The experimental set-valued index of refraction of dielectric and anelastic media

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Abstract

The dielectric parameter, in the generalised Debye form, of all studied substances (Cole and Cole, 1941) contains a rational power of the imaginary frequency which implies that the index of refraction is a multivalued function of the frequency; the same property, concerning the stress strain relation, also applies to anelastic media (Bagley and Torvik, 1983a,b). The multivalued index of refraction implies then that the free modes of dielectric and anelastic media are split into a number of modes which depend on the exponent of the imaginary frequency. In order to estimate if it is possible to observe this splitting, this note computes the parameters appearing in the generalised Debye form of the dielectric parameters of many substances and also the complex anelastic parameter of an anelastic medium. The analysis of the data indicates that, depending on the accuracy of the experimental data, with few exceptions, the splitting is observable.

Key words *dielectric – anelastic – splitting – index of refraction*

1. Introduction

The interpretation of the observed broadening of the spectral lines in all physical phenomena is generally made assuming an exponential decay of the modes, often without the support of a physical model and on the generic assumption of non linearity as the cause of the phenomenon.

The assumption of a frequency independent specific dissipation (constant Q model), which is observed in the decay of the free modes of many elastic materials, led to models of the stress-strain relations, represented with the in-

troduction of time in the relations, which imply the observed Q. The same models should naturally represent also the dispersion of the waves.

The most successful stress-strain relations in the representation of dispersion and dissipation in anelastic media are those which contain derivatives of rational order (Bagley and Torvik, 1983a,b; Körnig and Müller, 1989), which, in the frequency domain, lead to relations between stress and strain in which the imaginary frequency appears elevated to a rational power.

In dielectrics the most used representation of the frequency dependent dielectric parameter is that of Cole and Cole (1941) in which the relation between the applied electric field and the induction contains the imaginary frequency elevated to a rational power.

Caputo and Mainardi (1971) showed that this type of constitutive equation in the frequency domain implies, in the time domain, a relation containing a rational order differentiation. Therefore, also for dielectric media, in the time domain, the relation between the applied elec-

tric field and the induction is expressed by means of derivatives of rational order.

It is well known that the degeneracy of a partial differential equations system is removed when a symmetry of the geometry of the body is removed; this generates a splitting of the eigenfunctions which in turn leads to a splitting of the free modes of the media whose dynamics is represented by the differential equations.

When a body is perfectly elastic, the waves travelling in it have symmetry in the time domain; the same applies to electromagnetic waves travelling in vacuum. The removal of the degeneracy of the solutions, generated by withdrawing a symmetry in the geometry of the body, is also expected when withdrawing the symmetry of the waves in the time domain, which is the phenomenon implied by the decay. This will then generate an additional splitting of the eigenfunctions and of the free modes.

Due to the removal of symmetry in the time domain, a spectral line of the non dissipative system will split into a set of lines of the dissipative system. The observation of the splitting of the spectral line would therefore allow us to infer the values of the parameters appearing in the formula defining the index of refraction (Caputo, 1989). Since the parameters governing the dissipation in general represent a relatively small perturbation of the dynamic of the medium, the split spectral lines are expected to be very close to one another, so close that it seems difficult to separate them numerically in the experimental data where one would often observe only the classic broadening of the classic spectral lines.

In this note we shall discuss the splitting of the eigenvalues of the free modes in dielectric media and of its representation through a dielectric parameter which, in the frequency domain, is represented by a rational expression of the imaginary frequency elevated to a rational power. Concerning the physical implications of the introduction of the derivative of rational order, we may qualitatively consider this type of derivative as representative of a memory of the medium relative to its precedent physical conditions. From a quantitative and rigorous point of view the derivative of rational order implies a multivalued index of refraction and therefore (Ca-

puto, 1994b) a monochromatic electromagnetic wave, incident the medium with memory with direction normal to its surface, or originated by a source in the medium itself, will split into a set of waves with the same frequency, but different wavelengths and velocities, which interfere and have beats.

It has been shown by Cisotti (1911) that the memory introduced in the dielectric constant implies that the phenomenology of the medium is irreversible. It has also been shown that, in anelastic media, the stress strain relations with fractional order derivatives are causal (Bagley and Torvik, 1983a,b).

In an extensive study of energy storage in electric networks Jacquelin (1984, 1991) used the complex frequency dependent impedance represented by inserting fractional order derivatives in the relation between the induced electric current and the electric field. Jacquelin (1984, 1991) discussion is practically extended to almost all possible circuits; he illustrates his results in the frequency domain at steady state, when the Fresnel terms of the fractional order derivatives are nil and his technique, based on observations at many frequencies, relies on convolution for the retrieval of the response in the time domain.

In general, we may state that the formalism of this note is now spread to many linear approaches in the studies of dissipative and dispersive properties of many anelastic and dielectric media (e.g., Caputo and Mainardi, 1971; Bagley and Torvik, 1983a,b, 1986; Pelton et al., 1983; Le Mehaute and Crépy, 1983; Jacquelin, 1984; Körnig and Müller, 1989; Caputo, 1994a-c, 1999a; Mainardi, 1993).

2. The complex dielectric parameter

In the following we shall discuss the splitting of the spectral lines caused by the presence of the derivatives of fractional order in the constitutive relations of dielectric and anelastic media in one dimension since the extension to three dimensions implies formal complications but no additional physical insight.

One of the constitutive equations of the electromagnetic field is that relating the induction

vector D to the electric field vector E through the frequency dependent dielectric tensor ε_{ij} which, in many cases, is represented by a rational function of a rational power of the imaginary frequency.

In one dimension, in general, the relation in the frequency domain is written (*e.g.*, see Cole and Cole, 1941)

$$d + \gamma p^z d = \alpha e + \beta p^z e \qquad (2.1)$$

where d and e are the Laplace Transform (LT) of D and E respectively, p is the LT parameter, α, γ, and β are parameters, with the appropriate dimensions, $z = m/n$, with m and n positive, integer, prime with $m < n$. The values of these parameters are determined with experiments of various types. A relation analogous to (2.1) is also valid for anelastic media (Bagley and Torvik, 1983a,b, 1986; Caputo and Mainardi, 1971; Caputo, 1969) replacing d with the LT of strain and e with that of stress. In this note we consider both media. The index of refraction of the medium, the relaxation time and the $1/Q$ are obtained from formula (2.1) and its parameters (*e.g.*, Caputo, 1994a,b).

The time domain expression of the relation (2.1) is (Caputo and Mainardi, 1971)

$$\gamma \partial^z D / \partial t^z + D = \beta \partial^z E / \partial t^z + \alpha E \qquad (2.2)$$

where the derivative of order z is defined as follows:

$$\partial^z f / \partial t^z = (1 / \Gamma(1-z)) \int_0^t f'(u) du / (t-u)^z . \qquad (2.3)$$

The expression (2.1) with $p = i\omega$ contains a rational power of the imaginary unit which is a set valued function, since $z = m/n$, and has n, generally different, values. Therefore (2.1) defines a number of different Physically Acceptable (PA) values for the complex index of refraction associated to the dielectric parameter.

In turn, the PA values of the index of refraction will be associated with an equal number of different velocity fields for the signals and waves travelling in the medium (Caputo, 1994c, 1999b).

In general, the PA values of the index of refraction in each medium have values of the real

parts (Re), as well as the values of the imaginary parts (Im), which are very close to one another, therefore, for each frequency, there will be sets of waves with very close wavelength. Since all the waves of the sets have the same frequency, a Fourier analysis of the signal generated by the incident monochromatic wave will give the same frequency as the incident wave.

The variation of the amplitude of the wave along its path could be an observable effect of the splitting of the values of the index of refraction, but the information on the splitting would be very limited since it will be difficult to observe the wave along its path for a very long time and/or distance.

However one may perform an experiment based on the different wavelengths of waves with the same frequency. In the experiment the free modes of the medium, with appropriate shape and with the known index of refraction, are excited by a delta-like source and therefore the Fourier analysis of the signal recorded in the medium will give the frequencies forecast by the theory. To perform the latter experiment one should preferably select media whose index of refraction has split values most widely separated in the frequency domain.

The width of the band of the split lines gives an accurate estimate of the width of the unresolved lines and also the actual position of the central line which is not necessarily the line corresponding to the principal value (Caputo, 1994b,c, 1999b).

This note is addressed to the search for such media. We therefore searched the literature for media whose index of refraction was observed as a function of the frequency, we found the fitting of formula (2.1) to these values for each medium, computed the frequencies of the free modes of infinite plates made with these media and, finally, discussed the separation of the lines of the splitting.

3. The free modes of dielectric media with the shape of infinite plates

We therefore estimate the frequencies of the free modes of an infinite plate with thickness 1 cm, which implies that if this frequency is f,

the frequencies of the free modes of plates with thickness a will be f/a. The medium of the plate could indifferently be dielectric or anelastic.

It is seen (Caputo, 1994b) that the frequencies of the free modes of an infinite plate with thickness a, when the dielectric parameter is defined by (2.1), are given by the solutions to the equation

$$\beta p^{2+z} + \alpha p^2 + \gamma c^2 N_j^2 p^z + c^2 N_j^2 = 0 \qquad (3.1)$$

where c is the velocity of light in vacuum and N_j is

$$N_l = il\pi/h$$

with l integer and h thickness of the plate.

When the infinite plate is an anelastic medium the frequencies of the free modes are given by (Caputo, 1988)

$$\rho\gamma p^{2+z} + \rho p^2 + \beta N_j^2 p^z + \alpha N_j^2 = 0 \qquad (3.1')$$

where ρ is the density of the medium and α, β and γ are the coefficients in the relation between the stress and strain similar to (2.1) but with different physical dimensions.

The solution of eq. (3.1) and (3.1') are obtained substituting

$$y = p^{1/n} \qquad (3.2)$$

which transforms eq. (3.1) and (3.1') respectively into the following algebraic equations of order $2n + r$

$$\beta y^{2n+r} + \alpha y^{2n} + c^2\gamma N_j^2 y^r + c^2 N_j^2 = 0 \qquad (3.3)$$

$$\rho y^{2n+r} + \rho y^{2n} + \beta N_j^2 y^r + \alpha N_j^2 = 0. \qquad (3.3')$$

The number of distinct solutions of eq. (3.3) and (3.3') will therefore be at most $2n + r$ real or complex conjugate.

However, since p is a complex number ($p = a + ib$), are PA only the solutions p with negative real part; moreover each pair of the complex conjugate solutions will identify only one frequency of the free modes; then, since the solutions, in general, are complex conjugate, the number of PA solutions is about $(2n + r)/4$ depending on the parity of n and r.

4. The analysis of the experimental data on dielectric media

The experimental data on the dielectric parameter as a function of the frequency are scarce, only water has been extensively studied with experiments on its variation as a function of the frequency and of the temperature (Bäz, 1939; Slevogt, 1939; Pottel (Hasted, 1973), Kaatze and Uhlendorf, 1981; Kaatze et al., 1988).

We examined data on the dielectric parameter of 30 liquid and solid substances. The accuracy is highly variable and the data of only 12 substances gave satisfactory fitting to (2.1) because the number of data available for the substance was not sufficient to ensure a satisfactory fitting, which was the most frequent case, or because the data were not sufficiently accurate.

In all the data examined the observational error was not reported by the authors and we assumed it to be the reading error, that is, half of the last digit reported.

The values of the parameters appearing in formula (2.1) are presented in table I with the values determined by Cole and Cole (1941). As already mentioned, the new computation of the parameters appearing in (2.1) is necessary because, at the time when the paper by Cole and Cole (1941) was written, the computation facilities where not so efficient and did not allow the accuracy of the present.

In the following we shall briefly discuss the single substances studied by the various authors beginning with data gathered by Cole and Cole (1941). The most recent data concern water (Kaatze and Uhlendorf, 1981; Kaatze et al., 1988). Heptanol, Cyclohesanol and Formamide, which are in a circular of the National Bureau of Standard (NBS) are also discussed in papers which appeared in 1958 (Buckley and Maryott, 1958). The parameters of these substances are presented in table II. With the discussion of the parameters of the substances examined we also report on the solutions of (3.3) which gives the frequencies (Im part) of the eigenvalues of 1 cm thick plate made with the substance.

The value of $\alpha^{1/2}$ gives an approximate value of the velocity $c/\alpha^{1/2}$ of the electromagnetic waves in the medium, with c velocity of the light in vacuum; the values of $\alpha^{1/2}$ of all the substances

Table I. Comparison of the parameters calculated in this work with the parameters determined by Cole and Cole (1941). For water, methyl alcohol and ethyl alcohol the values in the upper line refer to the data of Bäz (1939), those in the lower line refer to the data of Slevogt (1939).

Substance	α	α C&C	τ	τ C&C	ε_∞	ε_∞ C&C	z	z C&C
Water	80.7	81	$0.115\ 10^{-10}$	$9.4\ 10^{-12}$	16.3	1.8	1	0.91
		81		–		1.8		1
Methyl Alcohol	33.8	31.8	$0.745\ 10^{-11}$	$6.4\ 10^{-11}$	5.89	2.0	1.1	0.87
		33.2		$5.9\ 10^{-11}$		2.0		1
Ethyl Alcohol	26.4	25.8	$0.162\ 10^{-9}$	$9.4\ 10^{-11}$	2.3	1.85	0.98	0.85
		26.0		$1.3\ 10^{-10}$		1.85		1
Vinsol 98 °C	8.91	8.48	$0.183\ 10^{-3}$	$1.2\ 10^{-4}$	3.252	3.42	0.35	0.39
Chlorinated Diphenyl n. 3 27.5 °C	4.4	4.23	$0.924\ 10^{-4}$	$7.2\ 10^{-5}$	2.8	2.8	0.7	0.75
Chlorinated Diphenyl n. 3 37.5 °C	4.1	4.06	$0.455\ 10^{-4}$	$3.5\ 10^{-6}$	2.97	2.8	0.9	0.87
Abietic Acid	3.7	86.8	$0.236\ 10^{-3}$	$2.2\ 10^{-6}$	2.82	3.7	0.6	0.6
Methylpentachlorobenzene 0 °C	5.3	5.36	$0.367\ 10^{-5}$	$3.2\ 10^{-7}$	3.07	3.10	0.92	0.89
Methylpentachlorobenzene – 18 °C	5.5	5.54	$0.153\ 10^{-4}$	$1.6\ 10^{-6}$	3.2	3.10	0.9	0.87
Methylpentachlorobenzene – 40 °C	5.9	5.94	$0.333\ 10^{-4}$	$1.4\ 10^{-5}$	3.08	3.10	0.8	0.79
Pentamethylchlorobenzene – 61 °C	7.4	7.34	$0.822\ 10^{-3}$	$5.9\ 10^{-6}$	3.58	2.94	0.9	0.82
Pentamethylchlorobenzene – 79 °C	8.6	8.60	$0.506\ 10^{-4}$	$5\ \ 10^{-5}$	2.97	2.92	0.75	0.74
Glycerol	59.9	63	$0.140\ 10^{-4}$	$6.9\ 10^{-5}$	7.22	5.0	0.85	0.79

Table II. Results of fitting the experimental data of substances of particular interest and not appearing in Cole and Cole (1941).

Substance	α	β	γ	z	Mean deviation	Number of data
Water	78.5	0.525×10^{-10}	1.0534×10^{-11}	0.99	0.0674	8
Formamide	109.8	3.042×10^{-10}	$4.727\ \times 10^{-11}$	0.98	2.15	12
Cyclohexanol	16.738	1.077×10^{-8}	$2.838\ \times 10^{-9}$	0.99	0.0916	18
Heptanol	13.985	1.523×10^{-8}	$4.829\ \times 10^{-9}$	0.99	0.073	16

considered in the following paragraph are in the range [2,9] which has a limited influence on the observation of the separation of the split lines.

In the following, CCCN is an abbreviation for Couple of Complex Conjugate Numbers, CM is an abbreviation for Corresponding Mode for a plate 1 cm thick, RE is an abbreviation for Reading Error, MSD is abbreviation for Mean Standard Deviation.

Water – Bäz (1939) and Slevogt (1939) measured the index of refraction and the absorption coefficient at 8 different frequencies, 4 in the range 0.1-3 GHz (Bäz, 1939) and 4 in the range 3-10 GHz (Slevogt, 1939). The fitting of these data to (2.1) gives $z = 1$ with a RE = 0.05 and a MSD = 0.7 which is not considered satisfactory.

Pottel (Hasted, 1973) gives 8 data in the frequency range 2-40 GHz. The fitting of these data to (2.1) gives $z = 0.99$ with MSD = 0.07, on a RE = 0.05, which is one order of magnitude smaller than that given by the data of Bäz (1939) and Slevogt (1939). These results, as already noted by others, makes it clear that water does not obey the theory of Debye rigorously.

In the case of the data of Pottel (Hasted, 1973) the PA solutions to eq. (3.3) are 77 CCCN. Of the CM only 5 have $Q > 20$, their separation is at least 2 MHz and should be observable.

Methanol and Ethanol – Bäz (1939) and Slevogt (1939) measured the index of refraction and the absorption coefficient of these substances at the same 8 frequencies reported for water. The fitting of the data to eq. (2.1) gives $z = 0.98$ with MSD = 0.41 on a RE = 0.05 for Ethanol and $z = 1.1$ for Methanol with MSD = 0.85 on a RE = 0.05. The value $z = 1.1$ is the only larger unity found in our investigation.

The PA solutions to eq. (3.3) for Ethanol are 37 CCCN. However the Q of the CM is so small that it would not be feasible to observe the separate lines.

The PA solutions to eq. (3.3) for Methanol are 8 CCCN, with Q ranging from 0.4 to 100. The minimum separation of the spectral lines is 2 GHz which should be observable.

Formamide – Budrun (NBS, 1958) studied this substance in the range 60 MHz-20 GHz. The number of data is 12 and the fitting to eq. (2.1) gives $z = 0.98$ with MSD = 2.15 on a RE = 0.01. The PA solutions to eq. (3.3) are 36 CCCN. The Q of the CM is 13 in one case and much less in the others which makes the observation of the separation of the split lines hardly feasible.

Heptanol – Lebrun (NBS, 1958) studied this substance in the range 500 KHz-10 GHz. The number of data is 16 and the fitting to eq. (2.1) gives $z = 0.99$ with MSD = 0.073 on a RE = 0.01. The PA solutions to eq. (3.3) are 75 CCCN. The Q of the CM is larger than 100 in 5 of them which are separated by at least 8 MHz which should make their separate observation possible.

Cyclohexanol – Arnoult (NBS, 1958) studied this substance in the range 500 KHz-10 GHz. The number of data is 18 and the fitting to eq. (2.1) gives $z = 0.99$ with MSD = 0.091 on a RE = 0.01. The PA solutions to eq. (3.3) are 75 CCCN. 6 of the CM have $Q > 70$ and are separated by at least 2 MHz which should make them observable.

Chlorinated Diphenyl – White and Morgan (1933) studied several types of Chlorinated Diphenylclorurates reporting graphically the dielectric parameter; we considered the type of Chlorinated Diphenyl indicated by the authors with number 3 which was studied at 4 different temperatures and in the frequency range 1 KHz-0.1 MHz for each value of the temperature. The values of z determined with the fitting of the data to eq. (2.1) increased with increasing temperature while $\alpha(\varepsilon_0)$ decreased; the value $z = 0.9$ considered in table I is for 37.5 °C.

The fitting of the data to eq. (2.1) has MSD $= 0.562 \cdot 10^{-2}$ on a RE = 0.005. The solutions to eq. (3.3), see figs. 1 and 2, are 14 CCCN and a real solution, 7 of them are PA. Of the CM, one has a too small Q to be observable, the others are separated by about 0.5 MHz and their Q is about 100 000 which should make them observable.

Methylpentachlorobenzene, Pentamethylchlorobenzene – White et al. (1940) studied these substances at 3 different temperatures and in the frequency range 1 KHz-0.1 MHz. For both sub-
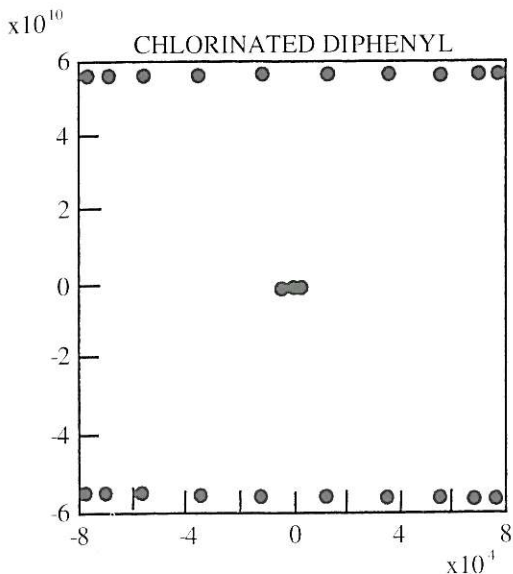
$x10^{10}$

Fig. 1. Distribution of the solutions to eq. (3.3) for Chlorinated Diphenyl at 37.5 °C. In abscissa is the real part and in the ordinate is the imaginary part.
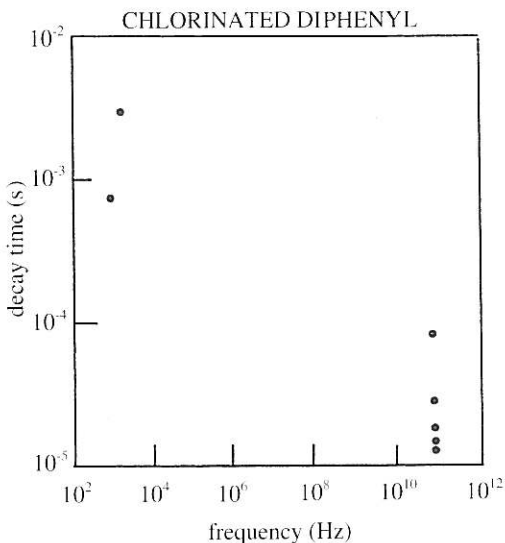
stances the values of z determined fitting the data to eq. (2.1) increased with increasing temperature while $\alpha(\varepsilon_0)$ decreased; the values reported in table I are for -40 °C (Methylpentachlorobenzene) and for -61 °C (Pentamethylchlorobenzene).

The fitting of the data of Pentamethylchlorobenzene to eq. (2.1) has MSD = 0.0617 on a RE = 0.05. The PA solutions of eq. (3.3) are 15, some of them are separated by about 1 MHz and have a Q of the order of 2500, which makes them observable.

The fitting of the data of Methylpentachlorobenzene to eq. (2.1) has MSD = 0.0112 on a RE = 0.005. The PA solutions of eq. (3.3), see fig. 3, are 3 CCCN; one of the CM has a very small Q, the others are separated by about 0.1 MHz and have a Q about 10 000, these modes should be observable.

Vinsol – Yager (1936) studied Vinsol at 3 different temperatures and 5 different frequencies in the range 1-100 KHz. Contrary to the preceding cases examined the values of z deter-
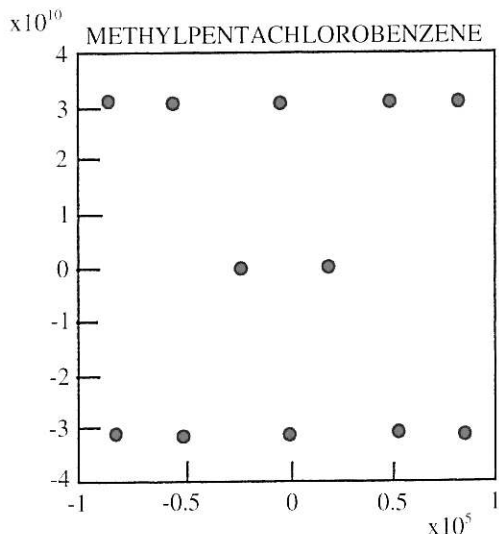

Fig. 2. Physically acceptable solutions to eq. (3.3) for Chlorinated Diphenyl at 37.5 °C. In abscissa is the frequency and in the ordinate is the decay time.

Fig. 3. Distribution of the solutions to eq. (3.3) for Methylpentachlorobenzene at -40 °C. In the abscissa is the real part and in the ordinate is the imaginary part.
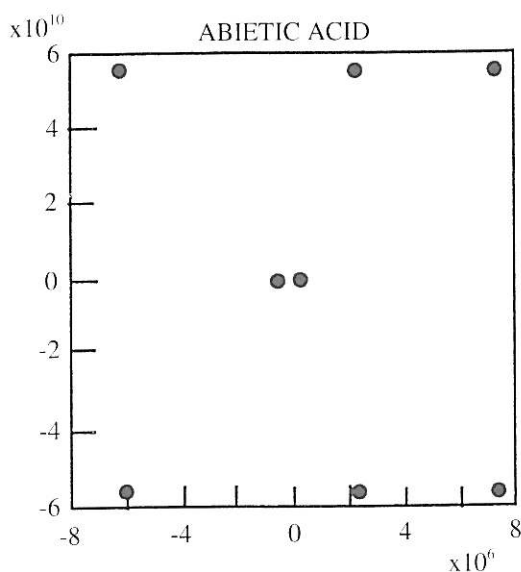
Fig. 4. Distribution of the solutions to eq. (3.3) for Abietic Acid. In the abscissa is the real part and in the ordinate is the imaginary part.
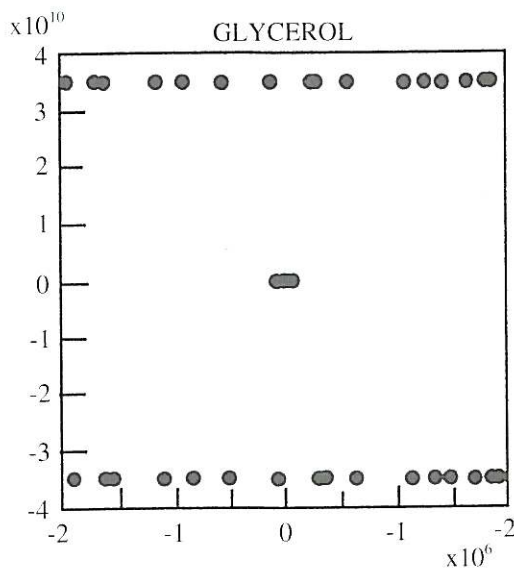

Fig. 5. Distribution of the solutions to eq. (3.3) for Glycerol. In the abscissa is the real part and in the ordinate is the imaginary part.

mined fitting these data to eq. (2.1) decrease with increasing temperature. We considered the set of data at 98 °C, with MSD = 0.371 10^{-2} and RE = 0.0005. The value $z = 0.35$ reported in table I is for 98 °C.

The solutions of eq. (3.3) for Vinsol are 23 couples of complex conjugate numbers but only 12 are PA; of the CM 2 have a very small Q and are not observable, the others are separated by about 14 MHz and have a $Q > 100$, they should be observable as separate spectral lines.

Glycerol, Abietic Acid – These substances have been studied by Yager (1936) at 5 different frequencies in the range 1-100 KHz for Glycerol and Abietic Acid and at 6 frequencies in the range 1 KHz-1 MHz for Glycerol; all at a temperature of 20 °C. The fitting of Abietic Acid to eq. (3.3) is shown in fig. 4, that of Glycerol data is shown in fig. 5. For Glycerol the reading error is 0.05 and MSD = 0.87, on a RE = 0.05, which is rather large; MSD = 0.013, on a RE = 0.0005, for Abietic Acid, MSD = 0.10, on a RE = 0.5, for Glycerol. The fitting of the data to eq. (2.1) gives $z = 0.85$ for Glycerol and $z = 0.6$ for Abietic Acid.

The PA solutions for Abietic Acid are 5; 2 CCCN and a real one. The separation of the frequencies of the CM is of the order of 10 MHz with a Q around 10 000 and therefore the separation should be observable.

The PA solutions to eq. (3.3) for Glycerol, see fig. 6, are 14 CCCN; 4 CM have a very small Q and are not observable; the other 10 are separated by about 1 MHz and have a Q of about 10 000 which makes them observable as separate spectral lines.

5. The analysis of the experimental data on anelastic media

Bagley and Torvik (1986) studied Nitrile Rubber 1479 at 10 °C obtaining 9 values of the storage and of the dissipation moduli in the frequency range 0.1-100 KHz. They supplied the values of the parameters appearing in eqs. (3.1′) and (3.3′) with $z = 0.64$. The PA of (3.3′) are 17 CCCN which define 17 CM. The frequencies may be divided into 4 groups, see fig. 7,
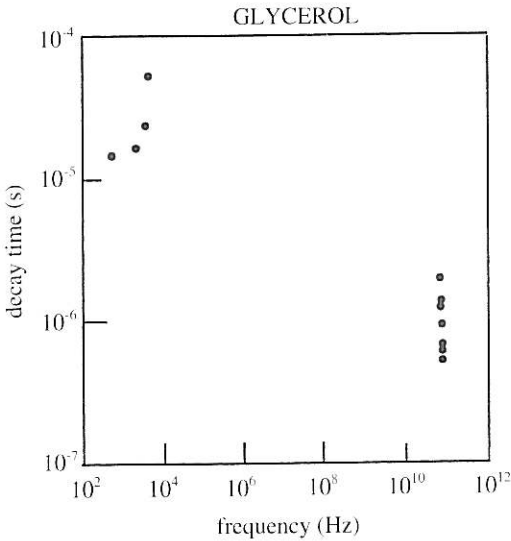
Fig. 6. Physically acceptable solutions to eq. (3.3) for Glycerol. In the abscissa is the frequency and in the ordinate is the decay time.
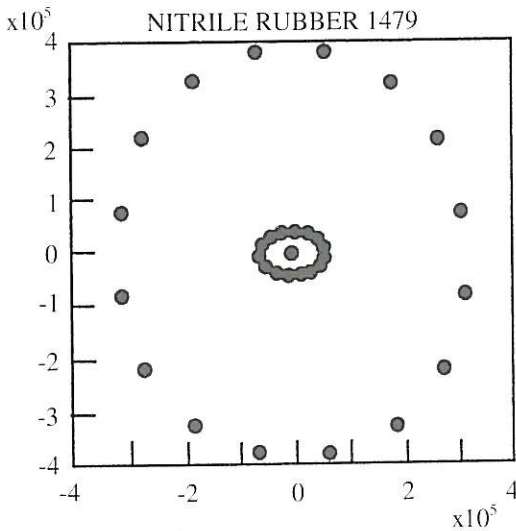

Fig. 7. Distribution of the solutions to eq. (3.3′) in the case of Nitrile Rubber. The data used for the solution of eq. (3.3′) are from Bagley and Torvik (1986). In the abscissa is the real part and in the ordinate is the imaginary part.

in one group there are the frequencies 2.5 and 70 Hz, in another group are the frequencies 124 and 125 Hz, in the third group frequencies 22, 32, 38 KHz, in the last group 10 frequencies in the range between 10-80 KHz located in the small circle in the centre of the figure.

The data of three more substances studied by Bagley and Torvik (1983a,b) are available and give results similar to those of Nitrile Rubber.

6. Conclusions

The analysis of the previous section indicates that, with the exception of Formamide Methanol and Ethanol, the dielectric substances examined may be tentatively used to observe the splitting of the free modes caused by the multivalued index of refraction. It is important to note that the relatively small values of β and γ obtained for the substances considered imply relatively small variations of the eigenvalues relative to the classic case when $\beta = \gamma = 0$ and/or $z = 0$ ($m = 0$).

In the case of the anelastic substances, the separation of the free modes, relative to the case of the corresponding elastic case ($\beta = \gamma = 0$ and/or $z = 0$ ($m = 0$)), is highly significant and we conclude that the anelastic substance examined should also allow us to observe the splitting of the free modes caused by the multivalued index of refraction.

Concerning the splitting of the free modes we must also consider the possibility that the value of z, found with different experiments, is always determined with some error which may have a very large effect on the number of lines in the splitting.

Although no hypothesis was necessary relative to the distribution of energy in the different split lines, we have tacitly assumed equipartition of energy among the lines. But this could be true only if the equipartition refers to the initial energy since the different eigenvalues, in general, imply different dispersion and different dissipation of energy which may have a strong effect on the distribution of energy. Certainly the lines with stronger attenuation would have more difficulties in appearing in the power spectrum analysis of the observed data.

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