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The set valued unified model of dispersion and attenuation for wave propagation in dielectric (and anelastic) media

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Abstract
Since the dispersion and attenuation properties of dielectric and anelastic media, in the frequency domain, are expressed by similar formulae, as shown experimentally by Cole and Cole (1941) and Bagley and Torvik (1983, 1986) respectively, we note that the same properties may be represented in the time domain by means of an equation of the same form; this is obtained by introducing derivatives of fractional order into the system functions of the media. The Laplace Transforms (LT) of such system functions contain fractional powers of the imaginary frequency and are, therefore, multivalued functions defined in the Riemann Sheets (RS) of the function. We determine the response of the medium (dielectric o anelastic) to a generic signal summing the time domain representation due to the branches of the solutions in the RSs of the LT. It is found that, if the initial conditions are equal in all the RSs, the solution is a sum of two exponentials with complex exponents, if the initial conditions are different in some of the RSs, then a transient for each of those RSs is added to the exponentials. In all cases a monochromatic wave is split into a set of waves with the same frequency and slightly different wavelengths which interfere and disperse. As a consequence a monochromatic electromagnetic wave with frequency around 1 MHz in water has a relevant dispersion and beats generating a tunnel effect. In the atmosphere of the Earth the dispersion of a monochromatic wave with frequency around 1 GHz, like those used in tracking artificial satellites, has a negligible effect on the accuracy of the determination of the position of the satellites and the positioning of the bench marks on the Earth. We also find the split eigenfunctions of the free modes of infinite plates and shells made of dielectric and anelastic media.

Key words *set-valued – dielectric – anelastic – wave – tunnel effect – fractional derivatives*

1. Introduction

Physics and mathematics have long given great emphasis to the modelling of energy dissipation and dispersion in the propagation of elastic waves and perturbations in solid anelastic media and of electromagnetic waves and perturbations in plasmas, liquids and solid dielectrics (*e.g.*, Heaviside, 1899; Cisotti, 1911; Cole and Cole, 1941; Bagley and Torvik, 1983).

Laboratory experiments (*e.g.*, Cole and Cole, 1941; Hasted, 1973; Bagley and Torvik, 1983, 1986; Körnig and Müller, 1989; Jacquelin, 1991) have confirmed that the introduction of memory mechanisms into the constitutive equations of the propagation of the above mentioned waves and perturbations adequately presents their phenomena of dispersion and energy dissipation.

A Q power law for seismic waves, which may be explained by a memory mechanism, has also been observed by many authors listed in table I of Caputo (1984).

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The most successful memory mechanisms used to represent dispersion and energy dissipation in anelastic media, in dielectrics and electric networks, are those of the derivatives of fractional order (Caputo, 1969; Caputo and Mainardi, 1971; Bagley and Torvik, 1983, 1986; Pelton *et al.*, 1983; Jacquelin, 1991) defined as follows

$$d^{w+z} f(t) / dt^{w+z} = \\ = (1-\Gamma(1-z)) \int_0^t f^{(w+1)}(v) dv / (t-v)^z \tag{1.1}$$

where $z = m/u$ (m, u positive integer and prime, $m < u$) and w is a positive integer. In the following, in order to simplify the discussion without altering the essence of the results we shall assume $w = 0$; this assumption is also in agreement with many experimental results on dissipation and dispersion in anelastic and dielectric media.

The Laplace Transform (LT) of (1.1) is (Caputo, 1969)

$$\mathrm{LT}[d^{w+z} f(t) / dt^{w+z}] = \\ = p^{w+z}\mathrm{LT}[f(t)] + p^z \sum_{n=0}^{w} p^{n-1} f^{(w-n)}(0) \tag{1.2}$$

where p is th LT parameter.

The fractional order derivative defined in (1.1) will be used extensively in this note to solve initial value problems of anelastic and dielectric media.

In this paper we first note that the index of refraction n of anelastic and dielectric media, whose constitutive equations include derivatives of fractional order, contains a fractional power of the imaginary frequency which gives a set valued function for n defined in u RSs. Then we find the response of the medium to a generic input summing the responses due to the eigenfunctions of the RSs corresponding to all the values of n. In particular, we discuss the solution depending on the initial conditions of the different RS with specific attention to the case of the propagation of monochromatic waves.

2. The problem

If Y is the LT of induction (or strain) y and X is the LT of the electric field (or stress) x applied to a dielectric (anelastic) medium (*e.g.*, Cole and Cole, 1941; Caputo and Mainardi, 1971; Caputo, 1996) the relation

$$Y = [\varepsilon_\infty + (\varepsilon_0 - \varepsilon_\infty)/(1+(\tau p)^z)]X = Z(p)X \tag{2.1}$$

where $p = i\Omega$, with Ω, frequency, represents well in the frequency domain the data of most laboratory experiments (*e.g.*, Cole and Cole, 1941; Hasted, 1973; Bagley and Torvik, 1983, 1986; Körnig and Müller, 1989, Jacquelin, 1991) and of field observations (Pelton *et al.*, 1983; Caputo, 1984, 1997; Körnig and Müller, 1989). ε_0 and ε_∞ are the values of the system function $Z(p)$ at infinite and zero frequency respectively, τ is a relaxation time.

The time domain representation of (2.1) is (Caputo and Mainardi, 1971)

$$(1+\tau^z d^z / dt^z) y = (\varepsilon_0 + \varepsilon_\infty \tau^z d^z / dt^z) x. \tag{2.2}$$

Given x, according to (1.1), relation (2.2) is an integro-differential equation in y.

In the following we shall use the relations (2.1) and (2.2) without specifying whether the medium represented is dielectric or anelastic because, depending on the dimensions of the parameters ε_0 and ε_∞, they represent the dispersive and dissipative properties of both types of media.

We shall here discuss the one dimensional case of (2.1) and (2.2) since the two-dimensional and the three-dimensional cases are more complicated to discuss but yield the same qualitative results.

To study the propagation of electromagnetic or elastic waves, or of the free modes or of the propagation of perturbations in media whose physical properties are represented by (2.1) one may use the explicit time domain differential representation (2.2) of the relation between x and y or (2.1) (which is the LT of (2.2). Here we will use (2.1).

Since we assume that $z = m/u$ is a rational number, (2.1) is a set valued function of $p = i\Omega$ with u values which define u different values of the system function and therefore of the index of refraction n and of the velocity fields for the medium. However Physically Acceptable (PA) are only those values of the index of refraction $n(\Omega = n_r(\Omega) - n_i(\Omega)$ which give positive velocity and positive Q, that is

$$n_i(\Omega) \geq 0, \quad n_r(\Omega) \geq 0. \tag{2.3}$$

In practice, it is common to select and use only the *principal* value of $n(\Omega)$, real or complex; however there is no physical reason for this choice nor for the selection of any other single PA value of the set valued $n(\Omega)$, since they are *a priori* all acceptable provided

$$n_i(\Omega) \geq 0, \quad n_r(\Omega) \geq 0.$$

In this note we shall use *all* the PA values of $n(\Omega)$ and seek the sum of the LT^{-1} of all the branches of (2.1), which gives the time domain representation of the sum of the waves, with the same frequency but different velocities and phases, obtained in the RSs branches resulting from the PA values of the index of refraction. The sum of all these waves includes the phenomena of dispersion and dissipation of energy.

3. The LT^{-1} of the system function. Case when the initial conditions are nil on all the RSs

This problem has already been discussed by Caputo (1994a, 1996), however the important effects of the initial values of each of the set-valued functions in the different RSs was not discussed in the previous work (Caputo, 1994a,b, 1995a,b), and will be tackled in this and the next paragraph considering general initial conditions.

The LT^{-1} of each of the u Riemann branches of (2.1) is found integrating it in all the u sheets of the Riemann branches. In the Appendices A and B it is shown that, when u and m have opposite parity, the integration is done along the path shown in fig. 1 (where the branch cuts along the negative real axis are required by the fact that the initial values of the solution may be different in each RS), when u and m have the same parity then the integration is made along the path of fig. 2a-c.

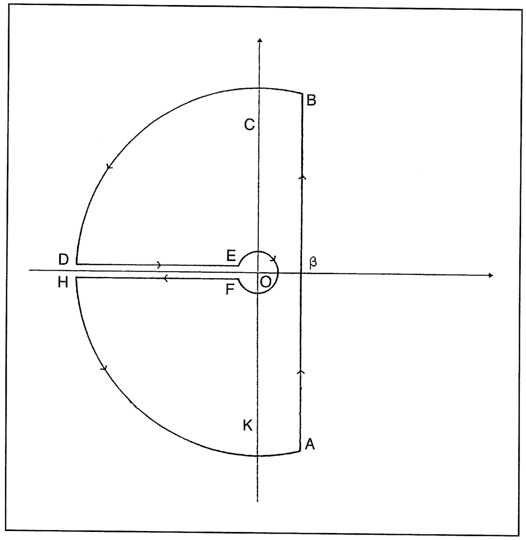

Fig. 1. Contour for the computation of the LT^{-1} of the set valued function $Y = [\varepsilon_\infty + (\varepsilon_0 - \varepsilon_\infty)/(1 + (\tau p))]X = Z(p)$ in all its Riemann sheets.

To take into account that the different branches may have different initial conditions, we note that according to (1.2), the LT of (2.2) also contains terms with the initial conditions for $x(t)$ and $y(t)$ and rewrite it as follows

$$Y = [\varepsilon_\infty + (\varepsilon_0 - \varepsilon_\infty)/(1 + \tau^z p^z)]X + [\tau^z (y(0) - \varepsilon_\infty x(0))/p^{1-z}]/[1 + \tau^z p^z] \tag{3.1}$$

where $p^z = (i\Omega)^z$, with Ω frequency, is a set valued function with u values.

The solution in each RS will be indicated with $y_h(h)$ where the index $h = 0, 1, \ldots, u-1$ identifies the h-th RS; as already stated the initial values $y_h(0)$ may be different in the u RSs.

In order to simplify the solution we shall first assume that all $y_h(0) = 0$ and $x(0) = 0$. We shall later see the effect of non zero initial values $y_h(0)$.

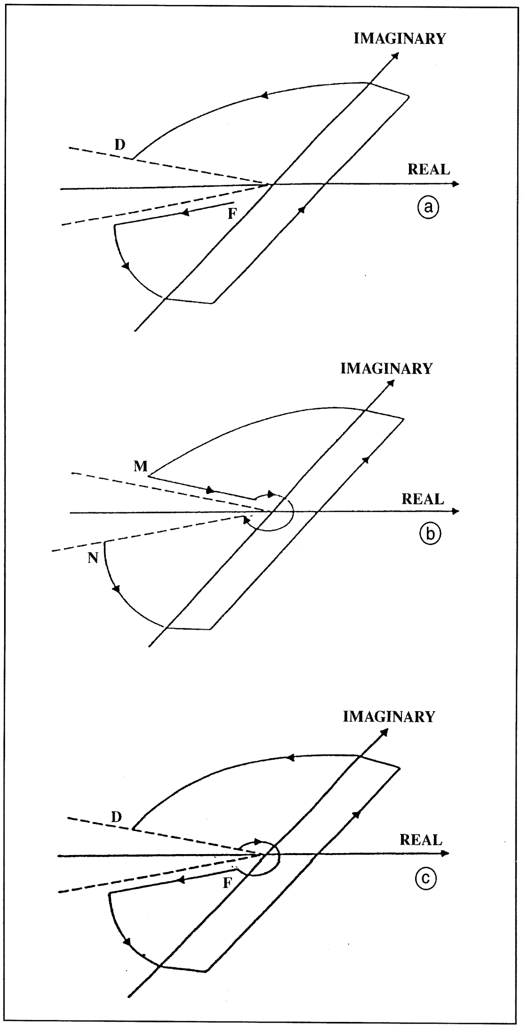

Fig. 2a-c. Contour for the computation of the LT^{-1} of the set valued function $Y = [\varepsilon_\infty + (\varepsilon_0 - \varepsilon_\infty) / (1 + (\tau p)]X = Z(p)$: a) in the first Riemann sheet ($h = 0$); b) in the last Riemann sheet ($h = u - 1$); c) in the first Riemann sheet for closing the circuit after running on all Riemann sheets from the first to the last and back. See also the text.

The procedure used to find these LT^{-1} is shown in Appendix A where one finds the transients in all the RSs and in Appendix B where one finds that there is a total of m poles in the set of the u RSs.

The poles of

$$\tau^{-z} / (p^z + \tau^{-z}) \qquad (3.2)$$

then give the following contribution in the LT^{-1} of (3.1)

$$\sum_{k=0}^{u-1} (u / m\tau)\{\exp(i(2k+1)\pi(u/m-1))\} \cdot$$

$$\cdot\{\exp((t/\tau)\exp((i(2k+1)\pi u/m)))\} \qquad (3.3)$$

where the first two factors, independent of t, are the residues (see formula (B.4) in Appendix B) of the PA poles possibly present in the k-th RS and the sum is extended to the RSs with PA poles, that is the poles which imply positive Q and positive velocity according to (2.3). Writing (3.3) as

$$F(t,k) = \sum_{k=0}^{u-1} (u / m\tau)\{\exp[i[(t/\tau) \cdot$$

$$\cdot \sin((2k+1)u\pi/m) + (2k+1)\pi(u/m-1)]]\} \cdot (3.4)$$

$$\cdot\{\exp[(t-\tau)\cos((2k+1)u\pi/m)]\}$$

it is seen that the frequency Ω_k, the phase Φ_k, and the Q_k^{-1} resulting in (3.4) are

$$\Omega_k = (1/\tau)\sin((2k+1)\pi u/m)$$

$$\Phi_k = (2k+1)\pi(u/m-1) \qquad (3.5)$$

$$Q_k^{-1} = (2/\tau)\cos((3k+1)\pi u/m).$$

Since the values of the index of refraction $n(\Omega)$ obtained from (2.1) repeat every u successive value of k, while the value of Q_k^{-1} in (3.5) repeats every successive m value of k, in order to have all the PA RSs (where both physically acceptable poles and velocity fields are in the same RS) one must give to k all values in the range $0 \le k \le um-1$; as an example, when $m/u = 4/5$, the PA values of the index of refraction n are in the RSs and obtained for $k = 0$ and $k = 6$.

However, if one is interested in knowing only the PA velocity fields and poles it is sufficient to vary k from 0 to $u - 1$; in fact when varying k from $u-1$ to $mu-1$ one would always find the RSs where the PA poles and velocity fields are coupled, but they have the same residues and velocity fields already found with one of the previous values of k.

Figure 3 shows the velocity of the waves of a medium with index of refraction given by the $Z(p)$ defined in (2.1) and the values of k associated to PA velocity fields.

The modulus of $1/(1+(i\Omega\tau)^z))$, where $p = i\Omega$ with Ω frequency, appearing in the system function $Z(p)$ defined in (2.1) is shown in fig. 4. This modulus is the transfer function of the filter implied by the relation (2.1).

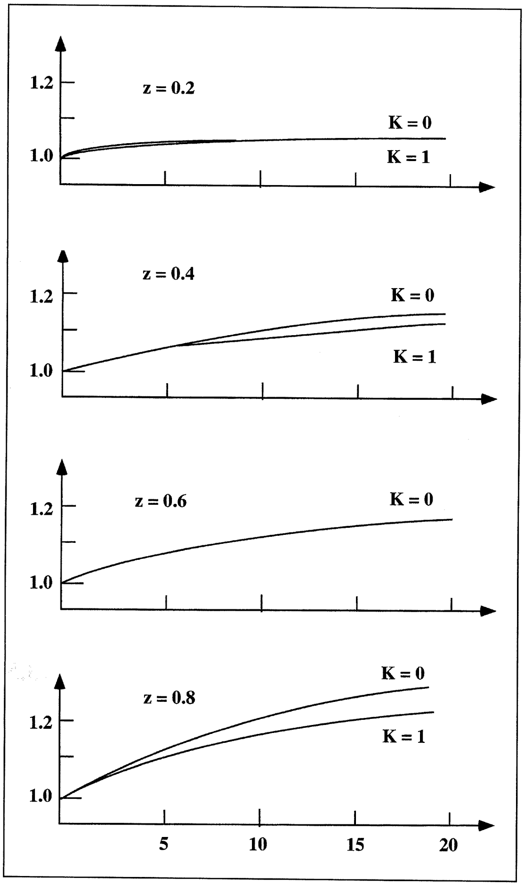

Fig. 3. Velocity of the waves as a function of the frequency Ω measured in units of the inverse of the relaxation time τ, obtained for various values of $z = m/u$ and the values of k which give physically acceptable velocity fields. The ordinate is in units of $\varepsilon^{-1/2}$. It is also assumed, for similarity with water, that $\varepsilon_\infty = 1.8$ and $\varepsilon_0 = 81$ (Hasted, 1973). The curves with $z = 0.2$ and $K = 0$, $K = 1$ are almost identical at low frequencies.

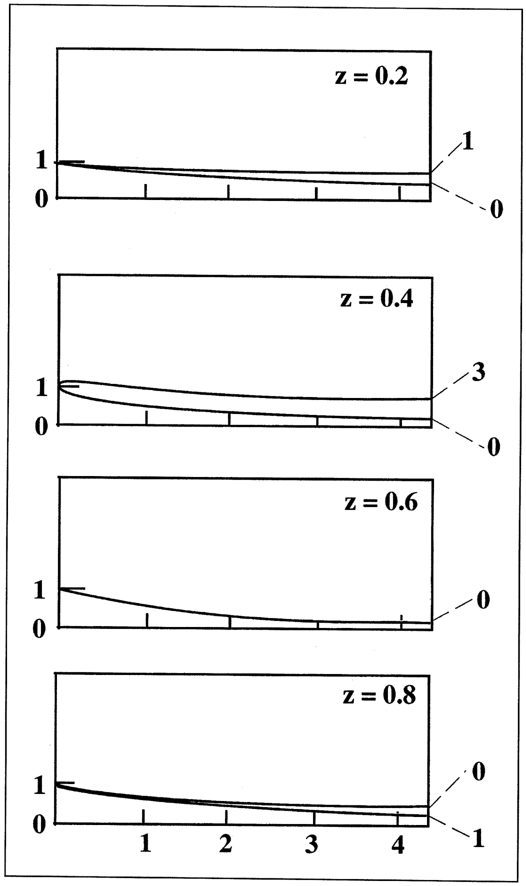

Fig. 4. Modulus of $1/(1 + (i\,\Omega\,\tau)^z)$, or weight function of the filter representing the frequency dependent part of the system function (1.2), as a function of the frequency Ω measured in units of the inverse of the relaxation time τ, for various values of $z = m/u$ and the values of k which give velocity fields physically acceptable. The abscissa shows the frequency in units of the inverse of the relaxation time τ.

When m and u have opposite parity there is no pole on the negative real axis of the RS. When m and u are both odd there are poles on the negative real axis of the RS which imply residues with zero frequency and dissipation $Q_k^{-1} = 2/\tau$.

Applying these results and using the formulae of the Appendices A and B, one finds that the integrated time domain representation of (3.1) is

$$y = \varepsilon_\infty x + (\varepsilon_0 - \varepsilon_\infty) x^* \sum_{k=0}^{u-1} F(t,k) \qquad (3.6)$$

where the poles considered for the residues are obviously those within the loop of each PA RS.

It is seen that, assuming $n = n_r - in_i$, in order to obtain $n_i \geq 0$ at all frequencies for a given value of k, that is in order to have a velocity field with $Q_k^{-1} \geq 0$, the value of k must give

$$\begin{aligned} &\sin((2k+0.5)\pi m/u) \geq 0, \\ &\cos((2k+0.5)\pi m/u) \geq 0. \end{aligned} \qquad (3.7)$$

However, in order to have $Q_k^{-1} \geq 0$ in (3.1) it must also be

$$\cos((2k+1)\pi u/m) \geq 0. \qquad (3.8)$$

Since n results from the two values of the square root of the dielectric parameter, which are symmetric with respect to the origin, there is always one value of n with a real part positive or nil at all frequencies for a given value of k.

The arguments θ of the poles are in $\theta = ((1+2k)\pi u/m) > \pi$ and there is no pole on the RS associated to $k = 0$ where $-\pi < \theta < \pi$. However there is always a RS which contains $\theta = \pi u/m > \pi$ since the upper limit of θ is $(2u-1)\pi$ in the RS $h = u-1$.

The residues are due to the poles of (2.1) which, as shown in Appendix B, are m; therefore not all the u RSs have a pole and, as already stated, not all of them are PA. For physical applications, the sum in (3.4) is extended only to the terms with k which satisfy (3.7) and (3.8), which are about $u/2$ and have frequency smaller or equal to $1/\tau$.

4. Case when the initial conditions are different in all the RSs

Sometimes, for physical reasons, the input must be considered nil at $t = 0$. Since the velocity of any signal must be finite, we may assume that $x'(t)$ be finite for $0 \leq t$ and, assuming causality, this implies that, when $x(t) = 0$ for $t \leq 0$, at $t = 0$ the observed initial value must be $y(0) = \sum_{h=0}^{u-1} y_h(0) = 0$; one may therefore assume that $y_k(0) = 0$ for all k or, more in general, one may assume that some $y_h(0) \neq 0$ but that, in any case, their sum be nil.

In the case when $y_h(0)$ and $x(0)$ are not nil then one must take into account the second term in the right hand side of (3.1) in all RSs. Moreover, we are not allowed to make the simplifications due to the cancelling of the integrals in dr in the sum of (A.3) and (A.4).

We note that according to the conclusions of Appendix A, m and u should have opposite parity and one integrates along the path of fig. 1; the case when m and n have the same parity is more complicated and postponed to another study.

Taking into account the initial conditions and the integrals in dr of (A.3) it is seen that the solution of (3.1) is

$$\begin{aligned} y = {} & \varepsilon_\infty x + (\varepsilon_0 - \varepsilon_\infty) x^* \sum_{k=0}^{u-1} F(t,k) + \\ & + \sum_{k=0}^{u-1} F(t,k) \cdot [\tau^{m/u} y_k(0) - \varepsilon_\infty \delta_{0k} x(0)] \cdot \\ & \cdot [((\tau/t)^{m/u} / \Gamma(1-m/u)) \exp(i2\pi k m/u)] + \\ & - \sum_{k=0}^{u-1} [\tau^{m/u} y_k(0) - \varepsilon_\infty \delta_{0k} x(0)] \cdot \\ & \cdot [((\tau/t)^{m/u} / \Gamma(1-m/u)) \exp(i2\pi k m/u] \cdot \\ & \cdot \int_0^\infty (\exp(-rt)) dr [1/(r^{m/u} \exp(i\pi(1+2k)m/u) + \\ & + \tau^{-m/u}) + 1/(r^{m/u} \exp(i\pi(-1+2k)m/u) + \tau^{-m/u})] \end{aligned} \qquad (4.1)$$

where $F(t, k)$ is defined in (3.4) and the factor

$((\tau/t)^{m/u}/\Gamma(1-m/u))\ \exp\ (i2km\pi/u)$ in the lines 1 and 2 of (4.1) comes from the LT^{-1} of $p^{m/u-1}$ formula (2.2) in the k-th RS, which is different in all RS (due to the presence of k in the exponentials) and represents a transient.

The first term in (4.1) reproduces the signal with amplitude ε_∞; the term with $\varepsilon_0-\varepsilon_\infty$ as factor in the first line of (4.1) and that in the second line are due to the residues of the poles of (3.2); the term with $(\varepsilon_0-\varepsilon_\infty)$ as factor is the contribution from the signal $x(t)$, while the terms with $\tau^{m/u}y_k(0)-\varepsilon_\infty\delta_{0k}x(0)$ as factor are the contributions from the initial values.

The terms in lines 3, 4 and 5 are the contributions due to the integrals on the negative real axis of the path of integration of fig. 1.

The sum is extended to the RSs which contain PA poles and velocity fields; the integral on the closed path of fig. 1 is nil when there are no poles inside the closed path; the RSs which contain unacceptable velocity fields or poles are disregarded.

When a source injects a wave with given frequency and direction in a medium with system function $Z(p)$ defined in (2.1), the PA velocity fields give rise to a set of waves represented by (3.4) with the same frequency but slightly different wavelengths and different Q represented by (3.5); fig. 5 shows the amplitude of such a set of waves in water as a function of the distance from the source (Caputo, 1995a).

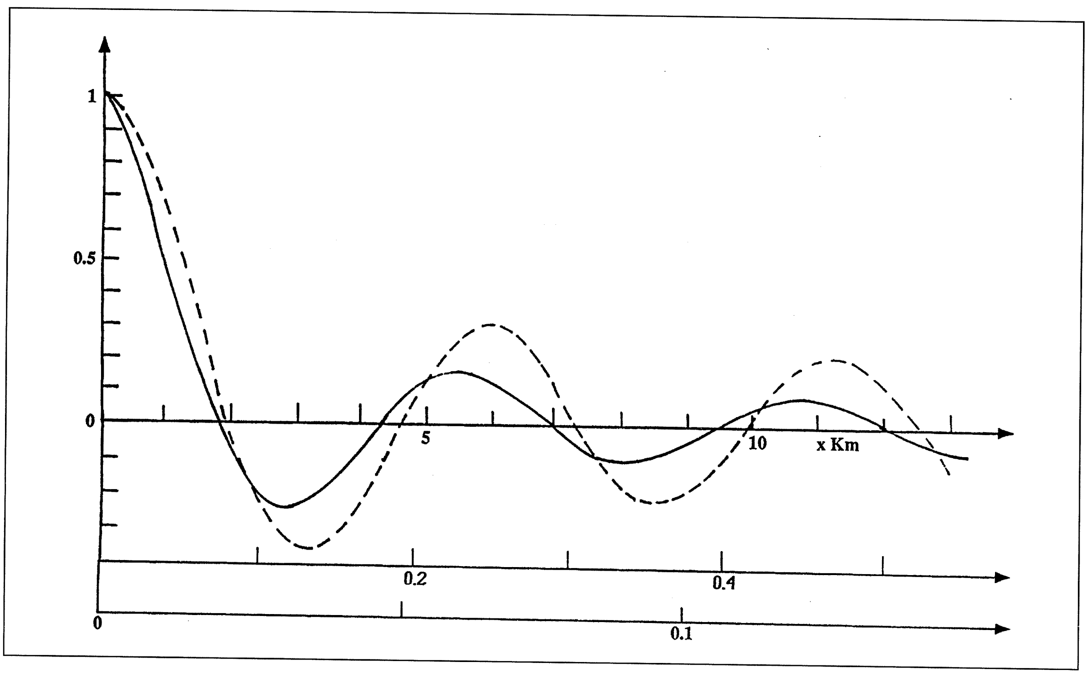

Fig. 5. Amplitude of a set of waves travelling in water as a function of the distance travelled from the source given in abscissa (from Caputo, 1995a). The ordinate is a percent of the total amplitude at the source where the single waves are assumed to have the same amplitude and phase. The solid line is the amplitude of the real set of decaying waves. The dashed line is the theoretical amplitude of the set of waves without the effect of their decay; note the apparent space decay of this amplitude due to the different phases of the single waves of the set along the path caused by their different wavelength. The top scale of the abscissa is for f = 10 MHz, the intermediate scale is for f = 50 MHz and the bottom scale is for f = 100 MHz; in all cases it is assumed that $\varepsilon_0 = 81$, $\varepsilon_\infty = 1.8$, $\tau = 1.525\ 10^{-11}$ and $z = 0.965$ (Hasted, 1973).

In the atmosphere of the Earth, modelled with an index of refraction resulting from (2.1) and with the data of Liebe (1985), a wave emitted with given frequency, phase and direction from a source at the surface of the Earth, splits into a set of waves with the same frequency and slightly different velocities. When the wave is emitted at 1 GHz frequency, which is about the frequency used in the Global Positioning System (GPS), the phases of the waves of the set, at about 60 km from the source, are spread in a time window of less than 0.01 ns (Caputo, 1994b); this spreading would cause an uncertainty of only 0.3 cm when locating positions of artificial satellites in space and a similar error in the location of the benchmarks on the ground.

5. The eigenfunctions

In this section we shall obtain the time domain representation of the free modes of anelastic and dielectric media which are governed, in the frequency domain, by a constitutive relation of the type (2.1). The splitting of the eigenvalues has already been discussed in Caputo (1996) in the cases when z is rational or irrational, here we shall focus on the solutions obtained considering the set of eigenfunctions in all the RSs in the case when z is rational.

It is shown that in the time domain the equations governing the displacements in bodies with shape of infinite plates or of spherical shells and made of dielectric or anelastic media whose dispersion properties are described by a constitutive equation of the type (2.2), may be separated (Caputo, 1993, 1996) and that the time component $g(t)$ of the eigenfunctions is a solution of the following equation in $g(t)$

$$\varepsilon_\infty \tau^z (d^{2+z} / dt^{2+z}) g + \varepsilon_0 (d^2 / dt^2) g + \tag{5.1}$$
$$+ \tau^z \gamma^2 (d^z / dt^z) g + \gamma^2 g = 0$$

where γ is a parameter identifying the eigenfunction.

In eq. (5.1) the terms with ε_0 or ε_∞ should be multiplied by the magnetic permittivity μ in the case of dielectric media and divided by density d in the case of anelastic media; in both cases it is here assumed $\mu = d = 1$.

In practice eq. (5.1) is an integral differential equation in the unknown $g(t)$. In order to solve it we may consider it in the LT domain and find according to (1.2)

$$G = [(\varepsilon_\infty \tau^z p^{1+z} + \varepsilon_0 p + \tau^z \gamma^2 p^{z-1}) g(0) +$$
$$+ (\varepsilon_0 + \tau^z \varepsilon_\infty p^z) g'(0) + \varepsilon_\infty \tau^z p^{z-1} g''(0)] / \tag{5.2}$$
$$/ [\varepsilon_\infty \tau^z p^{2+z} + \varepsilon_0 p^2 + \tau^z \gamma^2 p^z + \gamma^2].$$

A case of particular interest in applications is that when we may assume $\varepsilon_\infty = 0$ (Hasted, 1973) in which case equation (5.2) simplifies to

$$G = [(\varepsilon_0 p + \tau^z \gamma^2 p^{z-1}) g(0) +$$
$$+ \varepsilon_0 g'(0)] / [\varepsilon_0 p^2 + \tau^z \gamma^2 p^z + \gamma^2]. \tag{5.3}$$

The right hand member of eq. (5.3) is a set-valued function with u values defined in u RSs each being identified by an integer h $(0 \le h \le u-1)$.

We shall find here the sum of the set of u eigenfunctions $g_h(t)$, $(h = 0,1,\dots u-1)$, which are associated to the same value of the parameter γ, which is given by the sum of the LT^{-1} of the u Riemann branches $G_h(p)$ of $G(p)$ in the u RSs

$$\mathrm{LT}^{-1} \sum_{h=0}^{u-1} G_h(p) = \sum_{h=0}^{u-1} g_h(t), \quad -\pi < \theta < \pi$$
$$G_h(p) = G(r \exp(i2h\pi + \theta)) \tag{5.4}$$

r is here the modulus of p and θ its argument.

The computation of the LT^{-1} of (5.4) is made using its poles and integrating along the path of fig. 1 since, as seen in Appendix B, (5.3) has no poles on the negative real axis.

The poles are obtained from the roots of the equation

$$\varepsilon_0 q^{2u} + \tau^{m/u}\gamma^2 q^m + \gamma^2 = 0 \quad (5.5)$$

where q stands for $p^{1/u}$.

It has been seen (Caputo, 1994a) that when

$$(\varepsilon_0^{1/2} / \tau\gamma)^{-z} >> 1 \quad (5.6)$$

which is the case in many dielectric and anelastic media, then the roots of (5.5) are distributed in couples in each of the RSs, one in the positive and the other in the negative imaginary half plane; those in the positive imaginary half plane are set on a small circle around the point $i(\gamma / \varepsilon_0^{1/2})$, those in the negative imaginary half plane are set around a small circle with centre in $-i(\gamma / \varepsilon_0^{1/2})$.

We may write

$$G = \sum_{j=0}^{u-1} [(\varepsilon_0 p + \tau^{m/u}\gamma^2 p^{m/u-1}) g_h(0) + \quad (5.7)$$

$$+ \varepsilon_0 g'_h(0)] / [A_j / (q-\alpha_j) + B_j / (q-\beta_j)]$$

where each term of the summation is a function with u values. The branch of $G(p)$ in the h-th RS will be indicated by

$$G_h(p) = G(r \exp(i2h\pi + \theta)) = \sum_{j=0}^{u-1} [\varepsilon_0 g'_h(0) +$$

$$+ (\varepsilon_0 p + \tau^{m/u}\gamma^2 p^{m/u-1}) g_h(0)] \cdot$$

$$\cdot [A_j / (r^{1/u} \exp(i(2h+\theta)) / u - \alpha_j) + \quad (5.8)$$

$$+ B_j / (r \exp(i(2h+\theta)) / u - \beta_j)] +$$

$$-\pi < \theta < \pi.$$

The poles identified by the two fractions with the same j in the summation of (5.8) however are only two. In fact the poles of

$$A_j / (q-\alpha_j) + B_j / (q-\beta_\varphi)$$

$$\alpha_j = \rho_{\alpha j} \exp(i\theta_{\alpha j}), \quad (5.9)$$

$$\beta_j = \rho_{\beta j} \exp(i\theta_{\beta j}), \quad -\pi < \theta < \pi$$

are

$$\rho^u_{\alpha j} \exp(iu\theta_{\alpha j} + 2kiu\pi)$$

$$\rho^u_{\beta j} \exp(iu\theta_{\beta j} + 2kiu\pi). \quad (5.10)$$

When $0 < \theta_{\alpha j} < \pi$ or $0 < \theta_{\beta j} < \pi$ the poles are obtained for $k = 0$ which is the only value of k in the range $0, u-1$ which gives the argument of the pole in the range $\pi, (2u-1)\pi$; when $-\pi < \theta_{\alpha j} < 0$ or $-\pi < \theta_{\beta j} < 0$ the two poles for each j are obtained from the only one value of k ($k = 0$ or $k = 1$) which gives the argument of the pole in the range $-\pi, (2u-1)\pi$. The poles are in α^u_j, β^u_j.

We seek here the sum of the LT^{-1} of all the $G_h(p)$, that is

$$\sum_{h=0}^{u-1} LT^{-1} G(p) = \sum_{h=0}^{u-1} \sum_{j-0}^{u-1} LT^{-1} [[\varepsilon_0 g'_h(0) +$$

$$+ (\varepsilon_0 p + \tau^{m/u}\gamma^2 p^{m/u-1}) g_h(0)][A_j / (q-\alpha_j) +$$

$$+ B_j / (q-\beta_j)]] \quad (5.11)$$

and changing the order of summation

$$\sum_{h=0}^{u-1} LT^{-1} G(p) = \sum_{j=0}^{u-1} \sum_{h-0}^{u-1} LT^{-1} [[\varepsilon_0 g'_h(0) +$$

$$+ (\varepsilon_0 p + \tau^{m/u}\gamma^2 p^{m/u-1}) g_h(0)][A_j / (q-\alpha_j) +$$

$$+ B_j / (q-\beta_j)]]. \quad (5.12)$$

The computation of the right hand of (5.12) is simpler if we assume that all the $g_h(0)$ are nil, as we will do, without loosing much generality in the results.

Integrating over the path of fig. 1 in all the RS we may then write

$2\pi i$ (sum of the residues in all the RSs) =

$$= \sum_{h=0}^{u-1} \int_{\lambda-i\infty}^{\lambda+i\infty} G(p)\exp(pt)dp +$$

$$+\sum_{j=0}^{u-1}\sum_{h=0}^{u-1}\varepsilon_0 g'_h(0)\left[\int_0^{\infty}\exp(-rt)dr\cdot\right.$$

$$\cdot[A_j/(r^{1/u}\exp(i(2h+1)\pi/u)-\alpha_j)+ \quad (5.13)$$

$$-A_j/(r^{1/u}\exp(i(2h-1)\pi/u)-\alpha_j)+$$

$$+B_j/(r^{1/u}\exp(i(2h+1)\pi/u)-\beta_j)+$$

$$\left.+B_j/(r^{1/u}\exp(i2h-1)\pi/u)-\beta_j)]\right].$$

The discussion of the problem is simpler if we assume at first that $g'_h(0) = g'(0)$ for all values of h and therefore $g'(0)$ may be factored out of the summation in h. We shall now discuss this case.

Following the same procedure adopted in Appendix A for the function $\tau^{-m/u}/(p^{m/u}-\tau^{-m/u})$, it may be shown that the sum over h of the integrals in dr of (5.13) is identically nil for any fixed j. Equation (5.13) reduces then to

$2\pi i$ (sum of the residues in all the RSs) =

$$= 2i\pi\sum_{h=0}^{u-1} g_h(t) \quad (5.14)$$

with

$$\sum_{h=0}^{u-1} g_h(t) = \sum_{h=0}^{u-1}\varepsilon_0 g'(0)\sum_{j=0}^{u-1}\cdot \quad (5.15)$$

$$\cdot[A_j\alpha_j^{u-1}\exp(\alpha_j t)+B_j\beta_j^{u-1}\exp(\beta_j t)]$$

where $\sum_{h=0}^{u=1}$ indicates the sum over the PA RSs.

In the case considered here, $(\varepsilon_0^{1/2}/\tau\gamma)^{1/z} >> 1$, the poles α_j^u and β_j^u are very near $i(\gamma/\varepsilon_0^{1/2}$ or $-i(\gamma/\varepsilon_0^{1/2})$; remembering that in (5.1) we assumed unity magnetic permittivity in the case of dielectric media and unity density in the case of anelastic media, indicating with v the phase velocity of the medium, we see that the poles are very near to the classic ones

$$\pm\gamma/\varepsilon^{1/2} = \pm\gamma v \quad (5.16)$$

and, when τ approaches zero, the poles approach (5.16).

When the $g'_h(0)$ are not equal then in (5.13) the sum over h of the integrals in dr of (5.13) is not nil and the double summation of the integrals in dr appearing in (5.13), divided by $2\pi i$, should be subtracted in the right hand member of the solution (5.15). One obtains

$$\sum_{h=0}^{u-1} g_h(t) = \sum_{h=0}^{u-1}\varepsilon_0 g'_h(0)\sum_{j=0}^{u-1}\cdot$$

$$\cdot[A_j\alpha_j^{u-1}\exp(\alpha_j t)+B_j\beta_j^{u-1}\exp(\beta_j t)]-$$

$$+\sum_{j=0}^{u-1}\sum_{h=0}^{u-1}\varepsilon_0 g'_h(0)\left[\int_0^{\infty}\exp(-rt)dr\cdot\right.$$

$$\cdot[A_j/(r^{1/u}\exp(i(2h+1)\pi/u)-\alpha_j)+ \quad (5.17)$$

$$-A_j/(r^{1/u}\exp(i(2h-1)\pi/u)-\alpha_j)+$$

$$+B_j/(r^{1/u}\exp(i(2h+1)\pi/u)-\beta_j)+$$

$$\left.-B_j/(r^{1/u}\exp(i(2h-1)\pi/u)-\beta_j)]\right]/2i\pi$$

The presence of the terms $\varepsilon_0 g'_h(0)$ in (5.17) gives a better perspective of the type of solution reached because it contains explicitly the arbitrary initial conditions $g'_h(0)$ on the PA RSs. Which is an important aspect the nature of the solution found here. The solution (5.17) also shows the presence of the transients represented by the integrals and the presence of the terms $\exp(\alpha_j^u t)$, $\exp(\beta_j^u t)$ which imply that there are as many acceptable frequencies $\mathrm{Im}\,\alpha_j^u$, $\mathrm{Im}\,\beta_j^u$ as there are negative $\mathrm{Re}\,\alpha_j^u$, $\mathrm{Re}\,\beta_j^u$.

6. Conclusions

We have seen that the derivatives of fractional order which appear in the constitutive equations of anelastic and dielectric media give a system functions $Z(p)$, defined in (2.1), and, therefore, an index of refraction with set-valued properties.

This index of refraction implies that a monochromatic wave will split into a set of waves with slightly different velocities and wavelengths which will therefore disperse and interfere; the set of waves will also have beats.

The velocities of the waves of the set, resulting from 1/Re n are illustrated with an example in fig. 3, for each value of k which satisfies eqs. (3.7) and (3.8) which condition the selection of the physically acceptable solutions. They are decreasing functions of the frequency with asymptotic value $1/[\varepsilon_\infty^{1/2}]$; the zero frequency value of the velocity of these waves is $1/[\varepsilon_0^{1/2}]$. Both values are independent of z and k. In both cases unity magnetic permittivity has been assumed when the medium is dielectric or unity density when the medium is anelastic respectively.

We found the responses arising from the poles of the system function $Z(p)$ defined in (2.1) (or of the index of refraction), in the case when the initial values of the responses of the set are zero in all RSs.

When the initial conditions are different in some RSs, then transient terms arising from these RSs are present. However, these transients become rapidly negligible when the relaxation time τ of the system function is large. The contributions arising from the poles, which have frequency Ω_k, given by (2.3), when the input signal has frequency Ω^*, have beats with frequencies $\Omega_k \pm \Omega^*$; these beats are accompanied with a signal with amplitude ε_∞ and frequency Ω^*.

It is noted that (2.1) acts as a low pass filter on the input signals, with a weight function asymptotically decreasing with increasing frequency; this is illustrated in fig. 4 where only the curves whose k satisfies (3.7) and (3.8) are considered.

In this note we also found the time domain representation of the free modes of infinite plates and spherical shells in the case when the initial values of the first order derivatives of the eigenfunctions arising from the RSs are assigned. The time components of the eigenfunctions are represented by simple exponentials with complex exponents, however the solution verifies the presence of transients caused by the different initial values of the first order derivatives of the eigenfunctions of the RSs.

A wave originating in the medium with given frequency and direction will split into a set of waves with the same frequency and slightly different wavelengths which interfere and disperse. The dispersion of such waves in water has been studied by simply considering the velocity fields which have a positive Q^{-1} and assuming zero initial values and zero phases for all the waves of the monochromatic set finding that the amplitude of the set of waves has beats along the paths as shown in fig. 5.

The dispersion of such waves in the atmosphere of the Earth has a negligible effect when locating the artificial satellites in space or the position of bench marks on the Earth by means of electromagnetic waves with frequency around 1 GHz (Caputo, 1994b).

In all cases there are points along the path of the set of waves, with the same frequency and slightly different wavelength, where, because of the interference of the waves, the amplitude of their sum is nil. This phenomenon may be taken as the tunnel effect of the classic mechanics. This phenomenon is forecasted for homogenous substances and may probably be observed in the laboratory; for instance in water where, extrapolating the computations summarized in fig. 5, in a length of several meters and with a frequency of the order of the GHz, one should observe the locations of the beats and null amplitude.

REFERENCES

BAGLEY, R.L. and P.J. TORVIK (1983): A theoretical basis for the application of fractional calculus to viscoelasticity, *J. Rheol.*, **27** (3), 201-210.

BAGLEY, R.L. and P.J. TORVIK (1986): On the fractional calculus model of viscoelastic behaviour, *J. Rheol.*, **30** (1), 133-155.

CAPUTO, M. (1969): *Elasticità e Dissipazione* (Elasticity and Anelastic Dissipation), (Zanichelli Publisher, Bologna).

CAPUTO, M. (1984): Relaxation and free modes of a selfgravitating planet, *Geophys. J. R. Astron. Soc.*, **77**, 789-808.
CAPUTO, M. (1993): Free modes, splitting and alterations of electrochemically polarizable media, *Rend. Fis., Accad. Naz. Lincei*, **9** (4), 89-98.
CAPUTO, M. (1994a): The Riemann sheet solutions of linear elasticity, *Ann. Mat. Pura Appl.*, **4** (154), 335-342.
CAPUTO, M. (1994b): Distance measurements with electromagnetic waves, dispersion and tomography of the atmosphere, in *Proceedings of the Annual Meeting of National Group of Solid Earth Geophysics*, National Research Council, Rome, 459-470.
CAPUTO, M. (1995a): Distance measurements and splitting of electromagnetic waves in water caused by dispersion, *Rend. Fis. Accad. Naz. Lincei*, **9** (5), 103-113.
CAPUTO, M. (1995b): Mean fractional-order-derivatives differential equations and filters (abstract). Invited presentation at *American Mathematical Society Annual Meeting, San Francisco, California, U.S.A.*, January 1995. The extended version is in *Ann. Univ. Ferrara, Sez. Matematica, 7*, **41**, 73-83.
CAPUTO, M. (1996): Modern rheology and dielectric induction: multivalued index of refraction, splitting of eigenvalues and fatigue, *Ann. Geofis.*, **39** (5), 941-966.
CAPUTO, M. (1997): Rigorous time domain responses of polarizable media. I, *Ann. Geofis.*, **40** (2), 423-434.
CAPUTO, M. and F. MAINARDI (1971): Linear models of dissipation in anelastic solids, *Riv. Nuovo Cimento*, **1** (2), 161-198.
CISOTTI, V. (1911): L'ereditarietà lineare ed i fenomeni dispersivi, *Nuovo Cimento*, **6** (2), 234-244.
COLE, K.S. and R.H. COLE (1941): Dispersion and absorption in dielectrics, *J. Chem. Phys.*, **9**, 341-349.
HASTED, J.B. (1973): *Aqueous Dielectrics* (Chapman and Hall, London).
HEAVISIDE, O. (1899): *Electromagnetic Theory* (The Electrician Printing and Publishing Co.), vol. 2.
JACQUELIN, J. (1991): A number of models for CPA impedences of conductors and for relaxation in non-Debye dielectrics, *J. Non-Cryst. Solids*, **131**, 1080-1083.
KÖRNIG, H. and G. MÜLLER (1989): Rheological models and interpretation of postglacial uplift, *Geophys. J. R. Astron. Soc.*, **98**, 245-253.
LIEBE, H.J. (1985): An updated model for millimetre wave propagation in moist air, *Radio Sci.*, 20, **5**, 1069-1089.
PELTON, W.H., W.R. SILL and B.D. SMITH (1983): Interpretation of complex resistivity and dielectric data, Part 1, *Geophys. Trans.*, **29** (4), 297-330.

Appendix A. Introduction to the inversion of the system function $Z(p)$ defined in (2.1).

The function

$$\tau^{-m/u} / (p^{m/u} + \tau^{-m/u}) \tag{A.1}$$

since $z = m/u$ (m, u, $m < u$, positive integer and prime), is a multivalued function with u values defined in u Riemann sheets which will be identified by the integer $h(0 \le h \le u-1)$.
To solve the problems of the inversion it is necessary to find

$$\sum_{h=0}^{u-1} \mathrm{LT}^{-1}(\tau^{-m/u} /(p^{m/u} + \tau^{-m/u})) = \sum_{h=0}^{u-1} \mathrm{LT}^{-1}(\tau^{-m/u} / (r^{-m/u} \exp(i(\theta + 2h\pi)m / u) + \tau^{-m/u})) =$$

$$= \sum_{h=0}^{u-1} (1/2\pi) \int_{\lambda - i\infty}^{\lambda + i\infty} \tau^{-m/u} \exp(pt)dp / (p^{m/u} + \tau^{-m/u}) \tag{A.2}$$

$$r = |p|, 2h\pi + \theta = \arg p \quad \text{in the} \quad h\text{-th RS with} - \pi < \theta < \pi$$

where λ is chosen to be greater than the real part of the poles of (A.1).
(A.2) is obtained summing the integrals of (A.1) in each of the u RSs defined by h, along the path of fig. 1, which is to be considered in all the RSs, and which contain all the poles of (A.1). In turn the sum of these integrals is equal to the product of $2\pi i$ times the sum of the residues of (A.1) inside the loop of the paths of integration in each of the Riemann sheets.
It is shown in Appendix B that, when m and u have opposite parity, (A.1) has no poles on the negative real axis of the RS and one may integrate (A.1) along the path of fig. 1 in all the RSs.
According to Caputo (1969) it is seen that integrals of (A.1) along the path of fig. 1, in the generic RS identified by h, when the points A,B,D,H go to infinity in the radial direction and the radius of the smaller internal

circle is reduced to a size as small as desired, reduce to

$$\int_{\lambda-i\infty}^{\lambda+i\infty}\tau^{-m/u}\exp(pt)dp\,/\,(p^{m/u}+\tau^{-m/u})+$$

$$+\int_{0}^{\infty}[\tau^{-m/u}\exp(-rt)dr\,/(r^{m/u}\exp(i\pi(1+2h)m\,/\,u)+\tau^{-m/u})+ \tag{A.3}$$

$$-\tau^{-m/u}\exp(-rt)dr\,/\,(r^{m/u}\exp(i\pi(-1+2h)m\,/\,u)+\tau^{-m/u})].$$

In the RS identified by $h + 1$, the integral of (A.1) along the same path is

$$\int_{\lambda-i\infty}^{\lambda+i\infty}\tau^{-m/u}\exp(pt)dp\,/\,(p^{m/u}+\tau^{-m/u})+$$

$$+\int_{0}^{\infty}[\tau^{-m/u}\exp(-rt)dr\,/(r^{m/u}\exp(i\pi(3+2h)m\,/\,u)+\tau^{-m/u})+ \tag{A.4}$$

$$-\tau^{-m/u}\exp(-rt)dr\,/\,(r^{m/u}\exp(i\pi(-1+2h)m\,/\,u)+\tau^{-m/u})].$$

We see that in the sum of (A.3) and (A.4) two of the four integrals in dr cancel out, while the two integrals in dp are summed. When summing all the integrals in dr for all the RS we are left with the integral on the path with $\theta=-\pi$ of the RS $h = 0$ and the integral on the path with $\theta = (2n-1)\pi$ of the RS $h = u - 1$. We obtain

$$\sum_{h=0}^{u-1}\int_{\lambda-i\infty}^{\lambda+i\infty}\tau^{-m/u}\exp(pt)dp\,/\,(p^{m/u}+\tau^{-m/u})+$$

$$+\int_{0}^{\infty}[\tau^{-m/u}\exp(-rt)dr\,/\,(r^{m/u}\exp(i\pi(-1+2u)m\,/\,u)+\tau^{-m/u})+ \tag{A.5}$$

$$-\tau^{-m/u}\exp(-rt)dr\,/\,(r^{m/u}\exp(-i\pi m\,/\,u)+\tau^{-m/u})].$$

It is easily verified that the integrand of the last integral in dr in (A.5) is identically nil and (A.5) is simplified to

$$\sum_{h=0}^{u-1}\int_{\lambda-i\infty}^{\lambda+i\infty}\tau^{-m/u}\exp(pt)dp\,/\,(p^{m/u}+\tau^{-m/u}). \tag{A.6}$$

In the case when m and u are odd, as shown in Appendix B, there is a pole in the negative real axis of at least one RS. In this case the method of integration used when m and u have opposite parity is not valid. We may integrate instead on the new path indicated in fig. 2a-c, which begins in the RS $h = 0$, then proceeds to the RS $h = u - 1$ passing over all of them and finally returns to the RS $h = 0$.
The path begins in F in the RS h = 0 (see fig. 2a), proceeds to D, from there it steps up in the RS h = 1 where it repeats the same helicoidal path as in the RS $h = 0$.
After repeating successively the same helicoidal path in all RS $h = 2, 3, \ldots u - 2$ it reaches the point N with $Z = (2u - 3)R$ in the RS $h = u - 1$ then it goes to M with $\theta = (2u - 1)$ in the same RS $h = u - 1$ (see fig. 2b).
From M the path runs along the real axis towards the origin in the positive imaginary plane and it steps down to the RS $h = u - 2$ following another helicoidal path backwards but on a helix with a small radius and internal to that previously described.
Following the latter helix the path reaches again the point F in the RS $h = 0$ (see fig. 2c).
The integral of (A.1) along the paths described in fig. 2a-c, gives the same result (A.6) as the integration along the path of fig. 1 but it includes all the poles on the negative real axis of the RS.
However the paths of integration of fig. 2a-c may not be used when the initial values in the RSs differ. In fact the discontinuity introduced by the different initial values in the RS, when passing from one RS to the next, requires the branch cut along the negative real axis. When there are poles in the negative real axis (m and u odd) and in the RS the initial values are different a more detailed discussion is required which we postpone to another study.

Appendix B. The poles of the system function $Z(p)$ defined in (2.1).

In this appendix we show that, when m and u are odd there is a pole in the negative real axis of at least one RS. The poles of (A.1) in the Riemann sheets are found through the roots of the equation

$$p^{m/u} + \tau^{-m/u} = r^{m/u} \exp(i(\vartheta + 2h\pi)m / u + \tau^{-m/u} 0 = \qquad h = 0, 1, 2..., u-1 \tag{B.1}$$

where $r = |p|$ and θ is the argument of p.
The poles of (A.1) are m and obtained from

$$r\tau = 1 \qquad \exp(i(\theta + 2h\pi)m / u) = \exp(i\pi(1+2k)) = -1 \tag{B.2}$$

with k positive integer, which give

$$r = 1/\tau \qquad \theta = (1+2k)u\pi / m - 2h\pi. \tag{B.3}$$

The corresponding residues are then

$$(u / m\tau)\exp(i(2k+1)u\pi(u / m - 1)) \qquad k = 0, 1, 2, ..., k_2 \tag{B.4}$$

Where k_2 is the largest integer in k_1

$$k_1 < m - (m+u)/2u \tag{B.5}$$

obtained from the condition that θ be limited in the range from zero to the maximum value $(2u-1)\pi$ in the Riemann sheet corresponding to $h = u-1$. Since $m + u < 2u$ we have $k_2 = m-1$.
If u and m are odd, then the following equation, obtained from (B.3) with $\theta = (2l+1)$,

$$(1+2k)u = (2l+1)m \tag{B.6}$$

with k and l integer has solutions. In fact the set $(u, m, (m-u)/2))$, is prime since a divider of $(u, (m-u)/2)$ or of $(m, (u-m)/2)$ would also be a divider (u, m) which is a prime set. Then eq. (B.6) has solutions k, l and there will be at least one pole on the negative real axis, which implies a zero frequency residue.
If u and m have opposite parity, the eq. (B.6) has no solutions because one side of (B.6) is always even and the other one is always odd. Then there will be no poles on the negative real axis.