Field lines of gravity, their curvature and torsion, the Lagrange and the Hamilton equations of the plumbline

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Abstract
The length of the gravitational field lines/of the orthogonal trajectories of a family of gravity equipotential surfaces/of the plumbline between a terrestrial topographic point and a point on a reference equipotential surface like the geoid – also known as the orthometric height – plays a central role in Satellite Geodesy as well as in Physical Geodesy. As soon as we determine the geometry of the Earth pointwise by means of a satellite GPS (Global Positioning System: «global problem solver») we are left with the problem of converting ellipsoidal heights (geometric heights) into orthometric heights (physical heights). For the computation of the plumbline we derive its three differential equations of first order as well as the three geodesic equations of second order. The three differential equations of second order take the form of a Newton differential equation when we introduce the parameter time via the Marussi gauge on a conformally flat three-dimensional Riemann manifold and the generalized force field, the gradient of the superpotential, namely the modulus of gravity squared and taken half. In particular, we compute curvature and torsion of the plumbline and prove their functional relationship to the second and third derivatives of the gravity potential. For a spherically symmetric gravity field, curvature and torsion of the plumbline are zero, the plumbline is straight. Finally we derive the three Lagrangean as well as the six Hamiltonian differential equations of the plumbline, in particular in their star form with respect to Marussi gauge.

Key words *field lines of gravity – plumbline – orthometric heights*

1. Introduction

With the advent of artificial satellites, in particular the satellite Global Positioning System (GPS: «Global Problem Solver»), high precision geometric positioning of points of the surface of the Earth has been developed. An unsolved key problem is the transformation of heights in geometry space, namely the ellipsoidal heights, into heights in gravity space, namely the orthometric heights/the length of the plumblines with respect to the geoid. The field lines of gravity/the orthogonal trajectories of a family of gravity equipotential surfaces/the plumblines are derived from a set of first order differential equations as soon as we balance the horizontal/tangential field of the plumbline with the vertical field/normal field of an equipotential surface of gravity as described by Caputo (1967), in particular with an ellipsoidal gravity field of reference, for instance.

Section two accordingly focuses on a set-up of the differential equations of first and second order of the plumbline with special reference to the transformation from the parameter arc length s to the «dynamic time parameter t» ac-

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cording to the celebrated Marussi gauge (Marussi 1979, 1985; Hotine 1986, 1991). The arc length squared $ds^2 = \| \operatorname{grad} w \|^2 (dx^2 + dy^2 + dz^2)$ has been represented in terms of conformal coordinates/isometric coordinates (*e.g.*, Caputo, 1959) with the modulus of gravity squared $\| \gamma \|^2 = \| \operatorname{grad} w |^2 = \lambda^2$ as the factor of conformality squared λ^2. In particular we succeed in proving that the second order differential equations of the plumbline establish a geodesic in a three-dimensional Riemann manifold $\{\mathbb{M}^3, g_{kl}\}$, notably in the form of a Newton dynamical equation if the matrix g_{kl}, of the metric is «conformally flat», $g_{kl} = \lambda^2 (\boldsymbol{x}) \delta_{kl}$, as restricted to the Marussi gauge. In order to determine the departure of the plumbline from a straight line, we compute its curvature and torsion on the basis of the Frenet derivational equations. We aim at the proof that the curvature of the plumbline is a functional of the second derivatives of the gravity potential, its torsion of the third derivatives of the gravity potential, while straight if the gravity field was spherically symmetric. Section three is devoted to establishing the three Lagrangean differential equations of second order as well as the six Hamiltonian differential equations of first order, in particular in Marussi gauge. In contrast to Moritz (1994) we succeed in constructing non-degenerate «star Lagrangean» and «star Hamiltonian».

2. Curvature and torsion of the field lines of the gravity field, the plumbline

At the beginning let us set up the differential equations of the plumbline/the orthogonal trajectory with respect to a family of equipotential surfaces by means of Box I. The quality between horizontal and vertical fields establishes the first order differential equation (2.1) of the plumbline: The normalized tangent vector of the plumbline is identical to the normalized surface vector of an equipotential surface pointwise. The normalized surface vector of an equipotential surface agrees with the negative gravity vector, the gradient of the gravity vector. As soon as we differentiate the identity of the horizontal field of the plumbline and the vertical field of an equipotential surface once more, we arrive at (2.2), the second order differential equation of a plumbline of inhomogeneous type. The inhomogeneity is generated by the quadrupole moment in gravity space. As soon as we introduce the parameter t in order to replace the curve arc length S via the Marussi gauge (2.3) (Marussi 1979, 1985) we are led to the first order differential eqs. (2.4) and the second order differential eqs. (2.5), (2.6) of a plumbline/orthogonal trajectory of a family of equipotential surfaces. With respect to the Marussi gauge, the second order differential equations of a plumbline in $\{\mathbb{R}^3, \delta_{kl}\}$ coincide with the second order differential equations of a geodesic in Newton form in the Marussi manifold $\{\mathbb{M}^3, \gamma^2(\boldsymbol{x}) \delta_{kl}\}$ with $\gamma^2(\boldsymbol{x})$ as the factor of conformality. Gravity squared taken half operates as a potential, according to a proposal by Chandrasekhar *et al.* called superpotential: The gradient of the superpotential $\gamma^2/2$ operates as the force field balanced by the acceleration vector $\boldsymbol{x}^{\bullet\bullet}$.

Indeed we have to explain better the duality between a curve in $\{\mathbb{R}^3, \delta_{kl}\}$, where the Kronecker δ_{kl} relates to the canonical metric in a three-dimensional Euclidean space, and a curve in $\{\mathbb{M}^3, \gamma^2(\boldsymbol{x}) \delta_{kl}\}$. $\{\mathbb{M}^3, \gamma^2(\boldsymbol{x}) \delta_{kl}\}$ is an abbreviated notation for a three-dimensional Riemannian space parameterized by three conformal coordinates/isometric coordinates, whose canonical metric is given by the product of the factor of conformality γ^2, the modulus of gravity squared, and the Kronecker δ_{kl}. Such a Riemann manifold $\{\mathbb{M}^3, \gamma^2(\boldsymbol{x}) \delta_{kl}\}$ will be called a Marussi manifold. Indeed there are many Marussi manifolds dependent on the various representations of the gravity field of the Earth. While the differential equations of second order (2.6) generate a curve in the chart $\{\mathbb{R}^3, \delta_{kl}\}$, at the same time this curve can be considered as a geodesic in the Riemann manifold in terms of special coordinates, the ones of conformal/isometric type. This conception of the curve as a geodesic will become clearer in the next chapter. We should mention that the duality described earlier has already been applied by Goenner *et al.* (1994) in order to interpret Newton mechanics as geodesic flow on a Maupertuis' manifold.

Box I. Duality between horizontal and vertical fields in $\{\mathbb{R}^3, \delta_{ij}\}$ equipped with a Euclidean metric δ_{ij}.

Normalized tangent vector of the plumbline is identical to the surface normal vector of an equipotential surface

1st order differential equations

$$\frac{d\boldsymbol{x}}{dS} = -\operatorname{grad} w / \| \operatorname{grad} w \| \sim \frac{dx^k}{dS} = -\partial_k w / \sqrt{\delta^{lm} \partial_l w \partial_m w} \tag{2.1}$$

2nd order differential equations

$$\frac{d^2x^k}{dS^2} + \frac{\gamma, l}{\gamma^3}(\gamma^2\delta^{kl} - \gamma^k\gamma^l) = 0 \tag{2.2}$$

Marussi gauge

$$\|\boldsymbol{x}^\blacklozenge\| = \|\operatorname{grad} w\|, \quad \frac{d\boldsymbol{x}}{dS} = \|\boldsymbol{x}^\blacklozenge\|^{-1}\boldsymbol{x}^\blacklozenge \tag{2.3}$$

1st order differential equation of the plumbline in Marussi gauge

$$\boxed{\boldsymbol{x}^\blacklozenge = -\operatorname{grad} w} \tag{2.4}$$

2nd order differential equation of the plumbline in Marussi gauge

$$x^{\blacklozenge\blacklozenge k} = -(\partial_l \gamma^k)\, x^{\blacklozenge l} = (\partial_l \gamma^k)\, \partial_l w = (\partial_l \gamma^k)\, \gamma^l \tag{2.5}$$

$$\boxed{x^{\blacklozenge\blacklozenge k} - \frac{1}{2}\partial_k \gamma^2(x^m) = 0} \tag{2.6}$$

Secondly we are going to derive the Frenet equations of the plumbline, a curve in $\{\mathbb{R}^3, \delta_{kl}\}$, a three-dimensional Euclidean space completely covered by one chart of Cartesian coordinates $\{x^1, x^2, x^3\}$. As outlined by means of Box II, we establish by (2.7) the Frenet frame {normalized tangent vector, normalized normal vector, normalized binormal vector} called $\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3\}(\boldsymbol{x})$, which is subject to the coupling to the gravity field (2.8), thanks to the first order differential equations of the plumbline in Marussi gauge (2.4). The derivational equations of the Frenet frame (2.9), (2.10), (2.11) are built on the celebrated antisymmetric Ω-matrix which contains as structure elements the curvature κ (2.12i) and the torsion τ (2.12ii), also called first and second curvature of the plumbline. A straight-forward computation of curvature and torsion of the plumbline subject to the coupling of the gravity field (2.8), namely with respect to the representation (2.12), leads to (2.13) and (2.14). Here we took advantage of the cross product identity $\boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\|^2 \|\boldsymbol{b}\|^2 - \langle \boldsymbol{a} | \boldsymbol{b} \rangle^2$ where $\|\boldsymbol{a}\|$ indicates the Euclidean norm of the vector a as well as $\langle \boldsymbol{a} | \boldsymbol{b} \rangle$ the Euclidean scalar product/

Box II. Curvature and torsion of the plumbline in $\{\mathbb{R}^3, \delta_{ij}\}$ equipped with a Euclidean metric δ_{ij}.

The Frenet frame as the natural triad of the plumbine

$$f_1 := \frac{x^{\bullet}}{\|x^{\bullet\bullet}\|} \tag{2.7i}$$

$$f_2 := \frac{x^{\bullet\bullet} - \langle x^{\bullet\bullet} | f_1 \rangle f_1}{\|x^{\bullet\bullet} - \langle x^{\bullet\bullet} | f_1 \rangle f_1\|} \tag{2.7ii}$$

$$f_3 := \frac{x^{\bullet\bullet\bullet} - \langle x^{\bullet\bullet\bullet} | f_1 \rangle f_1 - \langle x^{\bullet\bullet\bullet} | f_2 \rangle f_2}{\|x^{\bullet\bullet\bullet} - \langle x^{\bullet\bullet\bullet} | f_1 \rangle f_1 - \langle x^{\bullet\bullet\bullet} | f_2 \rangle f_2\|} \tag{2.7iii}$$

subject to

$$(i)\ x^{\bullet i} = -\gamma^i, \quad (ii)\ x^{\bullet\bullet i} = \frac{1}{2}\gamma^2_{,i}, \quad (iii)\ x^{\bullet\bullet\bullet i} = -\frac{1}{2}\gamma^2_{,ij}\gamma^j \tag{2.8}$$

The derivational equations of the Frenet frame

$$f_1^{\bullet} = \kappa S^{\bullet} f_2 \tag{2.9i}$$

$$f_2^{\bullet} = -\kappa S^{\bullet} f_1 + \tau S^{\bullet} f_3 \tag{2.9ii}$$

$$f_3^{\bullet} = -\tau S^{\bullet} f_2 \tag{2.9iii}$$

$$\begin{bmatrix} f_1^{\bullet} \\ f_2^{\bullet} \\ f_3^{\bullet} \end{bmatrix} = \begin{bmatrix} 0 & \omega_{12} & 0 \\ -\omega_{12} & 0 & \omega_{23} \\ 0 & -\omega_{23} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{2.10}$$

$$\begin{bmatrix} f_1^{\bullet} \\ f_2^{\bullet} \\ f_3^{\bullet} \end{bmatrix} = \begin{bmatrix} 0 & \kappa S^{\bullet} & 0 \\ -\kappa S^{\bullet} & 0 & \tau S^{\bullet} \\ 0 & -\tau S^{\bullet} & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \tag{2.11}$$

Curvature κ and torsion τ

$$\kappa = \frac{\|x^{\bullet} \times x^{\bullet\bullet}\|}{\|x^{\bullet}\|^3}, \quad \tau = \frac{\langle x^{\bullet} | x^{\bullet\bullet} \times x^{\bullet\bullet}\rangle}{\|x^{\bullet} \times x^{\bullet\bullet}\|^2} \tag{2.12}$$

$$\kappa = \frac{1}{2\gamma^3}\sqrt{\gamma^2\|\operatorname{grad}\gamma^2\|^2 - \langle \gamma | \operatorname{grad}\gamma^2\rangle^2} \tag{2.13}$$

$$= \frac{\sqrt{(\gamma_1^2+\gamma_2^2+\gamma_3^2)(\gamma^2_{,1}+\gamma^2_{,2}+\gamma^2_{,3})^2 - (\gamma_1\gamma^2_{,1}+\gamma_2\gamma^2_{,2}+\gamma_3\gamma^2_{,3})^2}}{2(\gamma_1^2+\gamma_2^2+\gamma_3^2)^{3/2}}$$

$$\tau = \frac{\gamma^2_{,1j}\gamma^j(\gamma_2\gamma^2_{,3} - \gamma_3\gamma^2_{,2}) + \gamma^2_{,2j}\gamma^j(\gamma_3\gamma^2_{,1} - \gamma_1\gamma^2_{,3}) + \gamma^2_{,3j}\gamma^j(\gamma_1\gamma^2_{,2} - \gamma_2\gamma^2_{,1})}{\gamma^2\delta^{kl}\gamma^2_{,k}\gamma^2_{,l} - (\delta^{kl}\gamma_k\gamma^2_{,l})^2} \tag{2.14}$$

Corollary

$$\kappa_0 = 0, \quad \tau_0 = 0, \quad \text{if} \quad w_0(x, y, z) = w_0(r) \tag{2.15}$$

inner product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Obviously the curvature κ of the plumbline is proportional to gravity gradients or second derivatives of the gravity potential. This can be seen by means of (2.13) as soon as we apply (2.21), namely grad $\gamma^2/2$, $(\partial_k\gamma^2)/2 = \gamma_1\partial_k\gamma_1 + \gamma_2\partial_k\gamma_2 + \gamma_3\partial_k\gamma_3 = \partial_1 w\partial_k\partial_1 w + \partial_2 w\partial_k\partial_2 w + \partial_3 w\partial_k\partial_3 w$. In contrast, the torsion τ of the plumbline is proportional to the second derivatives of the gravity vector or the third derivatives of the gravity potential. Such a result is motivated by the identity $\gamma^2_{,ij}/2 = \gamma_{k,ij}\gamma_k + \gamma_{k,i}\gamma_{k,j} = (\partial_i\partial_j\partial_k w)\,\partial_k w + (\partial_i\partial_k w)\,(\partial_j\partial_k w)$. Finally as a corollary we report the result that curvature and torsion of the plumbline amount to zero if the gravity field has spherical symmetry. Or we may say that $\kappa_0 = 0$, $\tau_0 = 0$, if the gravity field $w_0\,(x, y, z) = w_0\,(r)$ depends only on the radial coordinate r. This result gave the motivation for a decomposition of curvature and torsion $\kappa = \kappa_0 + \delta\kappa$, $\tau = \tau_0 + \delta\tau$ subject to $\kappa_0 = \tau_0 = 0$ in terms of the gravity field $\gamma = \gamma_0 + \delta\gamma$, the normal gravity field $\gamma_0 = \gamma_0\,(r)$ and the disturbing gravity field $\delta\gamma\,(\lambda, \phi, r)$ which depends on the lateral variation of lengthy spherical coordinates {longitude λ, latitude ϕ}. Since the representations $\delta\kappa$, $\delta\tau$ are lengthy, we drop them here.

Instead, thirdly, we compute by means of Box III the plumbline in a spherically symmetric gravity field subject to Marussi gauge. Let us depart from the first order differential equations (2.18) of a plumbline subject to Marussi gauge. Indeed by means of (2.17) we restrict the gravity field to be spherically symmetric: the gravity potential $w\,(x, y, z) = f(r)$ has been chosen to be a function of the radial coordinate only. The appropriate coordinate system in which to solve the first order differential equations is the spherical coordinate system $\{\lambda, \phi, r\}$. By means of (2.19)-(2.29) we have used the forward transformations «Cartesian coordinates into spherical coordinates» in order to represent the first order differential equations of the plumbline subject to Marussi gauge in spherical coordinates, to prove (2.27) $\lambda^{\bullet} = 0$, (2.28) $\phi^{\bullet} = 0$ and (2.29) $r^{\bullet} = -f(r)$, a result collected in the corollary (2.30). Finally as an example we have chosen the potential and the gravity field (2.31)-(2.35) of a homogeneous, massive sphere in the inner zone A and the outer zone B in order to solve the ordinary differential equation of the radial component of the plumbline by means of (2.37) and (2.38). The function $r = r_0 \exp \dfrac{gm}{R^s}(t - t_0)$ (2.37) is a representation of the solution of (2.30iii) $r^{\bullet} = -f(r)$ in *case 1*, $R > r$, while $r = \sqrt[3]{r_0^3 + 3\,gm\,(t - t_0)}$ in *case 2*, $R < r$. For $R = r$, both solutions agree with each other. gm denotes the product of the gravitational constant g and the mass m of the homogeneous, massive sphere.

Finally we illustrate by fig. 1 the solution of the first order differential eqs. (2.30) of the plumbline subject to Marussi gauge, namely the bundle of straight lines with the mass center as the focal point for a spherically symmetric gravity field. For a more realistic gravity field in the crust of the Earth, Svensson in Grafarend (1986) has computed a sample plumbline in the Alpes by a Runge-Kutta numerical computation of the solution of the first order differential equation of the plumbline (2.4) in Marussi gauge and a gravity field given by a set of homogeneous massive

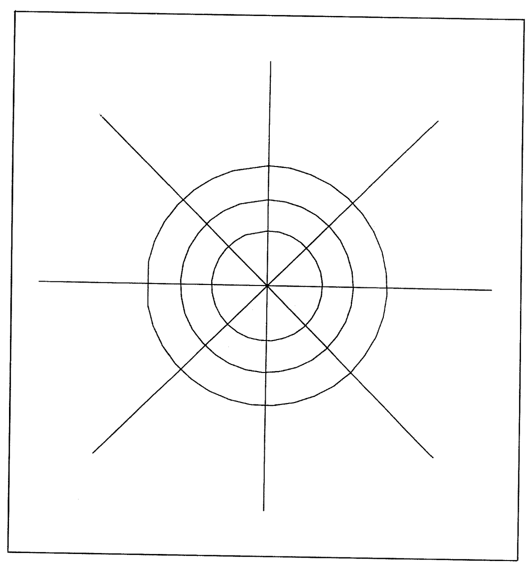

Fig. 1. A set of plumblines for a spherically symmetric gravity field γ_0; straight lines with the mass centre of the Earth as a focal point.

Box III. Computation of a plumbline in a spherically symmetric gravity field. Marussi gauge.

$$\boldsymbol{x}^{\bullet} = -\operatorname{grad} w \Leftrightarrow \begin{bmatrix} x^{\bullet} = -\dfrac{\partial w}{\partial x} \\ y^{\bullet} = -\dfrac{\partial w}{\partial y} \\ z^{\bullet} = -\dfrac{\partial w}{\partial z} \end{bmatrix} \tag{2.16}$$

spherically symmetric gravity field

$$w(x, y, z) = f(r) \text{ subject to } r^2 = x^2 + y^2 + z^2 \tag{2.17}$$

$$\partial_k w = \frac{dw}{dr}\frac{\partial r}{\partial x^k} = f'(r)\frac{x^k}{r}, \quad f'(r) := \frac{df}{dr} \tag{2.18}$$

forward transformation: Cartesian coordinates into spherical coordinates

$$\lambda = \arctan(y/x) + \left(-\frac{1}{2}\operatorname{sgn}(y) - \frac{1}{2}\operatorname{sgn}(y)\operatorname{sgn}(x) + 1\right)\pi, \quad \lambda \in \{\mathbb{R} \mid 0 \le \lambda < 2\pi\} \tag{2.19}$$

$$\phi = \arctan(z/\sqrt{x^2+y^2}), \quad \phi \in \{\mathbb{R} \mid -\pi/2 < \phi < +\pi/2\} \tag{2.20}$$

representation of the 1st order differential equation of the plumbline in spherical coordinates

$$d\tan\lambda = \frac{x^2+y^2}{x^2}\, d\lambda = \frac{-ydx + xdy}{x^2} \tag{2.21}$$

$$d\tan\phi = \frac{1}{(x^2+y^2)^{3/2}}((x^2+y^2)\,dz - zxdx - zydy) = \frac{r^2}{x^2+y^2}\, d\phi \tag{2.22}$$

$$x^{\bullet k} = -f'(r)\frac{x^k}{r} \Leftrightarrow \begin{bmatrix} x^{\bullet} = -f'(x)\, x/r \\ y^{\bullet} = -f'(r)\, y/r \\ z^{\bullet} = -f'(r)\, z/r \end{bmatrix} \tag{2.23}$$

$$\lambda^{\bullet} = \frac{1}{x^2+y^2}(-yx^{\bullet} + xy^{\bullet}) \tag{2.24}$$

$$\phi^{\bullet} = \frac{1}{(x^2+y^2)^{1/2}}\frac{1}{r}(-zxx^{\bullet} - zyy^{\bullet} + (x^2+y^2)\, z^{\bullet}) \tag{2.25}$$

$$r^{\bullet} = \frac{1}{r}(xx^{\bullet} + yy^{\bullet} + zz^{\bullet}) \tag{2.26}$$

$$\lambda^{\bullet} = \frac{1}{x^2+y^2}(yf'(r)\, x/r - xf'(r)\, y/r) = 0 \tag{2.27}$$

$$\phi^{\bullet} = -\frac{1}{(x^2+y^2)^{3/2}}\frac{1}{r^2}(zxf'(r)\, x/r + zyf'(r)\, y/r - (x^2+y^2)f'(r)\, z/r) = 0 \tag{2.28}$$

$$r^2 r^{\bullet} = -(x^2 f'(r) + y^2 f'(r) + z^2 f'(r)) = -r^2 f'(r) \tag{2.29}$$

Corollary

$$\lambda^{\bullet} = 0, \quad \phi^{\bullet} = 0, \quad r^{\bullet} = -f'(r), \quad \text{if} \quad w(x, y, z) = f(r) \tag{2.30}$$

Box III (*continued*).

Example: massive sphere

$$w_0(r) = \begin{cases} \dfrac{gm}{2R}\left(3 - \dfrac{r^2}{R^2}\right) \forall 0 \le r < R: \text{ zone A} \\ \dfrac{gm}{r} \forall R \le r < \infty: \text{ zone B} \end{cases} \tag{2.31}$$

or

in terms of the Heaviside function $H(R, r)$

$$w_0(r) = H(R-r)\frac{gm}{2R}\left(3 - \frac{r^2}{R^2}\right) + H(r-R)\frac{gm}{r} \tag{2.32}$$

$$\text{grad } w_0(r) = -e_r \begin{cases} \dfrac{gm}{R^3} r \forall 0 \le r < R: \text{ zone A} \\ \dfrac{gm}{r^2} \forall R \le r < \infty: \text{ zone B} \end{cases} \tag{2.33}$$

or

$$\text{grad } w_0(r) = -e_r H(R-r)\frac{gm}{R^3} r - e_r H(r-R)\frac{gm}{r^2} \tag{2.34}$$

$$-\partial_k w_0 = H(R-r)\frac{gm}{R^3} x^k + H(r-R)\frac{gm}{r^3} x^k = x^{\bullet k} \tag{2.35}$$

Corollary

$$\lambda^\bullet = 0, \quad \phi^\bullet = 0, \quad r r^\bullet = r^2 (H(R-r))\frac{gm}{R^3} + H(r-R)\frac{gm}{r^3} \tag{2.36}$$

if $w(x, y, z) = w_0(r)$

Case 1: $R > r: r^\bullet = \dfrac{gm}{R^3} r \Rightarrow \dfrac{dr}{r} = \dfrac{gm}{R^3} dt \quad \Rightarrow$

$$\Rightarrow \quad \ln r - \ln r_0 = \frac{gm}{R^3}(t - t_0) \quad \Rightarrow$$

$$\Rightarrow \quad \ln r/r_0 = \frac{gm}{R^3}(t - t_0) \quad \Rightarrow$$

$$\Rightarrow \quad \frac{r}{r_0} = \exp \frac{gm}{R^3}(t - t_0) \quad \Rightarrow$$

$$\boxed{r = r_0 \exp \frac{gm}{R^3}(t - t_0)} \tag{2.37}$$

Case 2: $r > R: r^\bullet = \dfrac{gm}{r^2} \Rightarrow r^2 dr = gmdt \quad \Rightarrow$

$$\Rightarrow \quad \frac{r^3}{3} - \frac{r_0^3}{3} = gm(t - t_0) \quad \Rightarrow$$

$$\Rightarrow \quad \frac{1}{3} r^3 = \frac{1}{3} r_0^3 + gm(t - t_0) \quad \Rightarrow$$

$$\boxed{r = \sqrt[3]{r_0^3 + 3gm(t - t_0)}} \tag{2.38}$$

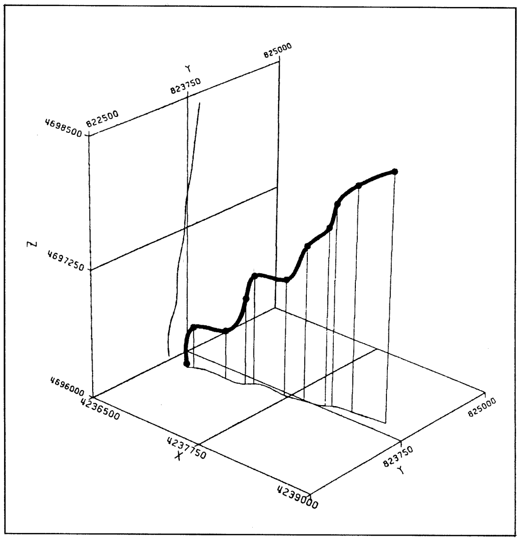

Fig. 2. Computation of a realistic plumbline at a mountain point in the Alpes according to Svensson in Grafarend (1986).

spheres around the plumbline representing the local gravity field. Reference is made to fig. 2. In addition, by fig. 3 we illustrate the computation of a realistic plumbline at the Swiss high mountain point «Jungfraujoch» performed by Hunziker (1960). An alternative procedure for the gravitational field in the crust is outlined in Engels and Grafarend (1993), Engels *et al.* (1996) and in Grafarend *et al.* (1995). Finally we refer to Grossman (1974, 1978) for the focal point of plumblines.

3. The Lagrangean portray *versus* the Hamilton portray of the field lines of the gravity field, the plumbline

First, let us derive the second-order differential equations of the plumbline from a general Lagrange functional. According to Box IV we look for a stationary functional (3.1) which varies the arc length between two fixed points in the three-dimensional Riemann mani-

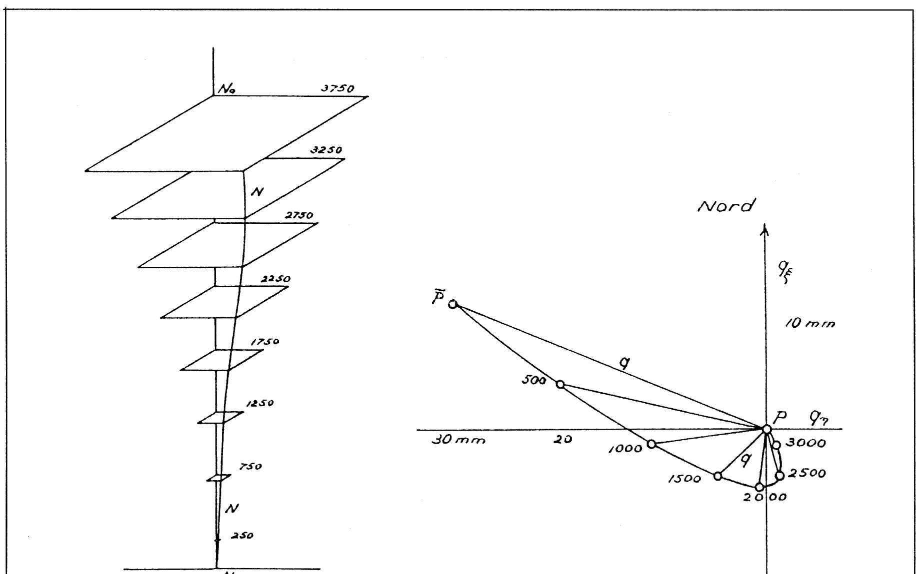

Fig. 3. Computation of a realistic plumbline at the mountain point «Jungfraujoch» at the Swiss Alpes by Hunziker (1960, p. 151), departure from a straight line projection onto the reference figure 12.2 mm/30.8 mm in the North/West direction.

Box IV. The field lines of the gravity field as geodesics, the general Lagrange portray.

«The stationary functional»

$$\delta \int_0^s ds = \delta \int_{\tau_1}^{\tau_2} \sqrt{2L^2}\, d\tau = \delta \int_{\tau_1}^{\tau_2} \sqrt{g_{kl} \frac{dx^l}{d\tau} \frac{dx^k}{d\tau}}\, d\tau = 0 \tag{3.1}$$

(fixed boundary points)

$$\Leftrightarrow$$

$$\delta \int_{\tau_1}^{\tau_2} L^2 \left(\mathbf{x}, \frac{d\mathbf{x}}{d\tau} \right) = \frac{1}{2} \delta \int_{\tau_1}^{\tau_2} \left(g_{kl} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} \right) d\tau = 0 \tag{3.2}$$

(fixed boundary points)

subject to a general Lagrangean of a conformally flat metric whose factor of conformality coincides with the modulus of the gravity squared.

$$2L^2 \left(\mathbf{x}, \frac{d\mathbf{x}}{d\tau} \right) := \frac{ds^2}{d\tau^2} = g_{kl} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} \tag{3.3}$$

$$g_{kl}(x^m) = \lambda^2 (x^m)\, \delta_{kl} \tag{3.4}$$

«the gravity eiconal»

$$\boxed{\lambda^2 (x^m) = \gamma^2 (x^m) = \delta^{kl} \partial_k w \partial_l w} \tag{3.5}$$

$$\forall k, l, m \in \{1, 2, 3\}$$

«The general Lagrangean»

$$\boxed{\begin{aligned} 2L^2 \left(\mathbf{x}, \frac{d\mathbf{x}}{d\tau} \right) &= \lambda^2 (x^1, x^2, x^3) \left\{ \left(\frac{dx^1}{d\tau} \right)^2 + \left(\frac{dx^2}{d\tau} \right)^2 + \left(\frac{dx^3}{d\tau} \right)^2 \right\} = \\ &= \gamma^2 (x^1, x^2, x^3) \left\{ \left(\frac{dx^1}{d\tau} \right)^2 + \left(\frac{dx^2}{d\tau} \right)^2 + \left(\frac{dx^3}{d\tau} \right)^2 \right\} \end{aligned}} \tag{3.6}$$

«The Euler-Lagrange equations»

$$\delta \int_{\tau_1}^{\tau_2} L \left(\mathbf{x}, \frac{d\mathbf{x}}{d\tau} \right) d\tau = 0 \Leftrightarrow \tag{3.7}$$

$$\Leftrightarrow \left[\frac{d}{d\tau} \frac{\partial L}{\partial \left(\frac{dx^k}{d\tau} \right)} \right] - \frac{\partial L}{\partial x^k} = 0 \tag{3.8}$$

Box IV (*continued*).

$$\sqrt{2}\frac{\partial L}{\partial\left(\frac{dx^k}{d\tau}\right)} = \frac{1}{2L}\frac{\partial L^2}{\partial\left(\frac{dx^k}{d\tau}\right)} = \frac{\sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2}}{\sqrt{\left(\frac{dx^1}{d\tau}\right)^2+\left(\frac{dx^2}{d\tau}\right)^2+\left(\frac{dx^3}{d\tau}\right)^2}}\frac{dx^k}{d\tau} \tag{3.9i}$$

$$\sqrt{2}\frac{\partial L}{\partial x^k} = \frac{1}{2L}\frac{\partial L^2}{\partial dx^k} = \frac{\sqrt{\left(\frac{dx^1}{d\tau}\right)^2+\left(\frac{dx^2}{d\tau}\right)^2+\left(\frac{dx^3}{d\tau}\right)^2}}{\sqrt{\gamma_1^2+\gamma_2^2+\gamma_3^2}}\frac{1}{2}\partial_k\gamma^2(x^m) \tag{3.9ii}$$

«transformation from the parameter τ to the parameter s (affine parameter)»

$$\left.\begin{aligned}\frac{dx^k}{d\tau} &= \frac{dx^k}{ds}\frac{ds}{d\tau} = x'^k\frac{ds}{d\tau} \\ \frac{ds}{d\tau} &= \sqrt{g_{kl}(x^m)\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}}\end{aligned}\right] \Rightarrow \qquad (3.10),\ (3.11)$$

$$g_{kl}\,x^l + [kl, m]\,x'^l x'^m = 0 \tag{3.12i}$$

$$\gamma^2 x^k + (\partial_l\gamma^2)\,x'^k x'^l - \frac{1}{2\gamma^2}\partial_k\gamma^2 = 0 \tag{3.12ii}$$

fold $\{\mathbb{M}^3, \gamma^2(\boldsymbol{x})\,\delta_{kl}\}$, the Marussi manifold, whose metric is chosen conformally flat, $g_{kl}(\boldsymbol{x}) = \lambda^2(\boldsymbol{x})\,\delta_{kl}$. Its factor of conformality $\lambda^2(\boldsymbol{x}) = \gamma^2(\boldsymbol{x}) = \delta^{kl}\delta_k w \delta_l w$ as the gravity eiconal (3.5) is fixed to the modulus of gravity squared where gravity is represented as the positive gradient of the scalar gravity potential. Accordingly, the standard Euler-Lagrange eqs. (3.8), (3.9) lead to the general differential equations of second order characterising the plumbline with respect to the length of arc via (3.10), (3.11) and (3.12).

Secondly, we are going to derive the second-order differential equations of the plumbline from a special Lagrangean L^* which is subject to Marussi gauge according to Box V. The star Lagrangean functional (3.13), namely $L^* := ds/dt$, leads us via (3.15) to the representation (3.16) when implementing the metric of a conformable flat Riemann manifold (3.4), the Marussi gauge (3.15) as well as the factor of conformality (3.16), called here the gravity eiconal. Based upon the star Lagrangean functional (3.17) first order variation via (3.18), (3.19) gives us the differential equations of second order (3.20) of the plumbline, namely as a geodesic in Newton form, where the gradient of the modulus of gravity squared and taken half appears as the gradient of the superpotential $\gamma^2/2$. Finally (3.21) offers a representation of the gradient of the superpotential in terms of first and second order gradients of the gravity potential. The transformation of the general Lagrangean into the star Lagrangean, forward as well as backward, is given by (3.22).

How can we derive the Hamiltonian portray, in particular the Hamiltonian equations of the plumbline, which have to appear as six differential equations of first order? Thirdly, Box VI outlines the procedure: As soon as we have defined by means of (3.23) the generalized mo-

Box V. The field lines of the gravity field as geodesics, the Lagrange portray with respect to Marussi gauge.

«The stationary functional»

$$\delta \int_0^s ds = \delta \int_{t_1}^{t_2} L^* dt = \delta \int_{t_1}^{t_2} \lambda(x^m) \sqrt{\delta_{kl}\, x^{\bullet k} x^{\bullet l}}\, dt = 0 \tag{3.13}$$

(fixed boundary points)

subject to a Lagrangean of a conformally flat metric whose factor of conformality coincides with the modulus of gravity squared as well as to the Marussi gauge.

$$L^* := \frac{ds}{dt} = \lambda^2 (x^m) \tag{3.14}$$

$$g_{kl} (x^m) = \lambda^2 (x^m)\, \delta_{kl} \tag{3.4}$$

«the Marussi gauge»

$$\lambda^2 = \delta_{kl} x^{\bullet k} x^{\bullet l} \tag{3.15}$$

«the gravity eiconal»

$$\lambda^2 (x^m) = \gamma^2(x^m) = \delta^{kl} \gamma_k \gamma_l = \delta^{kl} \partial_k w \partial_l w \tag{3.5}$$

$$\forall k,\, l,\, m \in \{1, 2, 3\}$$

«the star Lagrangean»

$$\boxed{L^* = \frac{1}{2}\, \delta_{kl}\, x^{\bullet k} x^{\bullet l} + \frac{1}{2}\, \lambda^2(x)} \tag{3.16}$$

«The Euler-Lagrange equations»

$$\delta \int_{t_1}^{t_2} L^* (\boldsymbol{x}, \boldsymbol{x})\, dt = 0 \Leftrightarrow \tag{3.17}$$

$$\Leftrightarrow \left(\frac{d}{dt} \frac{\partial L^*}{\partial x^{\bullet k}} \right) - \frac{\partial L}{\partial x^k} = 0 \tag{3.18}$$

mentum as the general velocity vector projected into the cotangent space $^*T_x\mathbb{M}^3$ at point $\boldsymbol{x}$ of the Riemann manifold $\{\mathbb{M}^3, g_{kl}\}$, its metric via (3.4) is given as conformally flat. In particular, the differential arc length squared $ds^2 = \lambda^2 [(dx)^2 + (dx^2)^2 + (dx^3)^2]$ up to the factor of conformality $\lambda^2 (\boldsymbol{x})$ looks like the metric of a flat space. As soon as we use (3.5) as the proper representation of the factor of conformality as the modulus of gravity, we gain (3.24) as the attached generalized momentum. The standard Legendre transformation (3.25) helps us to define via the general Lagrangean squared the Hamiltonian $H(\boldsymbol{x}(\tau), \boldsymbol{y}(\tau))$ in

Box V (*continued*).

$$\frac{\partial L^*}{\partial x^{\bullet k}} = x^{\bullet k}, \quad \frac{d}{dt}\frac{\partial L^*}{\partial x^{\bullet k}} = x^{\bullet\bullet k} \tag{3.19i}$$

$$\frac{\partial L^*}{\partial x^k} = \frac{1}{2}\partial_k \lambda^2 (x^m) = \frac{1}{2}\partial_k \gamma^2(x^m) \tag{3.19ii}$$

$$\Leftrightarrow$$

$$\boxed{x^{\bullet\bullet k} = \frac{1}{2}\partial_k \gamma^2(x^m)} \tag{3.20}$$

«the differential equation of the plumbline as a geodesic in Newton form; $\gamma^2/2$ is called superpotential»
«representation of the gradient of the factor of conformality/superpotential in terms of gravity gradients».

$$\boxed{\begin{aligned}\frac{1}{2}\partial_k \gamma^2(x^m) &= \delta^{k_1 l_1}\gamma_{k_1}\partial_k \gamma_{l_1} = \gamma_1 \partial_k \gamma_1 + \gamma_2 \partial_k \gamma_2 + \gamma_3 \partial_k \gamma_3 = \\ &= \partial_1 w \partial_k \partial_1 w + \partial_2 w \partial_k \partial_2 w + \partial_3 w \partial_k \partial_3 w\end{aligned}} \tag{3.21}$$

«The transformation of the general Lagrangean to the star Lagrangean subject to Marussi gauge»

$$L^* = \frac{2L^2}{\delta_{kl}\dfrac{dx^k}{d\tau}\dfrac{dx^l}{d\tau}} \Leftrightarrow L = \frac{1}{2}\sqrt{L^*}\,\delta_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau} \tag{3.22}$$

the parameter τ. The general Hamiltonian $H = g^{kl}y_k y_l$ should be compared with the degenerate Hamiltonian proposed by Moritz (1994). Implementing the metric (3.4), we are led to the elegant form of the non-degenerate Hamiltonian (3.26) and by the standard Hamiltonian variational calculus (3.27)-(3.29) to the six Hamiltonian equations (3.30) of first order.

The six Hamiltonian equations become much simpler as soon as we introduce the Marussi gauge. According to Box VII we introduce the «Marussi gauged momentum» $\boldsymbol{y}^*$ which coincides via (3.31)-(3.34) with the vector $\boldsymbol{x}^\bullet$ with respect to the parameter t, namely the negative gradient of the gravity potential. in the Hamilton portray $\boldsymbol{y}^* = -\operatorname{grad} w$ introduces the «action function» w, also called «Wirkungsfunktion», which plays a dominant role in the theory of refraction, wave fronts and rays. The surfaces $w = \text{const}$ correspond to wave fronts, their orthogonal trajectories, the plumblines to rays, Roemer (1994). Enjoy the experience to derive the star Hamiltonian $H^*(\boldsymbol{x}(t), \boldsymbol{y}^*(t))$, (3.35) from the star Lagrangean via the star Legendre Transformation, namely (3.36), the starting point for the standard Hamiltonian calculus (3.37): The star Hamiltonian eqs. (3.38), six in number, appear symmetrical. The time derivative of the conformal/isometric coordinates x^k equal the star momentum y_k^*, the time derivative of the star momentum equals the gradient of the superpotential $\lambda^2(x^1, x^2, x^3)/2 = \gamma^2(x^1, x^2, x^3)/2$. Finally by means of (3.38) we offer the transformation of the general Hamiltonian into the star Hamiltonian, of course, forward and backward. The elegant six Hamiltonian equations of first order have already been interpreted in phase space

Box VI. The field lines of the gravity field as geodesics, the general Hamilton portray.

«generalized momentum»

$$y_k := \frac{\partial L^2}{\partial \frac{dx^k}{d\tau}} = g_{kl} \frac{dx^k}{d\tau} \in {}^* \mathbb{T}_x \mathbb{M}^3 \tag{3.23}$$

subject to conformally flat metric whose factor of conformality coincides with the modulus of gravity squared

$$g_{kl}(x^m) = \lambda^2 (x^m) \, \delta_{kl} \tag{3.4}$$

$$\lambda^2 (x^m) = \gamma^2 (x^m) = \delta^{kl} \gamma_k \gamma_l = \delta^{kl} \partial_k w \, \partial_l w \tag{3.5}$$

$$y_k = \lambda^2 (x^m) \frac{dx^k}{d\tau} \forall \, {}^* \mathbb{T}_x \mathbb{M}^3 \tag{3.24}$$

«Legendre transformation, the general Hamiltonian $H(x(\tau), y(\tau))$ as the dual of the Lagrangean $L^2 \left(x, \frac{dx}{d\tau}\right)$»

$$H(x(\tau), y(\tau)) := y_k \frac{dx^k}{d\tau} - L^2 = \frac{1}{2} g^{kl} y_k y_l \tag{3.25}$$

subject to

$$g_{kl}(x^m) = \lambda^2 (x^m) \, \delta_{kl} \Leftrightarrow g^{kl}(x^m) = \frac{1}{\lambda^2 (x^m)} \delta^{kl} \tag{3.4}$$

$$\boxed{H = \frac{1}{2\gamma^2 (x^m)} (y_1^2 + y_2^2 + y_3^2)} \tag{3.26}$$

«The general Hamiltonian equations»

$$\delta \int_{\tau_1}^{\tau_2} \left(y_k \frac{dx^k}{d\tau} - H \right) d\tau = 0 \tag{3.27}$$

$$\Leftrightarrow$$

$$\frac{dx^k}{d\tau} = \frac{\partial H}{\partial y_k} = g^{kl} y_l \tag{3.28}$$

$$\frac{dy_k}{d\tau} = -\frac{\partial H}{\partial x^k} = -\frac{1}{2} \frac{\partial g^{lm}}{\partial x^k} y_l y_m \tag{3.29}$$

subject to a conformally flat metric

$$g^{kl} = \frac{1}{\lambda^2 (x^1, x^2, x^3)} \delta^{kl} \tag{3.4}$$

$$\boxed{\begin{aligned} \frac{dx^l}{d\tau} &= \frac{\partial H}{\partial y_k} = \frac{1}{\lambda^2} y_k \\ \frac{dy_k}{d\tau} &= -\frac{y_1^2 + y_2^2 + y_3^2}{2} \frac{\partial}{\partial x^k} \frac{1}{\lambda^2 (x^1, x^2, x^3)} \end{aligned}} \tag{3.30}$$

Box VII. The field lines of the gravity field as geodesics, the Hamilton portray with respect to Marussi gauge.

«Marussi gauged momentum»

$$y_k^* := \frac{\partial L^*}{\partial \dfrac{dx^k}{d\tau}} = g_{kl}\frac{dx^l}{ds} = g_{kl}\frac{dx^l}{dt}\frac{1}{\dfrac{ds}{dt}} \tag{3.31}$$

subject to conformally flat metric whose factor of conformality coincides with the modulus of gravity squared as well as to the Marussi gauge

$$g_{kl}(x^m) = \lambda^2(x^m)\,\delta_{kl} \tag{2.4}$$

$$\lambda^2(x^m) = \gamma^2(x^m) = \delta^{kl}\gamma_k\gamma_l = \delta^{kl}\partial_k w\,\partial_l w \tag{2.5}$$

$$ds = \lambda^2 dt \tag{3.32}$$

$$\boxed{y_k^* = \frac{dx^k}{dt} = x^{\blacklozenge k} = -\partial_k w} \tag{3.33}$$

$$\boxed{\mathbf{y}^* = \mathbf{x}^{\blacklozenge} = -\operatorname{grad} w} \tag{3.34}$$

«the Marussi gauged momentum coincides with the gradient of the gravity potential (*Wirkungsfunktion*, action function)»

«Legendre transformation, the star Hamiltonian $H^*(x(t), y(t))$ subject to the Marussi gauge as the dual of the star Lagrangean $L^*\left(x, \dfrac{dx}{dt}\right)$»

$$H^*(x(t), y(t)) := y_k^*\frac{dx^k}{dt} - L^* \tag{3.35}$$

$$\boxed{H^* = \frac{1}{2}\delta^{kl}y_k^* y_l^* - \frac{1}{2}\lambda^2} \tag{3.36}$$

«The Hamilton equations with respect to Marussi gauge»

$$\delta\int_{t_1}^{t_2}\left(y_k^*\frac{dx^k}{dt} - H^*\right)dt = 0 \tag{3.37}$$

$$\boxed{\begin{aligned}\frac{dx^k}{dt} &= \frac{\partial H^*}{\partial y_k^*} = \delta^{kl}y_l\\ \frac{dy_k^*}{dt} &= -\frac{\partial H^*}{\partial x^k} = \frac{1}{2}\frac{\partial}{\partial x^k}\lambda^2(x^1, x^2, x^3)\end{aligned}} \tag{3.38}$$

«The transformation of the general Hamiltonian subject to Marussi gauge»

$$H^* = \frac{H}{\delta_{kl}\dfrac{dx^k}{d\tau}\dfrac{dx^l}{d\tau}} - \frac{\lambda}{2} \Leftrightarrow H = \left(\delta_{kl}\frac{dx^k}{d\tau}\frac{dx^l}{d\tau}\right)\left(H^* + \frac{\lambda^2}{2}\right) \tag{3.39}$$

by Grafarend and You (1995, formulae (1.9)-(1.12)) on the basis of a six-dimensional phase space $\{\mathbb{M}^6, \Omega\}$. It would be tempting to solve the six Hamiltonian equations of first order for a spherically symmetric gravity field, but space restrictions do not allow us to do this.

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