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A «tidal» magnetic field?

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Abstract
It is shown that on the magnetization axis of a uniformly magnetized body of constant density the magnetic field intensity displays a «tidal» structure, *i.e.* the ratios among the differential magnetic field intensity in three orthogonal directions are the same as the ratios among the gravitational gradient tensor components pertaining to the same directions; it is also seen that the same characteristic ratios occur, both locally and non-locally, among the components of the magnetic field intensity and among the components of the gradient tensors of the two fields.

Key words *gravity – magnetics – Poisson relation*

1. Introductory remarks

The potentialities of the so-called «Poisson relation», which holds for uniformly magnetized bodies of constant density, in highlighting the connections between the gravitational and the magnetic fields of such bodies have been considered. This relation has frequently been used in the past (Grant and West, 1965) as a means to evaluate the magnetic field of uniformly magnetized bodies via their gravitational field. The reason is that, at least in principle, the potential field due to a distribution of monopoles is more easily handled than the potential field arising from a distribution of dipoles. Nevertheless, we are not concerned here with the application aspects of Poisson relation, but instead with more intrinsic relations between these two potential fields which stem from it.

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Since the magnetic field intensity $\boldsymbol{H}$ due to a magnetized body of volume V (fig. 1) with a magnetic dipole moment per unit volume $\boldsymbol{M}$ is

$$\boldsymbol{H}(\boldsymbol{r}) = \nabla \int_V (\boldsymbol{M}\cdot\nabla) \frac{dV}{|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{1.1}$$

we can write, in the case that the direction of magnetization α is the same throughout the body, being in this case $\boldsymbol{M}\cdot\nabla = M\,\partial/\partial\alpha$

$$\boldsymbol{H}(\boldsymbol{r}) = \nabla \frac{\partial}{\partial\alpha} \int_V \frac{M}{|\boldsymbol{r}-\boldsymbol{r}_0|}\, dV. \tag{1.2}$$

Since the gravitational force $\boldsymbol{g}$ due to a body of constant density ρ is

$$\boldsymbol{g}(\boldsymbol{r}) = G\rho\nabla \int_V \frac{dV}{|\boldsymbol{r}-\boldsymbol{r}_0|} \tag{1.3}$$

we finally have, for a uniformly magnetized body

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{M}{G\rho}\frac{\partial}{\partial\alpha}\boldsymbol{g} \tag{1.4}$$

which is the so-called Poisson relation.

2. Local and non-local implications of Poisson relation

We may write

$$\frac{\partial}{\partial \alpha} \boldsymbol{g} = U_{rs} \, l^r \qquad (2.1)$$

where U_{rs} is the gravitational gradients tensor of the body and l^r is the unit vector in the direction of magnetization; it thus follows

$$H_s = \frac{\mathcal{M}}{Gm} U_{rs} \, l^r \qquad (2.2)$$

where H_s denotes the components of the magnetic field intensity, $\mathcal{M}$ the magnetic dipole moment of the body and m its mass. We see therefore that the magnetic field of the body is closely related to the properties of its gravitational gradients tensor U_{rs}; in this connection let us consider, *e.g.*, a spherical body having a constant density which is uniformly magnetized along the z-axis of a Cartesian reference; since

$$U_{rs} = \frac{Gm}{r^3} \begin{Vmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3xy & 3y^2 - r^2 & 3yz \\ 3xz & 3yz & 3z^2 - r^2 \end{Vmatrix} \qquad (2.3)$$

at equidistant points $P_1\,(r, 0, 0)$, $P_2\,(0, r, 0)$, $P_3\,(0, 0, r)$ from the origin along the coordinate axes we have

$$U_{rs}(P_1) = \frac{Gm}{r^3} \begin{Vmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}$$

$$U_{rs}(P_2) = \frac{Gm}{r^3} \begin{Vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \qquad (2.4)$$

$$U_{rs}(P_3) = \frac{Gm}{r^3} \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{Vmatrix}.$$

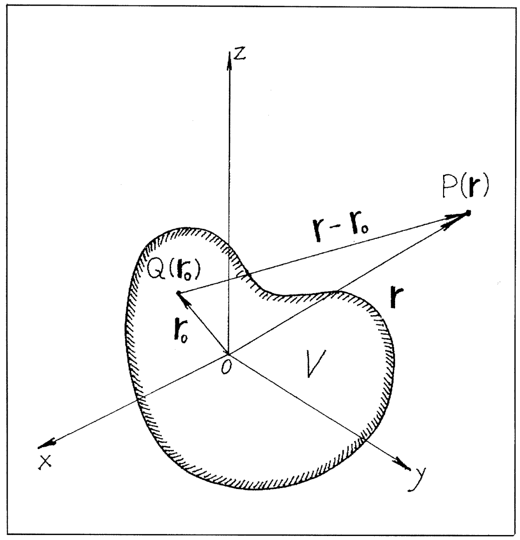

Fig. 1. Magnetics and gravity at P.

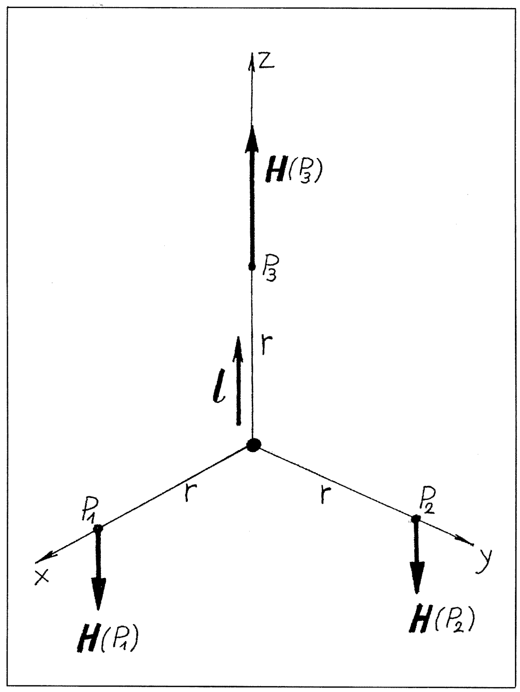

Fig. 2. Magnetic field intensity at the space points P_1, P_2, P_3.

It follows

$$H_z(P_1) = \frac{\mathcal{M}}{Gm} U_{zz}(P_1)$$
$$H_z(P_2) = \frac{\mathcal{M}}{Gm} U_{zz}(P_2) \qquad (2.5)$$
$$H_z(P_3) = \frac{\mathcal{M}}{Gm} U_{zz}(P_3).$$

Since

$$U_{zz}(P_1) = U_{xx}(P_3) = \frac{Gm}{\mathcal{M}} H_z(P_1)$$
$$\qquad (2.6)$$
$$U_{zz}(P_2) = U_{yy}(P_3) = \frac{Gm}{\mathcal{M}} H_z(P_2)$$

we obtain, considering the Laplace condition for U_{rs}, the following relation

$$H_z(P_1) + H_z(P_2) + H_z(P_3) = 0. \qquad (2.7)$$

It is also seen, due to rotational symmetry around the z-axis (fig. 2)

$$H_z(P_1) = H_z(P_2) = -H_z(P_3)/2. \qquad (2.8)$$

The interest of the relations (2.7)-(2.8) is mainly in the way they have been obtained here and in the characteristic ratios −1, −1, 2 among the components of $\boldsymbol{H}$ at the space positions P_1, P_2, P_3.

In a Cartesian reference we obtain from (2.2)

$$H_{ij} = \frac{\mathcal{M}}{Gm} U_{ij}/s\, l^s \qquad (2.9)$$

where H_{ij} is the gradienst tensor for the magnetic field intensity. If a spherical shape for the body is assumed

$$H_{ij} = -\frac{3\mathcal{M}}{r^5} \left\| \begin{matrix} \left(\frac{5x^2}{r^2} - 1\right)z & \frac{5xyz}{r^2} & \left(\frac{5z^2}{r^2} - 1\right)x \\ \frac{5xyz}{r^2} & \left(\frac{5y^2}{r^2} - 1\right)z & \left(\frac{5z^2}{r^2} - 1\right)y \\ \left(\frac{5z^2}{r^2} - 1\right)x & \left(\frac{5z^2}{r^2} - 1\right)y & \left(\frac{5z^2}{r^2} - 3\right)z \end{matrix} \right\|. \qquad (2.10)$$

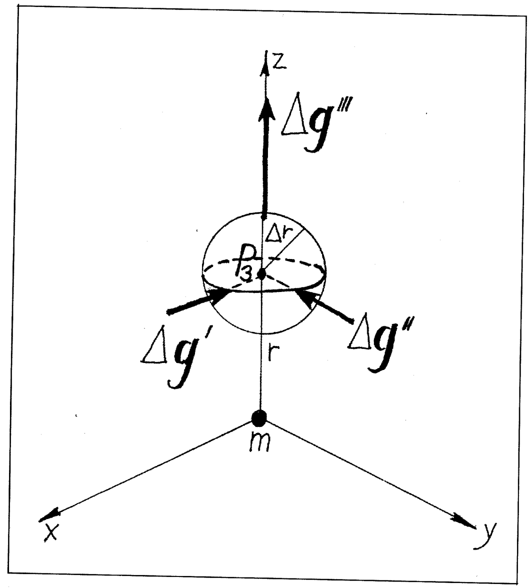

Fig. 3. Tidal gravitational field.

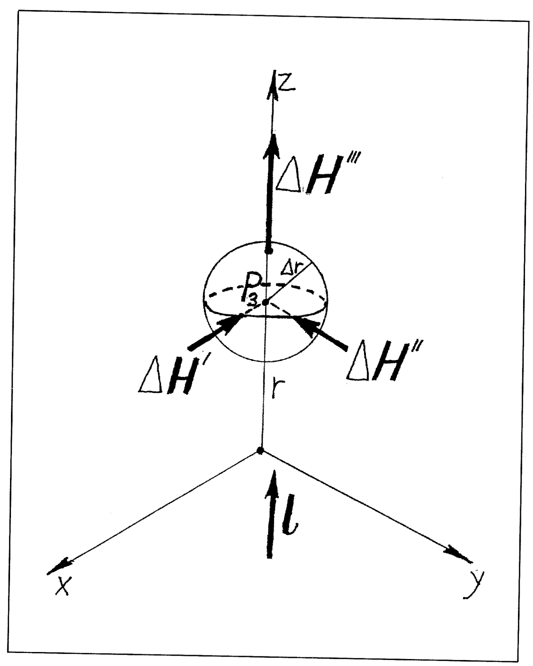

Fig. 4. Tidal magnetic field.

By comparison of (2.3) and (2.10) it follows, along the magnetization axis

$$H_{ij}(0, 0, r) = -\frac{3\mathcal{M}}{Gmr} U_{ij}(0, 0, r). \qquad (2.11)$$

We therefore see that, apart from the steepest decrease of H_{ij} with the distance with respect to U_{ij}, the gradient tensors for the gravitational and for the magnetic field of the body display the same structure along the magnetization axis. This means that, as is the case for the gravitational field of the body, which at the points $(\Delta r, 0, r)$, $(0, \Delta r, r)$, $(0, 0, r+\Delta r)$ of a small sphere with radius Δr centered at P_3 gives rise to the tidal forces

$$\Delta \boldsymbol{g}' = -\frac{Gm\,\Delta r}{r^3}(1, 0, 0)$$

$$\Delta \boldsymbol{g}'' = -\frac{Gm\,\Delta r}{r^3}(0, 1, 0) \qquad (2.12)$$

$$\Delta \boldsymbol{g}''' = -\frac{2\,Gm\,\Delta r}{r^3}(0, 0, 1),$$

we see that due to (2.11), we obtain at the same points, for the magnetic field (figs. 3 and 4)

$$\Delta \boldsymbol{H}' = -\frac{3\,\mathcal{M}\,\Delta r}{r^4}(1, 0, 0)$$

$$\Delta \boldsymbol{H}'' = -\frac{3\,\mathcal{M}\,\Delta r}{r^4}(0, 1, 0) \qquad (2.13)$$

$$\Delta \boldsymbol{H}''' = -\frac{6\,\mathcal{M}\,\Delta r}{r^4}(0, 0, 1).$$

We therefore see that the magnetic field has a «tidal» structure at points of the magnetization axis; it follows also that

$$U_{xx}(P_3) : U_{yy}(P_3) : U_{zz}(P_3) = -1 : -1 : 2 \qquad (2.14)$$

$$H_{xx}(P_3) : H_{yy}(P_3) : H_{zz}(P_3) = -1 : -1 : 2 \qquad (2.15)$$

Considering also eq. (2.8) we can thus conclude that the same characteristic ratios occur among components of the magnetic field intensity and among components of the gradients tensors H_{ij}, U_{ij} both in local and in non-local relations. This rather unusual behaviour seems not to take place by chance and deserves further investigations. A possible explanation could perhaps be found within the frame of a topological approach to the properties of gravitational and magnetic fields (Bocchio, 1989, 1990; Hide, 1997).

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