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Fixed point results concerning α-F-contraction mappings in metric spaces

Lakshmi Kanta Dey [a], Poom Kumam [b] and Tanusri Senapati [c]

[a] Department of Mathematics, National Institute of Technology Durgapur, India
(lakshmikdey@yahoo.co.in)
[b] Department of Mathematics, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bang Mod, Thung Khru, Bangkok 10140, Thailand. (poom.kum@kmutt.ac.th)
[c] Department of Mathematics, Indian Institute of Technology Guwahati, Assam, India
(senapati.tanusri@gmail.com)

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Abstract

In this paper, we introduce the notions of generalized α-F-contraction and modified generalized α-F-contraction. Then, we present sufficient conditions for existence and uniqueness of fixed points for the above kind of contractions. Necessarily, our results generalize and unify several results of the existing literature. Some examples are presented to substantiate the usability of our obtained results.

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1. Introduction and Preliminaries

Throughout this paper we denote by $\mathbb{R}_+$, $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}_0$, the set of positive real numbers, set of real numbers, set of natural numbers and set of nonnegative integers respectively. It is widely known that the Banach contraction principle [1] is the first metric fixed point theorem and one of the most powerful and versatile results in the field of functional analysis. Due to its significance and several applications, over the years, it has been generalized in different directions by

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several mathematicians (for example, see ([2, 3, 4, 5, 7, 10, 17, 18, 15, 16, 19]) and references therein).

Before stating our main results, at first we recollect some useful definitions and results in the comparable literature which will be needed throughout the study. So, we start by presenting the concept of α-admissible mappings and triangular α-admissible mappings as follows:

Definition 1.1 ([14])**.** A mapping $g : X \to X$ is said to be an α-admissible mapping if there exists a function $\alpha : X \times X \to \mathbb{R}^+$ such that for all $x, y \in X$

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(gx, gy) \geq 1.$$

Definition 1.2 ([11])**.** A mapping $g : X \to X$ is said to be a triangular α-admissible mapping if there exists a function $\alpha : X \times X \to \mathbb{R}^+$ such that

(1) for all $x, y \in X, \alpha(x, y) \geq 1 \Rightarrow \alpha(gx, gy) \geq 1$,
(2) for all $x, y, z \in X, \alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$.

Note 1.3. [11] Let g be a triangular α-admissible mapping. If (x_n) is any sequence defined by $x_{n+1} = gx_n$ and $\alpha(x_n, x_{n+1}) \geq 1$, then for all $n, m \in \mathbb{N}$, we get $\alpha(x_n, x_m) \geq 1$.

In 2012, Wardowski [19] introduced the concept of F-contractions which plays a crucial part in the recent trend of research in fixed point theory. After that, Wardowski and Dung [20] and Dung and Hang [6] extended the concept of F-contractions to F-weak contractions and generalized F-contractions respectively.

By mixing up the concept of α-admissible mappings with F-contractions [19] and F-weak contractions [20], Gopal et al. [8] introduced the concept of α-type F-contractions and α-type F-weak contractions as follows:

Definition 1.4 ([8])**.** Let (X, d) be a metric space and $g : X \to X$ be a mapping. Suppose $\alpha : X \times X \to \{-\infty\} \cup (0, \infty)$ be a function. The function g is said to be an α-type F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(gx, gy) > 0 \Rightarrow \tau + \alpha(x, y)F(d(gx, gy)) \leq F(d(x, y)).$$

Definition 1.5 ([8])**.** Let (X, d) be a metric space and $g : X \to X$ be a self-mapping. Let $\alpha : X \times X \to \{-\infty\} \cup (0, \infty)$ be a function. The function g is said to be an α-type F-weak contraction if there exists $\tau > 0$ such that for all $x, y \in X,\ d(gx, gy) > 0$ implies that

$$\tau + \alpha(x, y)F(d(gx, gy)) \leq F\left(\max\left\{d(x, y), d(x, gx), d(y, gy), \tfrac{d(x,gy)+d(y,gx)}{2}\right\}\right).$$

In the above definitions, the function F belongs to the family $\mathcal{F}$ of mappings from $(0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is a strictly increasing function, i.e., for all $x, y \in \mathbb{R}_+$ with $x < y$, $F(x) < F(y)$;
(F2) For each sequence (α_n) of positive numbers,

$$\lim_{n\to\infty} \alpha_n = 0 \iff \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F3) There exists a $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

In this sequel, the authors of [8] established some fixed point results and finally they presented an application to nonlinear fractional differential equations.

Subsequently, Piri and Kumam [13] established some new fixed point results by taking a weaker family of functions as well as by weakening the contraction condition given by:

Definition 1.6 ([13])**.** Let (X, d) be a metric space and let $g : X \to X$ be a mapping. The function g is said to be a modified generalized F-contraction of type (A) if there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(gx, gy) > 0 \Rightarrow \tau + F(d(gx, gy)) \leq F(N_g(x, y)),$$

where,

$$\begin{aligned} N_g(x,y) \quad = \quad & \max\Big\{d(x,y), \frac{d(x,gy)+d(y,gx)}{2}, \frac{d(g^2x,x)+d(g^2x,gy)}{2}, \\ & d(g^2x,gx), d(g^2x,y), d(gx,y)+d(y,gy), d(g^2x,gy)+d(x,gx)\Big\} \end{aligned}$$

and F satisfies the following conditions:

(1) F is strictly increasing,
(2) F is continuous.

In a similar fashion, they also defined modified generalized F-contraction of type (B) by considering different class of functions satisfying the above contractive condition along with the following properties:

(1) F is strictly increasing;
(2) There exists a $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

Using the notions of modified generalized F-contraction of type (A) and type (B), the authors presented some new fixed point results which generalized and extended several related results discussed in Wardowski [19], Piri and Kumam [12], Dung and Hang [6] and Wardowski and Dung [20].

For the sake of completeness of our paper, we need to recall the definition of α-complete metric spaces and α-continuous mappings.

Definition 1.7 ([9])**.** Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. The metric space (X, d) is said to be an α-complete metric space if and only if every Cauchy sequence with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}_0$, converges in X.

Definition 1.8 ([9])**.** Let (X, d) be a metric space. Let g be a self-map defined on X and $\alpha : X \times X \to [0, \infty)$ be a function. Then g is said to be an α-continuous mapping if for every $x \in X$ and sequence $(x_n) \in X$ with (x_n) converging to x,

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}_0 \Rightarrow gx_n \to gx.$$

Here, we provide an example of an α-continuous mapping which is not continuous.

Example 1.9. Let $X = [0, \infty)$ and $d(x, y) = |x - y|$, for all $x, y \in X$. We define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{for all } \ x, y \in [0,1]; \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

and the mapping $g : X \to X$ by

$$gx = \begin{cases} \frac{x}{2}, & \text{for all } \ x \in [0,1]; \\ 2x, & 1 < x \leq 3; \\ x^2, & \text{otherwise} \end{cases}$$

Clearly, g is not continuous as $x = 1$ and $x = 3$ are points of discontinuity but g is an α-continuous map.

Remark 1.10. Every complete metric space is α-complete and every continuous map is α-continuous but in both the cases, the converse does not hold in general.

In this article, by $\mathfrak{F}$, we denote the following family of functions given by

$$\mathfrak{F} = \{F/F : (0, \infty) \to \mathbb{R}\}$$

satisfying the following conditions:

(F') F is a strictly increasing function, i.e., for all $x, y \in \mathbb{R}_+$ with $x < y$, $F(x) < F(y)$;

(F'') There exists a $k \in (0,1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

The aim of this article is to present some new fixed point results in α-complete metric spaces and show that our obtained results generalize several existing results in the literature. For this, we introduce the concept of generalized α-type F-contractions and modified generalized α-type F-contractions. For simplicity, we call these contractions as generalized α-F-contractions and modified generalized α-F-contractions respectively. Finally, we construct some non-trivial examples to validate the potential of our results.

2. Main Results

We begin with this section by presenting the new concept of generalized α-F-contractions and modified generalized α-F-contractions respectively.

Definition 2.1. Let (X, d) be a metric space and $g : X \to X$ be a mapping. Let $\alpha : X \times X \to [0, \infty)$ be a function and $F \in \mathfrak{F}$. The function g is said to be a generalized α-F-contraction mapping if there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(gx, gy) > 0 \Rightarrow \tau + \alpha(x,y)F(d(gx,gy)) \leq F(M_g(x,y))$$

where,

$$\begin{aligned} M_g(x,y) \quad = \quad & \max\Bigg\{d(x,y), d(x,gx), d(y,gy), \frac{d(x,gy)+d(y,gx)}{2}, \\ & \frac{d(g^2x,x)+d(g^2x,gy)}{2}, d(g^2x,gx), d(g^2x,y), d(g^2x,gy)\Bigg\}. \end{aligned}$$

Definition 2.2. Let (X,d) be a metric space and $g : X \to X$ be a mapping. Let $\alpha : X \times X \to [0,\infty)$ be a function and $F \in \mathfrak{F}$. The function g is said to be a modified generalized α-F-contraction mapping if there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(gx, gy) > 0 \Rightarrow \tau + \alpha(x,y)F(d(gx,gy)) \leq F(N_g(x,y))$$

where,

$$\begin{aligned} N_g(x,y) &= \max\left\{ d(x,y), \frac{d(x,gy)+d(y,gx)}{2}, \frac{d(g^2x,x)+d(g^2x,gy)}{2}, \right. \\ &\qquad \left. d(g^2x,gx), d(g^2x,y), d(gx,y)+d(y,gy), d(g^2x,gy)+d(x,gx) \right\}. \end{aligned}$$

Remark 2.3. Every modified generalized F-contraction (respectively, generalized F-contraction) is a modified generalized α-F-contraction (respectively, generalized α-F-contraction).

The reverse implications do not hold. We illustrate this by presenting an example.

Example 2.4. Let $X = \{0,1,2,3,4\}$ and we define the distance function d as follows

$$d(x,y) = \begin{cases} 0, & \text{iff } x = y; \\ \frac{5}{2}, & (x,y) \in \{(0,3),(3,0)\}; \\ \frac{3}{2}, & \text{otherwise.} \end{cases}$$

Also, we define a mapping $g : X \to X$ by

$$g(0) = g(3) = 1; \quad g(1) = g(4) = 3; \quad g(2) = 0.$$

Therefore, we get

$$d(gx,gy) > 0 \iff [x \in \{0,3\} \wedge y \in \{1,4\}; x \in \{0,3\} \wedge y = 2; x \in \{1,4\} \wedge y = 2].$$

Now, we are interested to find $N_g(x,y)$. For this purpose, we consider the following cases:

Case-I. Let $x \in \{0,3\}$ and $y \in \{1,4\}$.

Then for any $(x,y) \in \{(0,1),(0,4),(3,1),(3,4)\}$, we get

$$d(gx,gy) = d(1,3) = \frac{3}{2}.$$

Let $(x,y) = (0,1)$. Then, we have

$$\begin{aligned} N_g(0,1) &= \max\{d(0,1), \frac{d(0,g1)+d(1,g0)}{2}, \frac{d(g^20,0)+d(g^20,g1)}{2}, \\ &\qquad d(g^20,g0), d(g^20,1), d(g0,1)+d(1,g1), d(g^20,g1)+d(0,g0)\} \\ &= \max\left\{\frac{3}{2}, \frac{5}{4}\right\} = \frac{3}{2}. \end{aligned}$$

For $(x, y) = (0, 4)$, we get

$$\begin{aligned} N_g(0,4) &= \max\Big\{d(0,4), \frac{d(0,g4)+d(4,g0)}{2}, \frac{d(g^20,0)+d(g^20,g4)}{2}, \\ &\quad d(g^20,g0), d(g^20,4), d(g0,4)+d(4,g4), d(g^20,g4)+d(0,g0)\Big\} \\ &= \max\left\{\frac{3}{2}, \frac{5}{4}, 2\right\} = 2. \end{aligned}$$

For $(x, y) = (3, 1)$, we obtain

$$\begin{aligned} N_g(3,1) &= \max\Big\{d(3,1), \frac{d(3,g1)+d(1,g3)}{2}, \frac{d(g^23,3)+d(g^23,g1)}{2}, \\ &\quad d(g^23,g3), d(g^23,1), d(g3,1)+d(1,g1), d(g^23,g1)+d(3,g3)\Big\} \\ &= \max\left\{\frac{3}{2}, 0\right\} = \frac{3}{2} \end{aligned}$$

and for $(x, y) = (3, 4)$, also have

$$\begin{aligned} N_g(3,4) &= \max\Big\{d(3,4), \frac{d(3,g4)+d(4,g3)}{2}, \frac{d(g^23,3)+d(g^23,g4)}{2}, \\ &\quad d(g^23,g3), d(g^23,4), d(g3,4)+d(4,g4), d(g^23,g4)+d(3,g3)\Big\} \\ &= \max\left\{\frac{3}{2}, \frac{3}{4}, 0, 3\right\} = 3. \end{aligned}$$

Case-II. Let $x \in \{0, 3\}$ and $y = 2$. Then for $(x, y) \in \{(0, 2), (3, 2)\}$,

$$d(gx, gy) = d(1, 0) = \frac{3}{2}.$$

Then, for $(x, y) = (0, 2)$, we have

$$\begin{aligned} N_g(0,2) &= \max\Big\{d(0,2), \frac{d(0,g2)+d(2,g0)}{2}, \frac{d(g^20,0)+d(g^20,g2)}{2}, \\ &\quad d(g^20,g0), d(g^20,2), d(g0,2)+d(2,g2), d(g^20,g2)+d(0,g0)\Big\} \\ &= \max\left\{\frac{3}{2}, \frac{3}{4}, \frac{5}{2}, 3, 4\right\} = 4. \end{aligned}$$

For $(x, y) = (3, 2)$, we get

$$\begin{aligned} N_g(3,2) &= \max\Big\{d(3,2), \frac{d(3,g2)+d(2,g3)}{2}, \frac{d(g^23,3)+d(g^23,g2)}{2}, \\ &\quad d(g^23,g3), d(g^23,2), d(g3,2)+d(2,g2), d(g^23,g2)+d(3,g3)\Big\} \\ &= \max\left\{\frac{3}{2}, \frac{3}{4}, 2, 3, 4\right\} = 4. \end{aligned}$$

Case-III. Let $x \in \{1, 4\}$ and $y = 2$. Then $(x, y) \in \{(1, 2), (4, 2)\}$,

$$d(gx, gy) = d(3, 0) = \frac{5}{2}.$$

Let $(x, y) = (1, 2)$. Then

$$\begin{aligned} N_g(1,2) &= \max\Big\{d(1,2), \frac{d(1,g2)+d(2,g1)}{2}, \frac{d(g^2 1,1)+d(g^2 1,g2)}{2}, \\ &\qquad d(g^2 1,g1), d(g^2 1,2), d(g1,2)+d(2,g2), d(g^2 1,g2)+d(1,g1)\Big\} \\ &= \max\left\{\frac{3}{2}, \frac{3}{4}, 3\right\} = 3. \end{aligned}$$

For $(x, y) = (4, 2)$, we have

$$\begin{aligned} N_g(4,2) &= \max\Big\{d(4,2), \frac{d(4,g2)+d(2,g4)}{2}, \frac{d(g^2 4,4)+d(g^2 4,g2)}{2}, \\ &\qquad d(g^2 4,g4), d(g^2 4,2), d(g4,2)+d(2,g2), d(g^2 4,g2)+d(4,g4)\Big\} \\ &= \max\left\{\frac{3}{2}, 3\right\} = 3. \end{aligned}$$

From the above cases, we observe that whenever $(x, y) \in \{(0, 1), (3, 1)\}$,

$$d(gx, gy) = N_g(x, y).$$

Since F is increasing, we can't find any $\tau > 0$ such that

$$\tau + F(d(gx, gy)) \leq F(N_g(x, y)).$$

This shows that g is not a modified generalized F-contraction. Hence, g can not be an F-contraction, F-weak contraction and generalized F-contraction.

Let us consider $F(x) = \ln x$ for all $x \in (0, \infty)$. Clearly, $F \in \mathfrak{F}$. Now, we define a function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} \frac{1}{2}, & (x,y) \in \{(0,1),(3,1)\}; \\ 1 & \text{otherwise.} \end{cases}$$

Then, we can find $\tau > 0$ such that

$$\tau + \alpha(x, y)F(d(gx, gy)) \leq F(N_g(x, y)),$$

whenever $d(gx, gy) > 0$. In particular, when $\alpha(x, y) = \frac{1}{2}$ one can choose $\tau \in (0, \frac{1}{5})$. Therefore g is a modified generalized α-F-contraction.

In the following, we present an example to show that the class of modified generalized α-F-contraction mappings is larger than that of generalized α-F-contraction mappings.

Example 2.5. Let $X = \{-1, 0, 1\}$ and g be a self-mapping on X defined by

$$g(-1) = g(0) = 0, \quad g(1) = -1.$$

We define a distance function d on X by

$$d(x,y) = \begin{cases} 0, & x = y; \\ \frac{1}{2}, & (x,y) \in \{(1,-1),(-1,1)\}; \\ 1 & \text{otherwise.} \end{cases}$$

So, (X, d) is a complete metric space. Now $d(gx, gy) > 0$ for $(x, y) = (0, 1)$ and $(x, y) = (-1, 1)$. Therefore we consider the following two cases.

Case-I. Let $(x, y) = (0, 1)$. Then, $d(g0, g1) = d(0, -1) = 1$

$$\begin{aligned} M_g(0,1) &= \max\Big\{d(0,1), d(0,g0), d(1,g1), \frac{d(0,g1)+d(1,g0)}{2}, \\ &\quad \frac{d(g^2 0,0)+d(g^2 0,g1)}{2}, d(g^2 0,g0), d(g^2 0,1), d(g^2 0,g1)\Big\} \\ &= \max\Big\{1,0,\frac{1}{2}\Big\} = 1. \end{aligned}$$

and

$$\begin{aligned} N_g(0,1) &= \max\Big\{d(0,1), \frac{d(0,g1)+d(1,g0)}{2}, \frac{d(g^2 0,0)+d(g^2 0,g1)}{2}, \\ &\quad d(g^2 0,g0), d(g^2 0,1), d(g0,1)+d(1,g1), d(g^2 0,g1)+d(0,g0)\Big\} \\ &= \max\Big\{1,0,\frac{1}{2},\frac{3}{2}\Big\} = \frac{3}{2}. \end{aligned}$$

Case-II. Let $(x, y) = (-1, 1)$. Then $d(g(-1), g1) = d(0, -1) = 1$ and

$$\begin{aligned} M_g(-1,1) &= \max\{d(-1,1), d(-1,g(-1)), d(1,g1), \\ &\quad \frac{d(-1,g1)+d(1,g(-1))}{2}, \frac{d(g^2(-1),-1)+d(g^2(-1),g1)}{2}, \\ &\quad d(g^2(-1),g(-1)), d(g^2(-1),1), d(g^2(-1),g1)\} \\ &= \max\Big\{1,\frac{1}{2},0\Big\} = 1. \end{aligned}$$

$$\begin{aligned} N_g(-1,1) &= \max\Big\{d(-1,1), \frac{d(-1,g1)+d(1,g(-1))}{2}, \\ &\quad \frac{d(g^2(-1),-1)+d(g^2(-1),g1)}{2}, d(g^2(-1),g(-1)), d(g^2(-1),1), \\ &\quad d(g(-1),1)+d(1,g1), d(g^2(-1),g1)+d(-1,g(-1))\Big\} \\ &= \max\Big\{1,0,\frac{1}{2},2,\frac{3}{2}\Big\} = 2. \end{aligned}$$

If we choose $F(x) = \ln(x)$ for all $x \in (0, \infty)$ and $\alpha(x, y) \geq 0$, then g can not be a generalized α-F-contraction, since

$$\begin{aligned} &\tau + \alpha(0,1)F(d(g0,g1)) \leq F(M_g(0,1)) \\ &\Rightarrow \tau + \alpha(0,1)\ln(1) \leq \ln(1) \\ &\Rightarrow \tau \leq 0. \end{aligned}$$

If we choose $N_g(0, 1)$ instead of $M_g(0, 1)$, one can check that g is a modified generalized F-contraction and hence modified generalized α-F-contraction.

In a similar fashion, for case-II, it can be shown that g is a modified generalized α-F-contraction but not generalized α-F-contraction.

Now, we are in a position to state our main results.

Theorem 2.6. *Let (X, d) be an α-complete metric space and $g : X \to X$ be a modified generalized α-F-contraction where $F \in \mathfrak{F}$. Assume that the following conditions hold:*

(1) *g is α-admissible, α-continuous mapping;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, gx_0) \geq 1$.*

Then g has a fixed point.

Proof. By the hypothesis, there exists a point $x_0 \in X$ such that $\alpha(x_0, gx_0) \geq 1$. Now we define a sequence (x_n) by $x_{n+1} = gx_n$, for all $n \in \mathbb{N}_0$. If for some $n \in \mathbb{N}$, $x_n = gx_n$, then x_n is a fixed point of g and the proof is complete. So we assume that there exists no such integer n for which $x_n = gx_n$.

Now $\alpha(x_0, gx_0) \geq 1 \Rightarrow \alpha(x_0, x_1) \geq 1$. Since g is an α-admissible mapping, for all $n \in \mathbb{N}_0$, we get $\alpha(x_n, x_{n+1}) \geq 1$. As $d(gx_{n-1}, gx_n) > 0$ and g is a modified generalized α-F-contraction, for some $\tau > 0$, we have

$$\begin{aligned} F(d(x_n, x_{n+1})) &= F(d(gx_{n-1}, gx_n)) \\ &\leq \tau + \alpha(x_{n-1}, x_n)F(d(gx_{n-1}, gx_n)) \\ &\leq F(N_g(x_{n-1}, x_n)). \end{aligned} \tag{2.1}$$

Now, by simple computations, we have

$$\begin{aligned} N_g(x_{n-1}, x_n) = \; & \max\left\{d(x_{n-1}, x_n), \frac{d(x_{n-1}, gx_n) + d(x_n, gx_{n-1})}{2}, \right. \\ & \frac{d(g^2x_{n-1}, x_{n-1}) + d(g^2x_{n-1}, gx_n)}{2}, \\ & d(g^2x_{n-1}, gx_{n-1}), d(g^2x_{n-1}, x_n), \\ & d(gx_{n-1}, x_n) + d(x_n, gx_n), \\ & \left. d(g^2x_{n-1}, gx_n) + d(x_{n-1}, gx_{n-1})\right\} \\ = \; & \max\left\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2}\right\}. \end{aligned}$$

If $\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$, then (2.1) shows that

$$\tau + \alpha(x_{n-1}, x_n)F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1}))$$

which is impossible. We must have

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n).$$

Therefore, (2.1) implies that

$$\begin{aligned} & F(d(x_n, x_{n+1})) \leq \alpha(x_{n-1}, x_n)F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau \\ & \Rightarrow F(d(x_n, x_{n+1})) < F(d(x_{n-1}, x_n)) \text{ as } \tau > 0 \\ & \Rightarrow d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \end{aligned} \tag{2.2}$$

This shows that (x_n) is a decreasing sequence of nonnegative real numbers. We claim that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. If possible, let $\lim_{n\to\infty} d(x_{n+1}, x_n) = \delta$ for some

$\delta > 0$. Therefore, for every $n \in \mathbb{N}$, we have $d(x_n, x_{n+1}) \geq \delta$. By (F') and (2.3), we have

$$\begin{aligned} F(\delta) \leq F(d(x_n, x_{n+1})) &\leq \alpha(x_{n+1}, x_n)F(d(x_n, x_{n+1})) \\ &< F(d(x_{n-1}, x_n)) - \tau \\ &< F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\ \ \vdots \\ &< F(d(x_0, x_1)) - n\tau. \end{aligned} \tag{2.3}$$

As $\lim\limits_{n\to\infty}(F(d(x_0, x_1)) - n\tau) = -\infty$, so we can find some $m \in \mathbb{N}$ such that $F(d(x_0, x_1)) - n\tau < F(\delta)$ for all $n > m$, which contradicts the above equation. Therefore, we must have $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Next, we claim that (x_n) is a Cauchy sequence. By (F''), there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty}(\alpha_n^k)F(\alpha_n) = 0, \tag{2.4}$$

where $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Again, from (2.3) and (2.4), we can obtain

$$\begin{aligned} &\lim_{n\to\infty}(\alpha_n^k)(F(\alpha_n) - F(\alpha_0)) \leq \lim_{n\to\infty} -(\alpha_n^k)n\tau \leq 0 \\ &\Rightarrow \lim_{n\to\infty}\{n\alpha_n^k\} = 0 \text{ as } \tau > 0. \end{aligned} \tag{2.5}$$

So, we can find some $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &n(\alpha_n)^k \leq 1, \quad \text{for all } \ n \geq n_0 \\ &\Rightarrow \alpha_n \leq \frac{1}{n^{\frac{1}{k}}}, \quad \text{for all } \ n \geq n_0. \end{aligned} \tag{2.6}$$

In view of (2.6), for all $m > n > n_0$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &< \Sigma_{j=1}^{\infty}\alpha_j \leq \Sigma_{j=1}^{\infty}\frac{1}{j^{\frac{1}{k}}}. \end{aligned}$$

As $\frac{1}{k} > 1$, the above series is convergent. This implies that $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$, i.e., (x_n) is a Cauchy sequence. Since, (X, d) is an α-complete metric space and (x_n) is a Cauchy sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we can find some $x \in X$ such that $x_n \to x$ whenever $n \to \infty$.

Now, we claim that x is a fixed point of g. Since $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}_0$, the α-continuity property of g implies that $gx_n \to gx$ as $n \to \infty$. Finally, we have

$$\begin{aligned} x_{n+1} &= gx_n \\ \Rightarrow \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} gx_n \\ \Rightarrow x &= gx. \end{aligned}$$

Hence x is a fixed point of g. □

Notice that the condition of α-continuity of g in Theorem 2.6 can actually be replaced by another weaker condition. In the sequel, we present the following result.

Theorem 2.7. *Let (X, d) be an α-complete metric space and let $g : X \to X$ be a modified generalized α-F-contraction, where $F \in \mathfrak{F}$. Assume that the following conditions hold:*

1. *g is α-admissible;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, gx_0) \geq 1$;*
3. *if (x_n) is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}_0$ and $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}_0$.*

Then g has a fixed point.

Proof. Following the proof of Theorem 2.6, we know that (x_n) is a Cauchy sequence with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}_0$ and it converges to some point $x \in (X, d)$. By the hypothesis (3), we have $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}_0$.

We claim that x is a fixed point of g. On the contrary, suppose that $gx \neq x \Rightarrow d(x, gx) > 0$. We can find a number $n \in \mathbb{N}$ such that

$$d(x_m, gx) > 0, \text{ for all } m \geq n \Rightarrow d(gx_{m-1}, gx) > 0.$$

So by the condition of the theorem and by the property of F, we can find some $\tau > 0$ such that

$$\begin{aligned}
&\tau + \alpha(x_{m-1}, x)F(d(gx_{m-1}, gx)) \leq F(N_g(x_{m-1}, x)) \\
&\Rightarrow F(d(gx_{m-1}, gx)) < F(N_g(x_{m-1}, x)), \text{ [as } \alpha(x_{m-1}, x) \geq 1; \ \ \tau > 0] \\
&\Rightarrow d(gx_{m-1}, gx) < N_g(x_{m-1}, x) \\
&\Rightarrow \lim_{m\to\infty} d(x_m, gx) < \lim_{m\to\infty} N_g(x_{m-1}, x). \qquad (2.7)
\end{aligned}$$

Now, we compute

$$\begin{aligned}
N_g(x_{m-1}, x) = \max\Big\{ & d(x_{m-1}, x), \frac{d(x_{m-1}, gx) + d(x, gx_{m-1})}{2}, \\
& \frac{d(g^2x_{m-1}, x) + d(g^2x, gx)}{2}, d(g^2x_{m-1}, gx_{m-1}), d(g^2x_{m-1}, gx), \\
& d(g^2x_{m-1}, gx) + d(x_{m-1}, gx_{m-1}), d(gx_{m-1}, x) + d(x, gx)\Big\}.
\end{aligned}$$

Using this in the above inequality, we get

$$\lim_{m\to\infty} d(x_m, gx) < \max\{d(x, x), d(x, gx)\}$$

which leads to a contradiction. Hence, our assumption was wrong. We must have $d(x, gx) = 0$, i.e., x is a fixed point of g. □

In the following theorem, we present a fixed point result for a modified generalized α-F-contraction where the function F satisfies only (F') property.

Theorem 2.8. *Let (X, d) be an α-complete metric space and let $g : X \to X$ be a modified generalized α-F-contraction where F is strictly increasing function on $(0, \infty)$. Assume that the following conditions hold:*

(1) *g is triangular α-admissible;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, gx_0) \geq 1$;*
(3) *if (x_n) is a sequence in X with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}_0$ and $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}_0$.*

Then g has a fixed point.

Proof. Following the proof of Theorem 2.6, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now, we prove that (x_n) is a Cauchy sequence.

If possible, suppose by contradiction that (x_n) is not a Cauchy sequence. Then for some $\epsilon > 0$, we can find sequences $p(n)$ and $q(n)$ of natural numbers such that

$$p(n) > q(n) > n, \qquad d(x_{p(n)}, x_{q(n)}) \geq \epsilon \text{ and } d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \tag{2.8}$$

for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \epsilon &\leq d(x_{p(n)}, x_{q(n)}) \\ &\leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &< d(x_{p(n)}, x_{p(n)-1}) + \epsilon \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(x_{p(n)}, x_{q(n)}) = \epsilon. \tag{2.9}$$

Again, from (2.8), we can find $n_0 \in \mathbb{N}$ such that

$$d(x_{p(n)}, gx_{p(n)}) < \frac{\epsilon}{4} \text{ and } d(x_{q(n)}, gx_{q(n)}) < \frac{\epsilon}{4}, \text{for all } \ n \geq n_0 \in \mathbb{N}. \tag{2.10}$$

Now, we claim that $d(gx_{p(n)}, gx_{q(n)}) > 0$. Indeed, if no, then there exists $m \geq n_0$ such that

$$d(gx_{p(m)}, gx_{q(m)}) = d(x_{p(m)+1}, x_{q(m)+1}) = 0.$$

From (2.10), it follows that

$$\begin{aligned} \epsilon &\leq d(x_{p(m)}, x_{q(m)}) \leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)}) \\ &\leq d(x_{p(m)}, x_{p(m)+1}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)+1}, x_{q(m)}) \\ &= d(x_{p(m)}, gx_{p(m)}) + d(x_{p(m)+1}, x_{q(m)+1}) + d(x_{q(m)}, gx_{q(m)}) \\ &\leq \frac{\epsilon}{4} + 0 + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{2} \end{aligned}$$

which is a contradiction. Therefore, we get

$$d(gx_{p(m)}, gx_{q(m)}) > 0, \quad \text{for all } \ m \in \mathbb{N}.$$

From (2.9), we get

$$\lim_{m\to\infty} d(gx_{p(m)}, gx_{q(m)}) = \lim_{m\to\infty} d(x_{p(m)+1}, x_{q(m)+1}) = \epsilon.$$

Since g is a modified generalized α-F-contraction, we can find $\tau > 0$ such that

$$\begin{aligned}&\tau + \alpha(x_{p(n)}, x_{q(n)})F(d(gx_{p(n)}, gx_{q(n)})) \leq F(N_g(x_{p(n)}, x_{q(n)})), \quad \text{for all} \quad n \geq n_0\\ &\Rightarrow \alpha(x_{p(n)}, x_{q(n)})F(d(gx_{p(n)}, gx_{q(n)})) \leq F(N_g(x_{p(n)}, x_{q(n)})) - \tau.\end{aligned}$$

Since, $\alpha(x_{p(n)}, x_{q(n)}) \geq 1$, for all $n \in \mathbb{N}_0; \tau > 0$ and F is strictly increasing, we have

$$\begin{aligned}&F(d(gx_{p(n)}, gx_{q(n)})) < F(N_g(x_{p(n)}, x_{q(n)}))\\ &\Rightarrow d(gx_{p(n)}, gx_{q(n)}) < N_g(x_{p(n)}, x_{q(n)}), \ \forall n \in \mathbb{N}\\ &\Rightarrow \lim_{n\to\infty} d(gx_{p(n)}, gx_{q(n)}) < \lim_{n\to\infty} N_g(x_{p(n)}, x_{q(n)}).\end{aligned} \tag{2.11}$$

Now, we observe that

$$\begin{aligned}\lim_{n\to\infty} N_g(x_{p(n)}, x_{q(n)}) = \ &\max\{\lim_{n\to\infty}\{d(x_{p(n)}, x_{q(n)}),\\ &\frac{d(x_{p(n)}, x_{q(n)+1}) + d(x_{q(n)}, x_{p(n)+1})}{2},\\ &\frac{d(x_{p(n)+2}, x_{p(n)}) + d(x_{p(n)+2}, x_{q(n)+1})}{2},\\ &d(x_{p(n)+2}, x_{p(n)+1}), d(x_{p(n)+2}, x_{q(n)}),\\ &d(x_{p(n)+2}, x_{q(n)+1}) + d(x_{p(n)}, x_{p(n)+1}),\\ &d(x_{p(n)+1}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1})\}\}.\end{aligned}$$

Using the triangle inequality and by some simple computations, one can easily check that

$$\lim_{n\to\infty} N_g(x_{p(n)}, x_{q(n)}) = \epsilon.$$

Using this in (2.11), we have

$$\epsilon = \lim_{n\to\infty} d(gx_{p(n)}, gx_{q(n)}) < \epsilon$$

which implies that our assumption was wrong. So (x_n) must be a Cauchy sequence with the property $\alpha(x_n, x_{n+1}) \geq 1$, hence it converges to some point $\tilde{x}$ in X as (X, d) is an α-complete metric space.

Next, we show that $\tilde{x}$ is a fixed point of g. By the hypothesis of the theorem, we have $\alpha(x_n, \tilde{x}) \geq 1$. Again, by the property of F, we obtain

$$\begin{aligned}&F(d(x_n, g\tilde{x})) \leq \tau + \alpha(x_{n-1}, x)F(d(gx_{n-1}, g\tilde{x})) \leq F(N_g(x_{n-1}, \tilde{x}))\\ &\Rightarrow d(x_n, g\tilde{x}) \leq N_g(x_{n-1}, g\tilde{x})\\ &\Rightarrow \lim_{n\to\infty} d(x_n, g\tilde{x}) \leq \lim_{n\to\infty} N_g(x_{n-1}, \tilde{x})\\ &\Rightarrow d(\tilde{x}, g\tilde{x})) = 0.\end{aligned}$$

This shows that $\tilde{x}$ is a fixed point of g. □

Now, we present an additional condition to ensure the uniqueness of fixed point.

Theorem 2.9. *Let g be a modified generalized α-F-contraction. If g has two fixed points $x, y \in X$ with $\alpha(x, y) \geq 1$, then we must have $x = y$.*

Proof. Given $x, y \in Fix(g)$ with $x \neq y \Rightarrow gx \neq gy \Rightarrow d(gx, gy) > 0$. For any $n \in \mathbb{N}$, we have $g^n x = x$ and $g^n y = y$. As g is an α-F-contraction with $d(gx, gy) > 0$, there exists some $\tau > 0$ such that

$$\begin{aligned}
&\tau + \alpha(x,y)F(d(gx,gy)) \leq F(N_g(x,y)) \\
&\Rightarrow \tau + \alpha(x,y)F(d(gx,gy)) < F(d(x,y)) \\
&\Rightarrow F(d(gx,gy)) < F(d(x,y)),\ [as\ \alpha(x,y) \geq 1;\ \tau > 0] \\
&\Rightarrow F(d(x,y)) < F(d(x,y)).
\end{aligned}$$

This contradiction shows that $x = y$. □

Remark 2.10. Notice that the above theorems establish the existence and then uniqueness of fixed point of the function g without assuming the continuity property of F as well as the continuity property of g.

Remark 2.11. Our results generalize several fixed point results in the existing literature. For instance, taking $\alpha(x, y) = 1$, we can obtain the main results of Piri and Kumam [13] and Dung and Hang [6] as a corollary of our main results. Most importantly, our results are the generalized versions of the fixed point results given by Gopal et al. [8]. Note that the authors of [8] established the existence of fixed points of α-type F-contractions with the hypothesis: either F or g is continuous function. Our results show that continuity property of F or g is not necessary for the existence of fixed points of such type mappings.

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