



Buckling and Pre Stressed Dynamics Analysis of Laminated Composite Plate with Different Boundary Conditions

Widad I. Majeed*

Zainab Abdul Kareem Abed**

*,**Department of Mechanical Engineering/ University of Baghdad

*Email: wedad.ibrahim@coeng.uobaghdad.edu.iq

**Email: zainabkareem2158@gmail.com

(Received 8 February 2018; accepted 4 July 2018)

<https://doi.org/10.22153/kej.2019.07.002>

Abstract

Critical buckling and natural frequencies behavior of laminated composite thin plates subjected to in-plane uniform load is obtained using classical laminated plate theory (CLPT). Analytical investigation is presented using Ritz- method for eigenvalue problems of buckling load solutions for laminated symmetric and anti-symmetric, angle and cross ply composite plate with different elastic supports along its edges. Equation of motion of the plate was derived using principle of virtual work and solved using modified Fourier displacement function that satisfies general edge conditions. Various numerical investigation were studied to exhibit a convergence and accuracy of the present solution for considering some design parameters such as edge conditions, aspect ratio, lamination angle, thickness ratio, orthotropic ratio, the results obtained gives good agreement with those published by other researchers.

Keywords: Buckling load, different boundary conditions, composite laminated plate, Free vibration, Rayleigh-Ritz method.

1. Introduction

The composite materials (C.M) reinforced by fiber are perfect for structural applications where high stiffness and strength to weight of ratios are necessary. Composite materials can be adapted to assemble the especial necessities of strength and stiffness by varying fiber orientations and lay-up. The capability to adapt a (C.M) to its work is very most important advantages of a (C.M) over a common material. In the past few decades the improvement and study of (C.M) in the mechanical, civil and aerospace structures design has developed.

The structures might be exposed to dynamic loads in difficult environmental conditions, so it is necessary to know the characteristic of vibrations. The cause of failure of the structure components when the natural frequencies of structure and the forcing frequency close to each other's which is

the resonance (when structure damping is considered), may be occur large torsion \ translation deflections and internal stresses. Many researchers have presented the stability of (C.M) subjected to buckling loads. [1]. Developed an exact solution on the base of the first order shear deformation theory (FSDT) to investigate the buckling behavior of symmetrical simply supported cross ply rectangular plates subjected to unidirectional linearly varying in-plane loads. [2]. Used a semi-analytical attitude to the buckling analysis of symmetric laminated plates with general edge conditions. The multi-term extended Kantorovich method was used to decrease the partial differential of the equations of buckling to a solution of ordinary differential equations. [3]. Using the finite element method to study buckling and vibration of composite laminated plates with variable fiber spacing. [4]. Extended a two variable refined theory to the free vibration analysis of ortho-

tropic plates using Navier's solution. [5]. Presented a study of buckling and post buckling behavior of simply supported composite plates subjected to non-uniform in-plane loading. The analytical solutions for laminated plate based on higher order shear deformation theory. The multiterm Galerkin method was used to solve the nonlinear partial differential equations governing post buckling behavior of plate. [6]. Investigated the natural frequencies and buckling of layered plates subjected to combined in-plane loadings with the aim of furnishing a few guidelines for the modeling and design of pre-stressed laminated panels. Different load conditions, stacking sequences, material properties and boundary conditions are considered. [7]. Presented a study of bending and free vibration analysis based on simple first order shear deformation theory (FSDT). Were the analytical solved by an exact method (Levy's method). [8]. Presented an exponential shear deformation theory which extended for buckling and free vibration analysis. The theory takes into account of transverse shear effects and parabolic distribution of the transverse shear strains through the thickness of the plate. Were the analytical solved by an exact method (Navier's method). [9]. Studied buckling behavior and free vibration of composite plates subjected to in plane parabolic, linear and uniform distributed loads using (CLPT). Analytical investigation is shows using Ritz method for eigenvalues problems of buckling loads solution for laminated plate. The edges conditions take into account are (SSSS, CCCC, SCSC, SFSF and CFCF). [10].The vibration and buckling of laminated beams studied by using a shear deformation and refined theory. The displacement field is estimated by using the Ritz technique. The functions used in the Ritz technique are chosen by way of either a hybrid polynomial-trigonometric series or a pure polynomial series.

2. Buckling and Pre Stressed Vibration Analysis of Laminated Plates

The governing equation is derived by using CPLT, [8]:

$$D_{11} \frac{\partial^4 w}{\partial x^4} + (2D_{12} + 4D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} + I_0 \frac{\partial^2 w}{\partial t^2} = N_x \frac{\partial^2 w}{\partial x^2} \quad \dots (1)$$

Where, stress resultants are expressed in displacement form from below:

$$(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy}) z dz = \sum_{k=1}^N \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \sigma_{xy}) z dz \quad \dots (2)$$

$$I_0 = \int_{-h/2}^{h/2} \rho dz \quad \dots (3)$$

Integrating Eq. (2), (3) through thickness of the plate, the stress resultant is associated to the displacement (w) by the relatives:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} k_x \\ k_y \\ k_{xy} \end{Bmatrix} \quad \dots (4)$$

$$D_{ij} = [\bar{Q}_{ij}] \int_{\frac{h}{2}}^{-\frac{h}{2}} z^2 dz \quad \dots (5)$$

The twisting moments and bending, transversal shear forces can be written in terms of the displacement function as, [11].

$$M_x = -D_{11} \frac{\partial^2 w}{\partial x^2} - D_{12} \frac{\partial^2 w}{\partial y^2} \quad \dots (6)$$

$$M_y = -D_{22} \frac{\partial^2 w}{\partial y^2} - D_{12} \frac{\partial^2 w}{\partial x^2} \quad \dots (7)$$

$$M_{xy} = -2D_{66} \frac{\partial^2 w}{\partial x \partial y} \quad \dots (8)$$

$$Q_x = -D_{11} \frac{\partial^3 w}{\partial x^3} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial y^2 \partial x} \quad \dots (9)$$

$$Q_y = -D_{22} \frac{\partial^3 w}{\partial y^3} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial x^2 \partial y} \quad \dots (10)$$

For a flexibly restricted rectangular plate shown in Fig.(1), the boundary conditions are :

$$k_{x0} w = Q_x \quad K_{x0} \frac{\partial w}{\partial x} = -M_x \quad \dots \text{at } x=0 \quad \dots (11-12)$$

$$k_{x1} w = -Q_x \quad K_{x1} \frac{\partial w}{\partial x} = -M_x \quad \dots \text{at } x=a \quad \dots (13-14)$$

$$k_{y0} w = Q_y \quad K_{y0} \frac{\partial w}{\partial y} = -M_y \quad \dots \text{at } y=0 \quad \dots (15-16)$$

$$k_{y1} w = -Q_y \quad K_{y1} \frac{\partial w}{\partial y} = -M_y \quad \dots \text{at } y=b \quad \dots (17-18)$$

Where k_{y0} k_{y1} and k_{x0} k_{x1} are the transitional stiffness of spring, K_{y0} K_{y1} and K_{x0} , K_{x1} are the rotations stiffness of spring. Eq.(11)-(18) express for different edge conditions, the classic homogeneously edge conditions can be direct obtained by putting the constants of spring equalize to an very small or large number.

By substituting Eq. (6-10) in Eq. (11-18), get the following equations:

$$k_{x0} w = -D_{11} \frac{\partial^3 w}{\partial x^3} - (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial y^2 \partial x} \quad \dots (19)$$

$$k_{x1} w = D_{11} \frac{\partial^3 w}{\partial x^3} + (D_{12} + 4D_{66}) \frac{\partial^3 w}{\partial y^2 \partial x} \quad \dots (20)$$

$$K_{x0} \frac{\partial w}{\partial x} = D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2} \quad \dots (21)$$

$$K_{x1} \frac{\partial w}{\partial x} = D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2} \quad \dots (22)$$

And similarly found other four equations in the y direction.

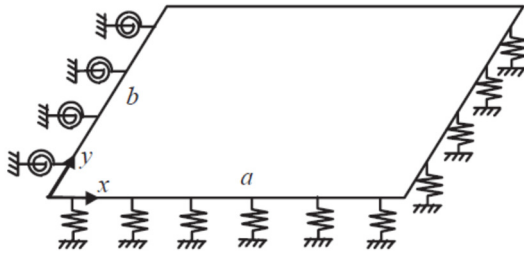


Fig. 1. Elastic restrained edges for a rectangular plate [12].

As mentioned by many plate and shell researches that exact solution for plate or shell with general boundary conditions is not available so we use Ritz method to get approximate solution from Hamilton's equation:

$$\delta \iiint (U - W + T) = 0 \quad \dots (23)$$

Where U is the strain energy, W is potential energy due to the external forces and T is the kinetic energy. δ the random variation.

Where

$$U = \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^a (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy}) dx dy dz +$$

Elastic energy of springs at edges

$$W = \frac{1}{2} \int_0^b \int_0^a \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 \right] dx dy$$

$$T = \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^b \int_0^a \left[\frac{\partial^2 w}{\partial t^2} w \right] dx dy dz \quad \dots (24)$$

Substituting Eqs.(2- 4),(19-22) in Eq.(24),yield to:

$$U - W = \frac{1}{2} \int_0^b \int_0^a \left[D_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + D_{22} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{66} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + 2D_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] dx dy +$$

$$\frac{1}{2} \int_0^b \left[k_{x0} w^2 + K_{x0} \left(\frac{\partial w}{\partial x} \right)^2 \right]_{x=0} dy + \frac{1}{2} \int_0^b \left[k_{x1} w^2 + K_{x1} \left(\frac{\partial w}{\partial x} \right)^2 \right]_{x=a} dy + \frac{1}{2} \int_0^a \left[k_{y0} w^2 + K_{y0} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=0} dx + \frac{1}{2} \int_0^a \left[k_{y1} w^2 + K_{y1} \left(\frac{\partial w}{\partial y} \right)^2 \right]_{y=b} dx - \frac{1}{2} \int_0^b \int_0^a \left[N_x \left(\frac{\partial w}{\partial x} \right)^2 \right] dx dy \quad \dots (25)$$

and

$$T = \frac{1}{2} \omega^2 \iint I_0 w_0^2 dx dy \quad \dots (26)$$

3. Admissible Functions

With Ritz method the allowable functions play an important part. The products of beam functions

are commonly selected like the displacement function and functions can be consequently like, [12].

$$w(x,y) = \sum_{m,n=1} A_{mn} X(x) Y(y) \quad \dots (27)$$

Where X(x), Y(y) are the specific variables for beams that include a similar edge conditions in the (y, x) direction, correspondingly.

The beam functions can be in general achieved like a linear collection of hyperbolic and trigonometric functions, which involve some unknown variables which are exist from the edge conditions. Then, every edge conditions essentially conduct to a various beam functions. In actual uses, this is obviously disadvantageous, beside the damage of computing the specific functions for a different boundary beam. with a view to avert this difficulty, an developed the series of Fourier technique has been suggested for beams with different boundaries at each ends in which the specific functions are written in the form of, [13].

$$w(x) = \sum_{m=0}^{\infty} a_m \cos \lambda_{am} x + P(x) \quad \dots (28)$$

$$\left(\lambda_{am} = \frac{m\pi}{a} \right), \quad 0 \leq x \leq a$$

P (x) is the function in Eq. (28) considers an arbitrarily continued function that, in any case of edge conditions, is constantly selected to satisfy the equations as following:

$$P'''(0) = W'''(0) = \alpha_0, \quad P'''(a) = W'''(a) = \alpha_1. \quad \dots (29-30)$$

$$P'(0) = W'(0) = \beta_0, \quad \text{and} \quad P'(a) = W'(a) = \beta_1. \quad \dots (31-32)$$

P (x) is here inserted to take care of the latent discontinuities of the function of displacement and its derivative at end points. Accurately, previously it is known that the smoothest a periodical function, the quicker its Fourier extension convergence. Thus, adding of the P (x) will have two instantaneous interests: (1) the series of Fourier extension is presently agreed with any edge conditions, and (2) the solution of the series of Fourier its accurateness of convergences.

Yet, P (x) have just been realized as a continuously function that satisfy Eq. (29) - (32), the function P (x) format is not a worry with regard to the convergences of the series solution. Therefore, it can be chosen in any required formula. Like a substantiation, supposes the P (x) is a the function of polynomial,

$$P(x) = \sum_{n=0}^4 C_n P_n \left(\frac{x}{a} \right). \quad \dots (33)$$

Where $P_n(x)$ is the Legendre function of order n , C_n is the coefficient of extension.

It is clarifies that P (x) desires to be minimum a 4th polynomial to jointly satisfy Eq.(29) - (32). Substituting Eq. (33) into Eq. (29) - (32) results yield to:

$$C_3 P_3'''(0) + C_4 P_4'''(0) = a^3 \alpha_0. \quad \dots (34)$$

$$C_3 P_3'''(1) + C_4 P_4'''(1) = a^3 \alpha_1. \quad \dots (35)$$

$$C_1 P_1'(0) + C_2 P_2'(0) + C_3 P_3'(0) + C_4 P_4'(0) = a \beta_0. \quad \dots (36)$$

$$C_1 P_1'(1) + C_2 P_2'(1) + C_3 P_3'(1) + C_4 P_4'(1) = a \beta_1. \quad \dots (37)$$

By using the above equations, the coefficients, C_1, C_2, C_3 and C_4 are straight acquired in terms of the edge constants, $\alpha_0, \alpha_1, \beta_0$. then β_1 . As the coefficient C_0 doesn't really seem in Eq. (34) -(37), it can be a random number theoretically. For example, C_0 is presently chosen to content

$$\int_0^a P(x) dx = 0 \quad \dots (38)$$

The last appearance for the $P(x)$ can be shown as $P(x) = \zeta_a(x)^T \bar{\alpha}$... (39)

Where

$$\bar{\alpha} = \{\alpha_0, \alpha_1, \beta_0, \beta_1\}^T \quad \dots (40)$$

Where

$$H_a = \begin{bmatrix} 1 + \frac{8k_{x0}a^3}{360D_{11}} & \frac{7k_{x0}a^3}{360D_{11}} & \frac{-k_{x0}a}{3D_{11}} & \frac{-k_{x0}a}{6} \\ \frac{7k_{x1}a^3}{360D_{11}} & 1 + \frac{8k_{x1}a^3}{360D_{11}} & \frac{-k_{x1}a}{3D_{11}} & \frac{-k_{x1}a}{6} \\ \frac{a}{3} & \frac{a}{6} & \frac{K_{x0}}{D_{11}} + \frac{1}{a} & \frac{-1}{a} \\ \frac{a}{6} & \frac{a}{3} & \frac{-1}{a} & \frac{K_{x1}}{D_{11}} + \frac{1}{a} \end{bmatrix} \quad \dots (43)$$

and

$$Q_{am} = \left\{ (-1)^m \frac{k_{x0}}{D_{11}} \quad (-1)^m \frac{k_{x1}}{D_{11}} \quad - \lambda_{am}^2 \quad (-1)^m \lambda_{am}^2 \right\}^T \quad \dots (44)$$

It must be reminded that a matrix H_a becomes single to a totally free Beam, [14],

. By using of Eqs. (39) and (42), Eq. (28) becomes as:

$$w(x) = \sum_{m=0}^{\infty} a_m \varphi_m^a(x) \quad \dots (45)$$

Where

$$\varphi_m^a(x) = \cos \lambda_{am}x + \zeta_a(x) H_a^{-1} Q_{am} \quad \dots (46)$$

Mathematically, Eq. (45) mention that every of the functions of the beam can be observed as a function in the functional space spanned by the base functions $\{\varphi_m^a(x): m = 0, 1, 2, \dots, \dots\}$. So, Eq.(27) can be consequently rewritten as:

$$w(x, y) = \sum_{m,n=0}^{\infty} A_{mn} \varphi_m^a(x) \varphi_n^b(y) \quad \dots (47)$$

Where:

$$\varphi_n^b(y) = \cos \lambda_{bn}y + \zeta_b(y) H_b^{-1} Q_{bn} \quad \dots (48)$$

The terms for $\zeta_b(y), H_b$ and Q_{bn} can be, correspondingly, obtained from Eqs. (41), (43) and (44) by easily changing the x- regarding parameters by the y- regarding.

and

$$\zeta_a(x)^T = \begin{Bmatrix} -(15x^4 - 60ax^3 + 60a^2x^2 - 8a^4)/360a \\ (15x^4 - 30a^2x^2 + 7a^4)/360a \\ (6ax - 2a^2 - 3x^2)/6a \\ (3x^2 - a^2)/6a \end{Bmatrix} \quad \dots (41)$$

The results in Eq. (39) - (41) are already derived from an additional simple but little common approach, [12].

So as to obtain the unknown of edge constants, $\alpha_0, \alpha_1, \beta_0$. and β_1 . substitution of Eq. (28) and eq. (39) into the edge conditions Eq. (19)-(22) that results in

$$\bar{\alpha} = \sum_{m=0}^{\infty} H_a^{-1} Q_{am} a_m \quad \dots (42)$$

4. Eigen Value Problem

Orthotropic plate are consider, the material directions of width identify with the plate directions. Uniaxial in-plane compressive force N_x along the both sides of edge (x) is subjected.

To calculate the critical buckling load; the natural frequency ω is set to zero, to find the natural frequency without the action of buckling load; N_x is set to zero, and to calculate the natural frequency under buckling load action; N_x is left as a known after finding the critical buckling load N_{cr} Previously. Performing the required mathematical processes (differentiations and then integrations) of Eq.(25) and(26) and then putting the mechanical energy in the following equation:

$$\frac{\partial E}{\partial A_{mn}} = 0 \quad \dots (49)$$

Eq. (48) gives homogenous equations as follow1;

$$\begin{aligned} f(A_{mn}, N_{cr}) &= 0 \quad \text{for buckling problem} \\ f(A_{mn}, \omega) &= 0 \quad \text{for vibration problem} \\ f(A_{mn}, N_{xx}, \omega) &= 0 \quad \text{for vibration under bucklin...} \end{aligned} \quad (50)$$

Eq. (50) solving as an Eigen-value problem which is written as below:

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,(m*n)} \\ \vdots & \ddots & \vdots \\ a_{(m*n),1} & \dots & a_{(m*n),(m*n)} \end{bmatrix} \begin{Bmatrix} A_{11} \\ \vdots \\ A_{mn} \end{Bmatrix} = 0 \quad \dots (51)$$

Where a_{ij} are the coefficients of the nonzero

unknowns A_{mn} . Finding the determinant of the first term of Eq.(51) and equating it to zero will lead to get the natural frequencies ω and critical buckling load N_{cr} . When M and N are more than 1, The N_{cr} and ω are determined by solving eigenvalue problem. For different edge conditions and M & N more than 1, the solution becomes more difficult and needs computer programming to determine N_{cr} . In this study MATLAB R2015a is used to solve the eigenvalue problem to find the natural frequency under buckling action. For numerical study, ANSYS APDL programming is used.

5. Results and Discussion

MATLAB (version15) programming is used to investigate and solved the buckling load and natural frequency under buckling of composite laminated plate (CLP) with elastic edge condition. The plates are described by a four-letter symbol for example SCSC denotes a plate with simply supported edge at $y=b$, $y=0$, clamped at $x=a$, $x=0$. To see the validity of the derived equations and performance of computer programming for buckling and vibration analysis of (CLP), orthotropic plate numerical results are compared with those found by Firas Hamzah Taya, 2014 [9]. And I. Shufrin, O. Rabinovitch, M. Eisenberger, 2008 [2]. Table (1, 2) shows a good agreement results for different edge conditions. It show that the clamped plate along two or four edges can hold buckling load more than plate with simply supported boundary conditions, especially in table (1) for the case where the plate is FSFS. In the case where the plate is simply supports or mixed with free edges, it is weak to hold large load compared with clamped plates. While show results for laminated composite plate with different edge conditions, stacking sequence, aspect ratio and modulus ratio give good agreement when compared with J. N. Reddy, 2003[15] and I. Shufrin, O. Rabinovitch, M. Eisenberger, 2008[2]. As shown in Table (3, 4, 5). Table (6) present the results of anti-symmetric

cross and angle ply with different aspect and modulus ratio and give a good agreement when compared with results obtained by J. N. Reddy, 2003. Table (7) shows the results of the natural frequency of laminated plate under buckling for different load ratio and compared with the results obtained by Firas Hamzah Taya, 2014, give very close results. It shows that the natural frequency is less than that found without loading because the stiffness reduction. In table (8,9,11) show the results of the natural frequency under buckling for different load ratio and compared with the results obtained by numerical program ANSYS, also in table (10) show the natural frequency under load ratio ($d=0.5$) with effect of aspect , modulus ratio and Fig.2. shows the mode shapes of the case.

6. Conclusions

Buckling of rectangular laminated plate with general elastic restraints along the edges is obtained by using modified Fourier function; also free vibration of this plate under in plane loading is investigated using Rayleigh–Ritz method. The results are compared with the results obtained by other researchers; the comparison showed good agreement between them. The effect of edge conditions, aspect ratio, lamination system, angle of lamination and load correction factor on buckling and vibration characteristics are studied. From the result it is concluded that, the buckling load decreases rapidly with increasing aspect ratio till it is about 1.5, after that takes constancy or close values for higher aspect ratio. The edge conditions affect the critical buckling load and fundamental natural frequency. Clamped edges conditions offer high stiffness, results in high critical buckling load and natural frequency. Clamped edges make the plate holds larger load than simply supported edges. The natural frequency changes reversely with buckling load ratio. Therefore, this investigation has actually showed that this function can be used to get buckling and vibration characteristics of laminated plate with various boundary conditions.

Table 1,

Non-dimensional buckling load ($\bar{N} = N_{cr} b^2 / E_2 h^3$), for [0 90 0] plates of different Boundary conditions, ($E_1/E_2 = 10$, $G_{12} = 0.6E_2$, $\nu_{12} = 0.25$, $a = b$).

References	Type of boundary conditions				
	SSSS	CCCC	SCSC	FSFS	FCFC
Present work	11.550	40.38	35.900	8.049	32.544
Firas[9]	11.491	40.507	36.255	7.991	32.982

Table 2,

Non-dimensional buckling load ($\bar{N} = N_{cr} 12(1 - \nu_{12}\nu_{21})/E_1 h^3$), for [30 -30 30] plates of different boundary conditions, ($E_1/E_2 = 2.45, G_{12} = 0.48E_2, \nu_{12} = 0.23$ a=b).

References	Type of boundary conditions			
	SSSS	CCCC	CSCS	SCSC
Present work	25.77	66.81	47.33	39.32
I. Shufrin[2]	26.67	65.26	49.18	40.93
Discrepancy%	3.4	2.3	3.9	4

Table 3,

Non-dimensional buckling load ($\bar{N} = N_{cr} b^2 / D_{22} \pi^2$), for [0 90 90 0] plates (SSSS) of different aspect, modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

References	a/b	$E_1/E_2=5$	10	20	25	40
Present work		13.94	18.225	22	23.1	25
Reddy[16]	0.5	13.9	18.126	21.87	22.87	24.59
Present work		5.66	6.353	7	7.13	7.5
Reddy	1	5.65	6.347	6.96	7.12	7.4
Present work		5.238	5.28	5.317	5.326	5.34
Reddy	1.5	5.233	5.27	5.31	5.318	5.33

Table 4,

Non-dimensional buckling load ($\bar{N} = N_{cr} b^2 / D_{22} \pi^2$), for [0 90]_{2s} laminated plates (CCCF) of different aspect, modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

E_1/E_2	References	a/b=1	1.5	2
3	Present work	6.7	3.48	2.47
	I. Shufrin	6.4	3.3	2.34
	Discrepancy%	4.7	5	5
10	Present work	8.08	3.96	2.6
	I. Shufrin	7.84	3.78	2.48
	Discrepancy%	2.9	4	4

Table 5,

Non-dimensional buckling load ($\bar{N} = N_{cr} b^2 / D_{22} \pi^2$), for [0 90]_{2s} laminated plates (CSCS) of different aspect, modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

E_1/E_2	References	a/b=1	1.5	2
3	Present work	6.671	6.379	6.12
	I. Shufrin	6.659	6.295	5.84
	Discrepancy%	0.179	1.3	4.5
10	Present work	6.584	6.096	5.71
	I. Shufrin	6.557	6.056	5.46
	Discrepancy%	0.41	0.656	4.3

Table 6,

Non-dimensional buckling load ($\bar{N} = N_{cr} b^2 / E_2 h^3$), for anti-symmetric laminated plates (SSSS) with effect of different modulus ratio, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$).

Ply Orientations	References	$E_1/E_2=10$	25	40
[0 90] ₄	Present work	11.174	23.523	35.874
	Reddy	10.864	22.622	34.381
	Discrepancy%	2.7	3.8	4.1
[45 - 45] ₄	Present work	18.2	42.81	67.38
	Reddy	17.637	41.16	64.68
	Discrepancy%	3	3.8	4

Table 7,

Dimensionless natural frequency ($\bar{\omega} = \omega_0 a^2 \sqrt{\rho_c/E_2}/h$), for [0 90 0] plates under buckling of different load ratios, ($E_1/E_2 = 10, G_{12} = 0.6E_2, \nu_{12} = 0.25, a = b$).

D	References	SSSS	CCCC	CSCS	SFSF	CFCF
0	Present work	10.656	22.255	21.065	8.892	20.106
	Firas	10.649	22.323	21.119	8.886	20.143
0.25	Present work	9.228	19.27	18.243	7.701	17.413
	Firas	9.223	19.33	18.29	7.443	17.444
0.5	Present work	7.535	15.736	14.895	6.287	14.217
	Firas	7.53	15785	14.933	5.973	14.243
0.75	Present work	5.328	11.127	10.532	4.446	10.053
	Firas	5.324	11.161	10.559	3.995	10.071

Table 8,

Dimensionless natural frequency ($\bar{\omega} = \omega_0 a^2 \sqrt{\rho_c/E_2}/h$), for [0 90 90 0] plates under buckling of different load ratios, ($E_1/E_2 = 40, G_{12} = 0.5E_2, \nu_{12} = 0.25, a = b$).

d	References	SSSS	CCCC	SCSC	SFSF
0	Present work	18.817	41.216	38.668	6.916
	Ansys	18.703	40.662	38.099	6.914
0.25	Present work	16.296	35.695	33.488	6.916
	Ansys	16.196	35.434	33.169	6.902
0.5	Present work	13.306	29.145	27.343	5.812
	Ansys	13.224	29.14	27.244	5.813
0.75	Present work	9.408	20.609	19.335	4.11
	Ansys	9.351	20.777	19.394	4.115

Table 9,

Dimensionless natural frequency ($\bar{\omega} = \omega_0 a^2 \sqrt{\rho_c/E_2}/h$), for [45 - 45]₄ plates under buckling of different load ratios, ($E_1/E_2 = 10, G_{12} = 0.5E_2, \nu_{12} = 0.25, a = b$).

D	References	SSSS	CCCC	SCSC	SFSF
0	Present work	14.09	21.632	17.914	5.252
	Ansys	13.111	21.165	17.533	4.671
0.25	Present work	11.613	18.734	15.514	4.548
	Ansys	11.395	18.378	15.258	4.072
0.5	Present work	9.482	15.296	12.667	3.713
	Ansys	9.341	15.05	12.529	3.35
0.75	Present work	6.704	10.816	8.957	2.626
	Ansys	6.634	10.678	8.917	2.39

Table 10,

Dimensionless natural frequency ($\bar{\omega} = \omega_0 a^2 \sqrt{\rho_c/E_2}/h$), [30 -30 30] plates of different boundary conditions, ($E_1/E_2 = 2.45, G_{12} = 0.48E_2, \nu_{12} = 0.23, a=b$).

D	References	SSSS	CCCC	SCSC
0	Present work	7.311	13.02	9.949
	Ansys	7.237	13.04	9.913
0.25	Present work	6.332	11.289	8.619
	Ansys	6.272	11.359	8.595
0.5	Present work	5.171	9.223	7.04
	Ansys	5.124	9.334	7.027
0.75	Present work	3.657	6.526	4.98
	Ansys	3.625	6.653	4.978

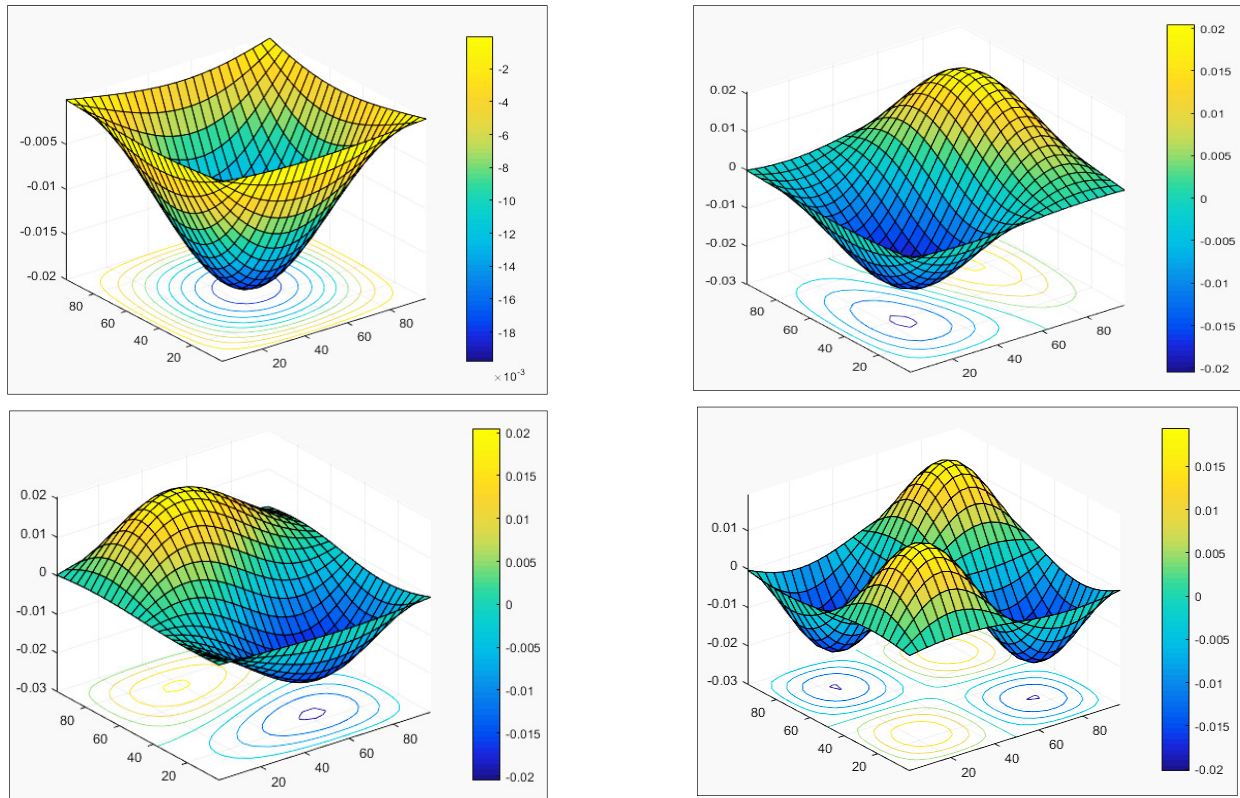


Fig. 2. Mode shape for free vibration of (SSSS) for [30 -30 30] laminated square plate a-first b-second c-third d-fourth modes.

Table11,

Dimensionless natural frequency ($\bar{\omega} = \omega_o a^2 \sqrt{\rho_c/E_2/h}$), for [0 90 90 0] (SSSS) plates with effect of aspect and modulus ratios, ($G_{12} = 0.5E_2, \nu_{12} = 0.25$) (natural frequency without load) d=0.5.

References	a/b	$E_1/E_2=10$	25	40
Present work	0.5	6.275	9.654	12.125
		(8.875)	(13.652)	(17.147)
Anslys	1	6.275	9.63	12.066
		(8.875)	(13.62)	(17.064)
Present work	1	7.426	10.774	13.306
		(10.502)	(15.237)	(18.817)
Anslys	1.5	7.411	10.731	13.22
		(10.48)	(15.176)	(18.702)
Present work	1.5	10.157	13.964	16.938
		(14.364)	(19.748)	(23.954)
Anslys	1.5	10.161	13.964	16.934
		(14.367)	(19.747)	(23.948)

Nomenclature

Symbol	Discretion	Units
A	Length of a plate	M
B	width of a plate	M
H	Plate thickness	M
A	vector of the expansion or Rayleigh–Ritz coefficients	
A_{mn}	expansion or Rayleigh–Ritz coefficients	

a_m	expansion or Rayleigh–Ritz coefficient	
D_{ij}	flexural rigidity	
M, N	numbers of expansion terms used in x- and y direction, respectively	
$M_x, M_y,$ M_{xy}	Moment result per unit length	N.m/m
Q_x, Q_y	Transverse shear force result	N
K_{x0}, K_{x1}	rotational stiffness at	Rad.N/m

	x = 0 and a, respectively	
K_{y0}, K_{y1}	rotational stiffness at y = 0 and b, respectively	Rad.N/m
k_{x0}, k_{x1}	translational stiffness at x = 0 and a, respectively	N/m
k_{y0}, k_{y1}	translational stiffness at y = 0 and b, respectively	N/m
$P(x)$	a simple polynomial function	
x,y,z	Cartesian coordinate system	M
Π	Total potential energy of the System	N.m
U	Strain energy of deforma- tion	N.m
V_c	the elastic potential en- ergy	N.m
$W(x)$	flexural displacement of a beam	M
$W(x,y)$	flexural displacement of a plate	M
$X(x),$ $Y(y)$	beam characteristic function	
α_0, α_1	$W''''(a), W''''(0)$	
β_0, β_1	$W'(0), W'(a)$	
$\varphi_m^a(x)$	admissible functions in x direction	
$\varphi_n^b(y)$	admissible functions in y direction	
S, C, F	Simply- clamped- free	

7. References

- [1] Zhong, Hongzhi, and Chao Gu. "Buckling of symmetrical cross-ply composite rectangular plates under a linearly varying in-plane load." *Composite structures* Vol. 80 No.1 (2007): PP. 42-48.
- [2] Shufrin, I., O. Rabinovitch, and M. Eisenberger. "Buckling of symmetrically laminated rectangular plates with general boundary conditions—A semi analytical approach." *Composite Structures* Vol. 82 No.4 (2008) PP: 521-531.
- [3] Kuo, Shih-Yao, and Le-Chung Shiau. "Buckling and vibration of composite laminated plates with variable fiber spacing" *Composite Structures* Vol. 90 No.2 (2009) PP: 196-200.
- [4] Kim, Seung-Eock, Hui-Tai Thai, and Jaehong Lee. "Buckling analysis of plates using the two variable refined plate theory." *Thin-Walled Structures* Vol. 47 No. 4 (2009) PP: 455-462.
- [5] Kumar Panda, Sarat, and L. S. Ramachandra. "Buckling and postbuckling behavior of cross-ply composite plate subjected to nonuniform in-plane loads." *Journal of engineering mechanics* Vol. 137 No. 9 (2011) PP: 589-597.
- [6] Carrera, E., et al. "Effects of in-plane loading on vibration of composite plates." *Shock and Vibration* Vol. 19 No. 4 (2012) PP: 619-634.
- [7] Thai, Hui-Tai, and Dong-Ho Choi. "A simple first-order shear deformation theory for laminated composite plates." *Composite Structures* Vol. 106 (2013) PP: 754-763.
- [8] Sayyad, ATTESHAMUDDIN S., and Y. M. Ghugal. "Buckling and free vibration analysis of orthotropic plates by using exponential shear deformation theory." *Latin American Journal of Solids and Structures* Vol. 11 No. 8 (2014) PP: 1298-1314.
- [9] Firas Hamzah Taya., "Buckling and Vibration Analysis of Laminated Composite Plate", M.Sc. Thesis, University of Baghdad, Mech. Dep. 2014.
- [10] Mantari, J. L., and F. G. Canales. "Free vibration and buckling of laminated beams via hybrid Ritz solution for various penalized boundary conditions." *Composite Structures* Vol. 152 (2016) PP: 306-315
- [11] Khov, Henry, Wen L. Li, and Ronald F. Gibson. "An accurate solution method for the static and dynamic deflections of orthotropic plates with general boundary conditions." *Composite Structures* Vol. 90 No. 4 (2009) PP: 474-481.
- [12] Li, Wen L. "Vibration analysis of rectangular plates with general elastic boundary supports." *Journal of Sound and Vibration* Vol. 273 No. 3 (2004) PP: 619-635
- [13] Li, Wen L. "Free vibrations of beams with general boundary conditions." *Journal of Sound and Vibration* Vol. 237 No. 4 (2000) PP: 709-725.
- [14] Li, W. L., and M. Daniels. "A Fourier series method for the vibrations of elastically restrained plates arbitrarily loaded with springs and masses." *Journal of Sound Vibration* Vol. 252 (2002) PP: 768-781
- [15] Reddy, Junuthula Narasimha. *Mechanics of laminated composite plates and shells: theory and analysis*. CRC press, 2004.

تحليل الإبعاج والاهتزاز لصفائح مركبة طبقية مع ظروف أسناد عامة

وداد مجيد ابراهيم* زينب عبد الكريم عبد**

**قسم الهندسة الميكانيكية/ كلية الهندسة/ جامعة بغداد

*البريد الإلكتروني: wedad.ibrahim@coeng.uobaghdad.edu.iq

**البريد الإلكتروني: zainabkareem2158@gmail.com

الخلاصة

تمت دراسة الاهتزاز والانبعاث للصفائح الرقيقة المركبة المكونة من الطبقات المعرضة لاحمال منتظمة ضمن مستوى الصفيحة باستخدام نظرية الصفائح الكلاسيكية (CLPT). في الجانب النظري التحليلي تم اشتقاق معادلات الحركة باستخدام طريقة رنر للحصول على مجموعة معادلات متجانسة وحلها كمسألة eigenvalue لحل مسألة حمل الانبعاث لصفائح متناظرة وغير متناظرة مكونة من طبقات متعامدة وغير متعامدة الزوايا. والدوال المستخدمة في هذا البحث يمكن ان تمثل بدوال مثلثية ودوال عشوائية مستمرة وذلك لضمان سلاسة المطلوبة لعمل الدالة الرئيسية. ولعل الاله من ذلك، ان هذه الدراسة قد طورت أسلوب عام لاشتقاق مجموعة كاملة من الدوال المقبولة التي يمكن تطبيقها لشروط اسناد الحافات المختلفة. لقد تم دراسة عدة أمثلة عديدة لاثبات دقة نتائج الحل وتقاربها مع الأخذ بنظر الاعتبار تغيير في بعض معايير التصميم مثل شروط الحدود نسبة الارتفاع وزاوية التصفيح ونسبة السماكة ونسبة ال-orthotropy حيث تم مقارنة النتائج مع باحثين اخرين واعطت تقارباً جيداً جداً.