exp(-(uR)^{1/z} t) du/(u^2 +$$
$$+2u\cos\pi z + 1) + (\varepsilon/\gamma)\, t^{z-w-1}/\Gamma(z-w)$$
$$R = \alpha/\gamma. \qquad (3.4)$$

In the following, to simplify the presentation and to conform to the work of CC we will assume $z = w$; formula (3.4) is then

$$D_0 = (\varepsilon/\gamma)\delta(t) + B(\sin\pi z/\pi z)\int_0^\infty (uR)^{1/z}$$
$$\exp(-(uR)^{1/z} t) du/(u^2 + 2u\cos\pi z + 1)$$
$$B = (\sigma/\alpha - \varepsilon/\gamma). \qquad (3.5)$$

The values of the integral in (3.5), assuming $T = R^{1/z}\, t$ as independent variable and z as parameter are shown in fig. 1.

a) *The response to a step function input*

With $e = 1/p$ in eq. (3.3), we obtain the response to a step field

$$D_a = \varepsilon/\gamma + B(\sin\pi z/\pi z)$$
$$\int_0^\infty (1 - \exp(-(uR)^{1/z} t)) du/(u^2 + 2u\cos\pi z + 1). \qquad (3.6)$$

$D_a(0) = \varepsilon/\gamma$ and $D_a(\infty) = \sigma/\alpha$ give meaning to the ratios ε/γ and σ/α; specifically the ratio $\varepsilon/\gamma = \varepsilon_\infty$ is the step response to $E(\infty) = 1$ or the response at infinite frequency according to eq. (3.2) and $\sigma/\alpha = \varepsilon_0$ is the asymptotic value to $E(\infty) = 1$ or the response at zero frequency.

Formula (3.6) is useful for the computation of the parameters $D_a(0) = \varepsilon/\gamma$ and $D_a(\infty) = \sigma/\alpha$ appearing in CC formula where it is assumed that $\alpha = 1$, $p = i\omega$. The third parameter appearing in CC formula is $1/R = \tau^z = \gamma/\alpha$. The recording of eq. (3.6) gives directly $D_a(0) = \varepsilon/\gamma$ and $D_a(\infty) = \sigma/\alpha$, while the fitting of eq. (3.6) to the data gives $\alpha/\gamma = R$, B and z from which τ is obtained.

b) *The response to a box input*

A case of practical interest is the response to an applied E with a form of a box of unit amplitude and duration T (fig. 2(b)); from eq. (3.5) it is seen that this response is

$$D_{b1}(t) = \varepsilon/\gamma + B(\sin\pi z/\pi z)$$
$$\int_0^\infty (1 - \exp(-(uR)^{1/z} t)) du/(u^2 + 2u\cos\pi z + 1)$$
$$\text{for} \quad 0 < t < T \qquad (3.7)$$

$$D_{b2} = B(\sin\pi z/\pi z)\int_0^\infty (\exp(-(uR)^{1/z}(t-T)) +$$
$$-\exp(-(Ru)^{1/z} t)) du/(u^2 + 2u\cos\pi z + 1)$$
$$\text{for} \quad T < t,$$

which is shown in fig. 3.

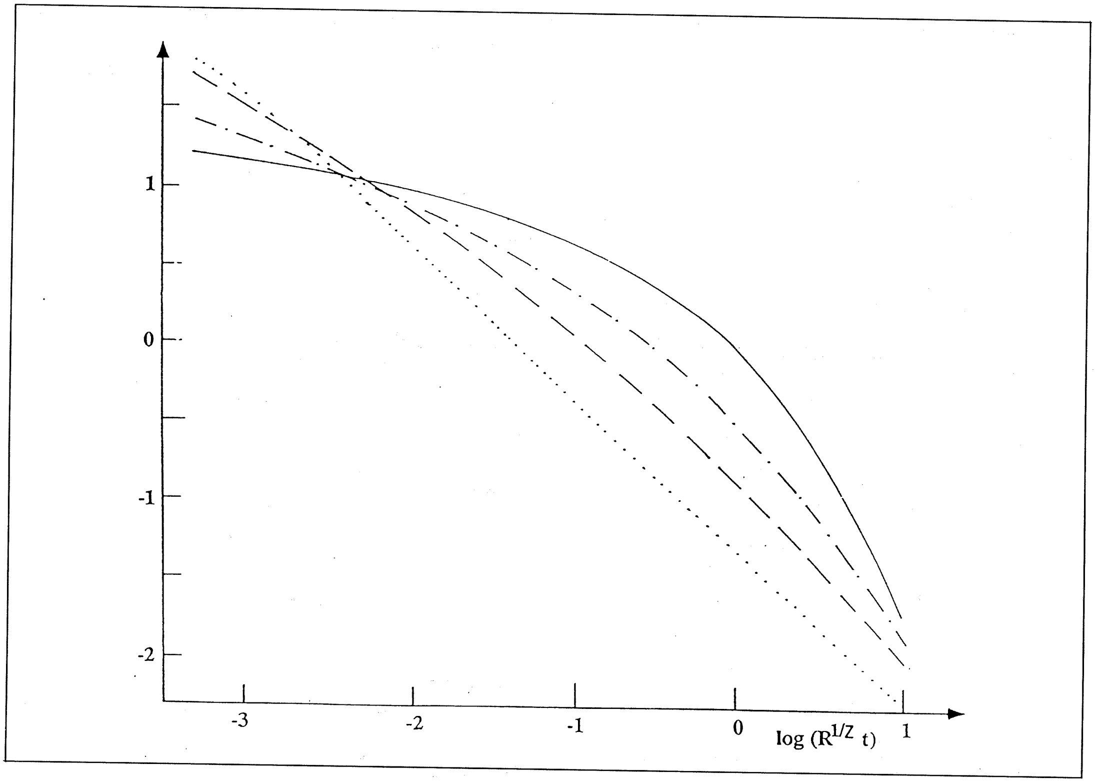

Fig. 1. Derivative of the induction response to a unit step field (eq. (3.5)) for z = 0.2 (dotted curve), z = 0.4 (dashed curve), z = 0.6 (dotted dashed curve) and z = 0.8 (solid curve). The ordinate is in units of $(\varepsilon_0 - \varepsilon_\infty)$ $(\sin \pi z)/\pi z$.

In order to verify the values of the parameters B, z, and τ resulting from the fitting of the theoretical curves to the observed one it is useful to recall that theoretically we have

$$\lim_{t\to\infty} D_{b1}(t) = \sigma/\alpha, \quad D_{b1}(\infty) - D_{b1}(0) = B,$$

$$D_{b1}(T) - D_{b2}(T) = \varepsilon/\gamma$$

$$D_{b1}(T) - D_{b1}(0) = D_{b2}(T), \quad D_{b1}(0) = \varepsilon/\gamma. \quad (3.8)$$

Considering that $D_{b2}(2T) \neq 0$, from the third of (3.8) it follows that the load response curve for $t < T$ and the initial part of the unload curve for $T < t < 2T$ may not be superimposed by means of a rotation and a translation.

If it may be considered that T is sufficiently larger than τ so $D_{b1}(T)$ practically coincides with the asymptotic value of the response to a step load then, from $D_{b1}(\infty) = D_{b1}(T)$, it follows that:

$$D_{b1}(T) = \sigma/\alpha$$

$$D_{b2}(T) = \sigma/\alpha - \varepsilon/\gamma.$$

In practice however many IP experts use the observed data for $t \geq T$ only (*i.e.*, Patella *et al.*, 1987). In this case one may retrieve the theoretical values of τ, B and z using the following method.

A double infinity of Theoretical Curves (TC) for $z = q\Delta z$ and $\tau = m\Delta\tau$, with q and m integer, is

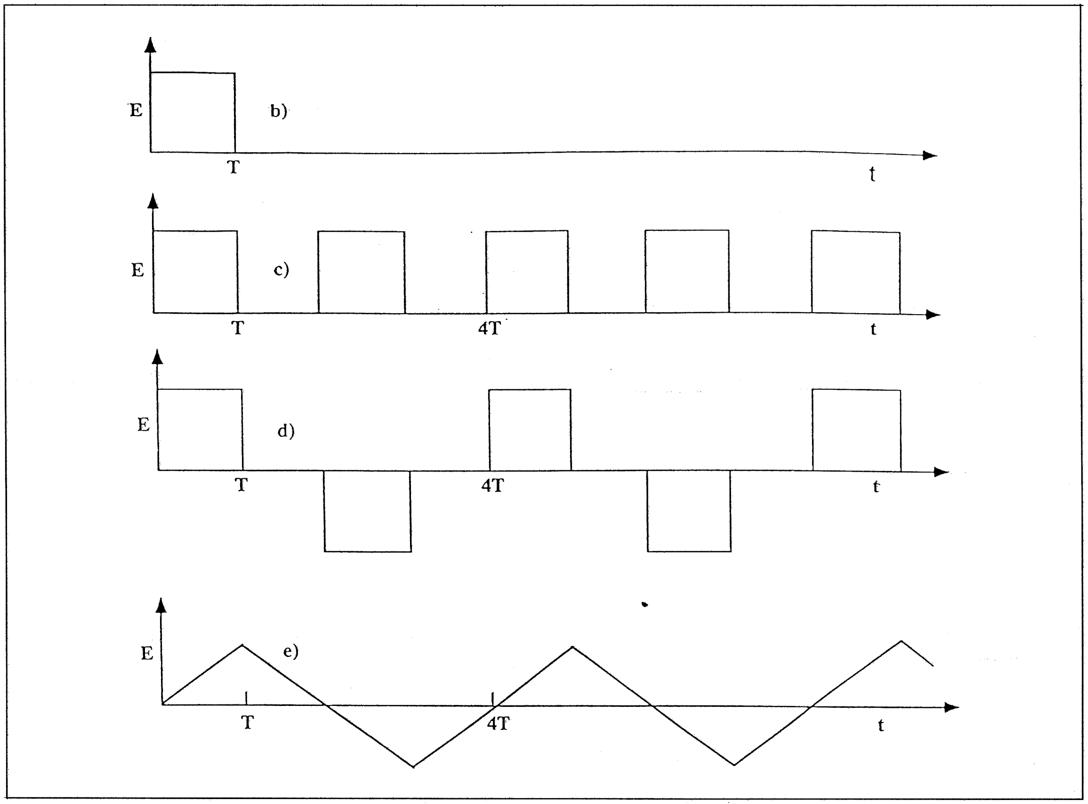

Fig. 2. Shapes of the electric field E considered as input in eq. (3.2); b) a box of duration T; c) a set of positive boxes of duration T and separated by T; d) a set of boxes of positive and negative sign of duration T and separated by T; e) a set of saw teeth alternatively of positive and negative sign (only for the case $\alpha = 0$).

computed from the second of (3.7) for $t \geq T$ and the values defining the points of each curve are stored in the memory of the computer.

Then the value of B is assumed for each theoretical curve such that for $t = T$ the theoretical curve has the same value as the observed one. At this stage an automatic search is made with the following procedure in order to find which theoretical curve has the closest match to the observed one.

To this purpose a generic point in the grid $z = q\Delta z$ and $\tau = m\Delta\tau$ is selected, and the Mean Square Difference (MSD) computed between the assumed TC and the Curve of the Observed Data (COD); then by trial and error the computer finds which of the four possible directions of the grid in the point selected has the largest decrease of the MSD between the TC and the COD curves.

The procedure is then automatically repeated until a relative minimum of the MSD is reached.

But one may reasonably increase the number of TC in the set of those matching the COD by taking into account the experimental errors. In fact one may estimate the mean square error of the experimental curve (MSE) and, during the search for the minimum

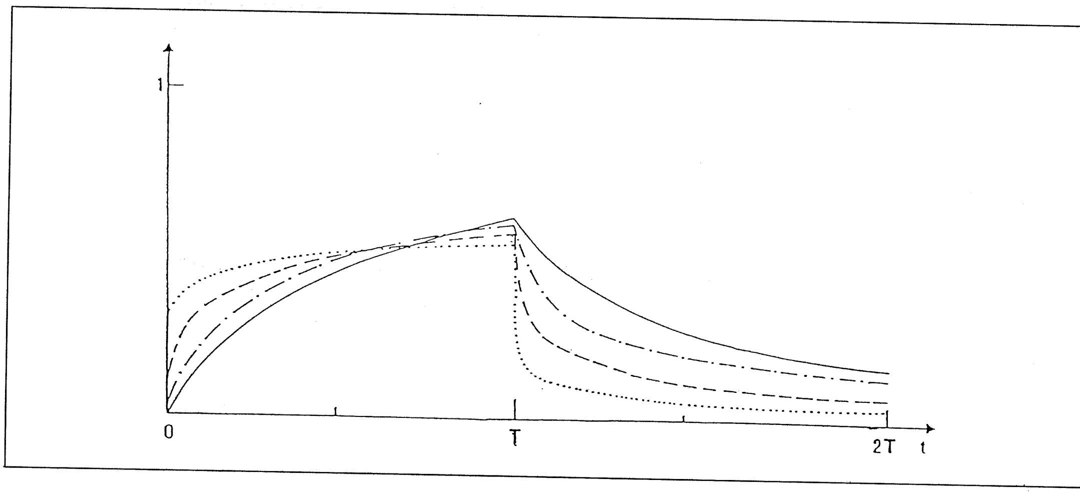

Fig. 3. Induction in a medium with constitutive eq. (3.2) and caused by a box of unit amplitude and duration T (see fig. 2(b)). The time is in units of the relaxation time $\tau = R^{-1/z} = (\alpha/\gamma)^{-1/z}$. The curves are for $z = 0.2$ (dotted curve), $z = 0.4$ (dashed curve), $z = 0.6$ (dotted dashed curve), $z = 0.8$ (solid curve). The constant term ε/γ for $0 < t < T$ is omitted. The ordinate is in units of $(\varepsilon_0 - \varepsilon_\infty)$ $(\sin \pi z)/\pi z$.

of the MSD, a number of TC which have MSD ≤ MSE is obtained. In principle, one should consider as acceptable all the TC which have MSD ≤ MSE.

Obviously, as was already found (Gasperini and Caputo, 1979), it is possible that one finds more than one relative minimum for the MSD and, therefore, other sets of TC with MSD ≤ MSE which belong in a domain of the grid $z = q\Delta z$ and $\tau = m\Delta\tau$ not connected with the domain of the grid already found; it is the priviledge of the experimenter to foresee which of the curves selected by the method are the most appropriate to model the medium under study.

The method of finding with a random walk all the theoretical curves with MSD ≤ MSE and accepting them as physically acceptable (known as the Hedgehog method) has been successfully applied in geophysics especially in the studies of the Earth's surface waves where, instead of the 2D space (that of the grid $z = q\Delta z$ and $\tau = m\Delta\tau$), one uses a space with a larger number of dimensions where the models of the thickness and velocity of the layers forming the asthenosphere and lithosphere are represented.

The variant of the Hedgehog method suggested here consists in guiding the computer to the minimum MSD.

Experimentally, however, the input E is not exactly a box and, for instance, its rise and decay times are not nil. To take this into account one may then compute analytically the response curve to the actual input obtaining a formula similar to (3.7) and proceed as previously suggested.

c) *The response to a set of positive boxes input*

Another case of practical interest, in order to see the alteration of the medium, is that when a set of successive positive boxes, with the same sign and duration T and separated by T are applied (fig. 2(c)). Convolving this function with eq. (3.5) we find, in the case when we consider the n-th box, that the response is a box of amplitude ε/γ, given by the $\delta(t)$ term of eq. (3.5), to be added to the integral

$$D_{cn} = B\,(\sin \pi z/\pi z)$$

$$\int_0^\infty (\exp(-vt))\left(\sum_0^{n-1} \exp(2ivT)\right)(\exp(vT)-1)+$$

$$+(1-\exp(-v(t-2(n-1)T)))\,du/(u^2+2u\cos\pi z+1)$$

$$v=(uR)^{1/z}, \qquad 2(n-1)T<t<2(n-1)T+T \tag{3.9}$$

which has the sign of $\sigma/\alpha-\varepsilon/\gamma=\varepsilon_0-\varepsilon_\infty$.

By differentiating eq. (3.9) with respect to t it is seen that, for t within the box, the function (3.9) increases.

Assuming $t = 2(n-1)T$ and $t = 2nT$ which give the ordinates of the initial points of the n-th and of the $(n+1)$-th box respectively, it is seen that $|D_{cn}(2(n-1)T)| < |D_{c(n+1)}(2nT)|$ for all n.

That is the modulus of the ordinate at the initial points of the boxes increases. The same property applies also to the end points.

It is also seen that for n increasing the amplitude of the boxes converge to a finite value.

We may then conclude that if the sign of $\sigma/\alpha-\varepsilon/\gamma=\varepsilon_0-\varepsilon_\infty$ in eq. (3.5) is positive the response is formed by successive boxes increasing in amplitude, if that sign is negative the successive boxes of the response are of decreasing amplitude; the laboratory data show that the sign of $\sigma/\alpha-\varepsilon/\gamma=\varepsilon_0-\varepsilon_\infty$ is positive. This indicates how the medium temporarily alters its attitude to be polarized.

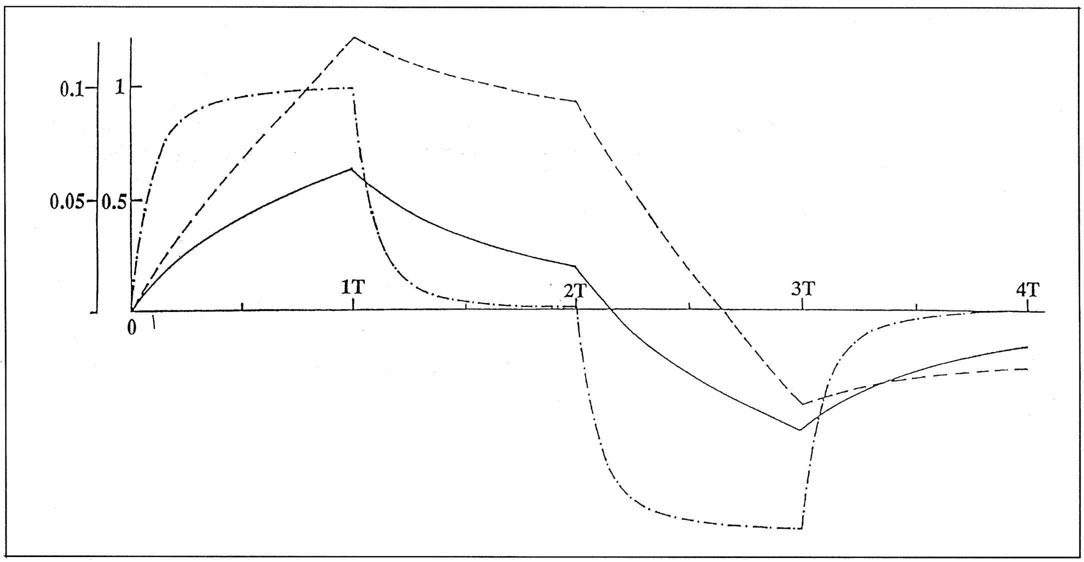

Fig. 4. Induction in a medium with a dielectric constant characterized by eq. (3.2). The input is an electric field of the type (d) of fig. 2 with $z = 0.9$ and the relaxation time is $\tau = 0.1\ T$ (dotted dashed line), $\tau = T$ (solid line) and $\tau = 10T$ (dashed line). The abscissa is in units of T and the ordinate is in units of $(\varepsilon_0-\varepsilon_\infty)$ $(\sin \pi z)/\pi z$. The scale to the far left is for the dashed line, the other is for the two other lines. In order to have total induction the curves shown must be added to a curve of the type (d) of fig. 2 with amplitude given by the ratio ε/γ, which generates discontinuities at the times $t = T$, $t = 2T$, $t = 3T$, but leaves the curved shape of the lines unchanged.

In an inverse experiment the three ratios $\alpha/\gamma = \tau^{-z}$, σ/α, ε/γ and the exponent z characterizing (3.5) may be retrieved fitting the curves determined experimentally to curves of the type shown in fig. 4. Considering the n-th box, as we mentioned, the response curve is a box of amplitude ε/γ to be added to a function of the type (3.9); amplitude of the box gives the ratio ε/γ, the fitting the curves of the type shown in fig. 4 gives B, z and the relaxation time τ.

d) *The response to a set of alternating positive and negative boxes input*

In exploration the unit amplitude boxes of duration T of the set considered are actually alternatively positive and negative and each box is separated by the next by a time interval of duration T during which the applied field is nil (see fig. 2(d)).

For the applications the response should therefore be computed for four successive time intervals T. Convolving the signal with eq. (3.5) we obtain a repetition of the signal with amplitude ε/γ as a result of the $\delta(t)$ term of eq. (3.5) to be added to

$$D_{dn} = B_1 \int_0^\infty (\exp(-vt))(\exp(vT)-1)((1-\exp(2vT)) \left(\sum_{j=0}^{n-1} \exp(4jT)\right) + \exp((4n-2)vT)) + (\exp(vt) - \exp(4nT)))\, du/(u^2 + 2u\cos\pi z + 1),$$

$$\text{for } 4nT < t < (4n+1)T; \qquad (3.10)$$

$$D_{dn} = B_1 \int_0^\infty (\exp(-vt))(\exp vT - 1)((1 - \exp 2vT) \left(\sum_{j=0}^{n-1} \exp 4jT\right) + \exp 4nT)\, du/(u^2 + 2u\cos\pi z + 1),$$

$$\text{for } (4n+1)T < t < (4n+2)T;$$

$$D_{dn} = B_1 \int_0^\infty (\exp(-vt))((\exp(vT)-1)((1-\exp(2vT)) \left(\sum_{j=0}^{n} \exp(4jT)\right) + \exp(4nT)) + -\exp(vt) + \exp((4n+2)T))\, du/(u^2 + 2u\cos\pi z + 1),$$

$$\text{for } (4n+2)T < t < (4n+3)T;$$

$$D_{dn} = B_1 \int_0^\infty (\exp(-vt))(\exp(vT)-1)((1-\exp(2vT)) \left(\sum_{j=0}^{n} \exp(4jT)\right) du/(u^2 + 2u\cos\pi z + 1)$$

$$B_1 = B\sin\pi z/\pi z,$$

$$\text{for } (4n+3)T < t < (4n+4)T.$$

Formulae (3.10) reproduce the shape of the response, shown in fig. 4 obtained in exploration and give another method of retrieving the parameters of eq. (3.2) when the set of inputs of the type of fig. 2(d) are applied.

4. The case of $\alpha = 0$

In rheology, expression (3.2) is used in the one-dimensional case as the parameter relating strain to stress because it gives a rheological model which fits many laboratory data (Bagley and Torvik, 1983a, 1983b, 1986).

Formula (3.2) is more general than that of CC because of the presence of the parameter α appearing in the denominator and of the presence of two fractional derivatives of different order ($z \neq w$); the two formulae coincide when $\alpha = 1$ and $z = w$.

The case $\alpha = 0$ is of great interest in the linear rheology of the very slow motions occurring in the Earth, because it gives the model best fitting the observed geological phenomena (Körnig and Müller, 1989); this case, which has a very simple mathematical solution, would escape CC formulation.

I shall consider the case $\alpha = 0$ now to compute the responses and estimate the alteration of the medium when applying a step input (a) and a set of saw teeth input of the type shown in fig. 2(e).

With $e = 1$ in eq. (3.2), I obtain

$$D_{00} = (\varepsilon/\gamma)\,\delta(t) + (\sigma/\gamma\Gamma(z))\,t^{z-1}, \quad (4.1)$$

which is the Green function.

a) *Response to a step input*

For the retrieval of the parameters σ/γ, ε/γ and z characterizing the medium it is sufficient to observe the response to a step input; in fact from eq. (3.2), with $\alpha = 0$, substituting $e = 1/p$ we obtain the linear relation in ln t

$$D_{0a}(0) = \varepsilon/\gamma$$

$$\ln (D_{0a}(t) - D_{0a}(0)) = \ln (\sigma/\gamma\Gamma(1+z)) + z \ln t \quad (4.2)$$

where the initial value $D_{0a}(0) = \varepsilon/\gamma$ and the values of z and σ/γ are obtained by fitting the observed data.

e) *The response to a set of saw teeth input*

To see the alterations of the medium we consider the case when E is represented by a function $S(t)$ nil for $t < 0$ and formed by an infinite set of saw teeth with nil initial and mean value, with period $4T$ and slope b (fig. 2(e)). $S(t)$ also shows the deviations from linear behaviour; the convolution with eq. (4.2) gives the saw teeth term with period $4T$ and amplitude ε/γ, plus the following function $D_{0e}(t)$

$$D_{0e}(t) = bt^{z+1}/(z+1), \quad \text{for } 0 < t < T, \quad (4.3)$$

$$D_{0e}(t) = b(t^{z+1} - 2(t-T)^{z+1})/(z+1), \quad \text{for } T < t < 3T,$$

$$D_{0e}(t) = b(t^{z+1} - 2(t-T)^{z+1} + 2(t-3T)^{z+1})/(z+1), \quad \text{for } 3T < t < 5T,$$

$$D_{0e}(t) = b(t^{z+1} + \sum_{m=1}^{n} (-1)^m 2(t-(2m-1)T)^{z+1})/(z+1), \quad \text{for } (2n-1)T < t < (2n+1)T.$$

The function $D_{0e}(t)$ has maxima, at $t = (4q-3)T$, with q positive integer, which decrease with increasing q indicating an alteration of the polarization properties of the medium which decreases its attitude to be polarized.

It is noted that the decreases of the moduli of the amplitudes of the successive maxima and minima are converging to zero.

The variation of the maximum from the $(q+1)$-th cycle to the $(q+2)$-th is

$$T^{z+1}(-(4q+5)^{z+1} + (4q+1)^{z+1} + 2((4q+4)^{z+1} - (4q+2)^{z+1}))/(z+1), \quad (4.4)$$

which is positive; the sum of these variations

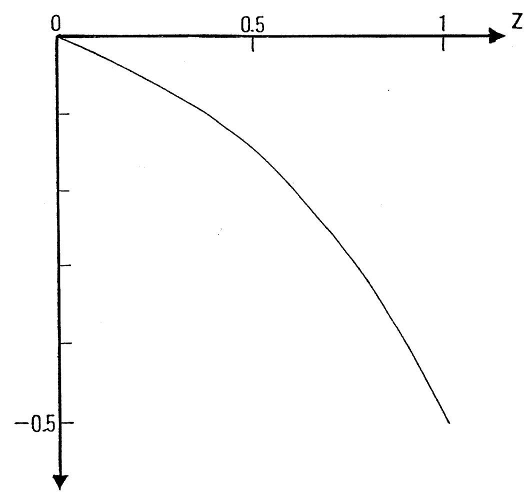

Fig. 5. Sum of the successive decreases of the induction of a medium represented by constitutive eq. (3.2) with $\alpha = 0$ and caused by an infinite set of saw teeth (see fig. 2(e)), as a function of the order of fractional differentiation z. The values of the curve have been obtained dividing the values of formula (4.5) by T^{z+1} where T is the period of the saw teeth.

after an infinite number of cycles is obtained assuming q as a continuous variable and integrating eq. (4.4) from zero to infinity; I find

$$T^{z+1}(-1+5^{z+2}-2\,(4^{z+2}-2^{z+2}))/4\,(2+z)\,(1+z), \qquad (4.5)$$

which is positive, limited for $0 < z < 1$ and plotted in fig. 5.

5. Conclusions

The time domain representation of the usually accepted CC formula is generalized with the introduction, in the formula, of two fractional derivatives of different order; the resulting fractional order differential equation is solved obtaining its Green function (response to a delta input) which allows the solutions to a large variety of inputs to be found.

The mathematical approach used obtains the mathematically rigorous and closed formulae representing the response of the medium to a step input, to a box, to a set of positive boxes and to a set of alternatively positive and negative boxes in the time domain which may be used to compute and check the parameters appearing in the CC formula relative to the medium under consideration.

It is shown how the introduction of the memory formalism represented by derivatives of fractional order in the classic relation $D = \varepsilon E$ is representative of a temporary different attitude of the medium to be polarized; special attention is given to the transients of many signals of interest in geoelectric prospecting.

When $\alpha = 0$ it is seen that if periodic positive (negative) signals are applied, the medium decreases temporarily its attitude to IP, when the periodic signals are alternatively positive and negative this attitude increases.

The rate of change of the temporary variation of the attitude to IP is proportional to $B = (\varepsilon_0 - \varepsilon_\infty)\,(\sin \pi z)/\pi z$, is a function of the number of elapsed cycles and converges to zero when steady state is reached.

In the case $\alpha = 0$ the number of arbitrary parameters is only 3, namely ε_0 and ε_∞ and z; the shape of the response to a step seems to fit the field data well and the retrieval of the parameters describing the medium seems possible through the response to this simple input.

Finally the new method to retrieve CC parameters from the observation of the response to an input box allows the variety of parameters which represent the IP phenomena of the medium and are physically acceptable to be discussed.

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