APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 20, no. 1 (2019), 75-79
doi:10.4995/agt.2019.9817
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On monotonous separately continuous functions

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Communicated by O. Valero

Abstract

Let $\mathbb{T} = (\mathbf{T}, \leq)$ and $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$ be linearly ordered sets and X be a topological space. The main result of the paper is the following:
If function $\boldsymbol{f}(t,x) : \mathbf{T} \times X \to \mathbf{T}_1$ is continuous in each variable ("t" and "x") separately and function $\boldsymbol{f}_x(t) = \boldsymbol{f}(t,x)$ is monotonous on $\mathbf{T}$ for every $x \in X$, then $\boldsymbol{f}$ is continuous mapping from $\mathbf{T} \times X$ to $\mathbf{T}_1$, where $\mathbf{T}$ and $\mathbf{T}_1$ are considered as topological spaces under the order topology and $\mathbf{T} \times X$ is considered as topological space under the Tychonoff topology on the Cartesian product of topological spaces $\mathbf{T}$ and X.

2010 MSC: *54C05.*

Keywords: *separately continuous mappings; linearly ordered topological spaces; Young's theorem.*

1. Introduction

In 1910 W.H. Young had proved the following theorem.

Theorem A (see [9]). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be separately continuous. If $f(\cdot, y)$ is also monotonous for every y, then f is continuous.*

In 1969 this theorem was generalized for the case of separately continuous function $f : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 2)$:

Theorem B (see [5]). *Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ $(d \in \mathbb{N})$ be continuous in each variable separately. Suppose $f(t_1, \ldots, t_d, \tau)$ is monotonous in each t_i separately $(1 \leq i \leq d)$. Then f is continuous on $\mathbb{R}^{d+1}$.*

Note that theorems A and B were also mentioned in the overview [2]. In the papers [6, 7] authors investigated functions of kind $\boldsymbol{f} : \mathbf{T} \times X \to \mathbb{R}$, where

Received 12 March 2018 – Accepted 10 September 2018

$(\mathbf{T}, \leq)$ is linearly ordered set equipped by the order topology, (X, τ_X) is any topological space and the function $\boldsymbol{f}$ is monotonous relatively to the first variable as well continuous (or quasi-continuous) relatively to the second variable. In particular in [7] it was proven that each separately quasi-continuous and monotonous relatively to the first variable function $\boldsymbol{f} : \mathbb{R} \times X \to \mathbb{R}$ is quasi-continuous relatively to the set of variables. The last result may be considered as the abstract analog of Young's theorem (Theorem A) for separately quasi-continuous functions.

However, we do not know any direct generalization of Theorem A (for separately continuous and monotonous relatively to the first variable function) in abstract topological spaces at the present time. In the present paper we prove the generalization of theorems A and B for the case of (separately continuous and monotonous relatively to the first variable) function $\boldsymbol{f} : \mathbf{T} \times X \to \mathbf{T}_1$, where $(\mathbf{T}, \leq)$, $(\mathbf{T}_1, \leq_1)$ are linearly ordered sets equipped by the order topology and X is any topological space.

2. Preliminaries

Let $\mathbb{T} = (\mathbf{T}, \leq)$ be any linearly (ie totally) ordered set (in the sense of [1]). Then we can define the strict linear order relation on $\mathbf{T}$ such that for any $t, \tau \in \mathbf{T}$ the correlation $t < \tau$ holds if and only if $t \leq \tau$ and $t \neq \tau$. Together with the linearly ordered set $\mathbb{T}$ we introduce the linearly ordered set

$$\mathbb{T}_{\pm\infty} = (\mathbf{T} \cup \{-\infty, +\infty\}, \leq),$$

where the order relation is extended on the set $\mathbf{T} \cup \{-\infty, +\infty\}$ by means of the following clear conventions:

(a): $-\infty < +\infty$;
(b): $(\forall t \in \mathbf{T}) \ \ (-\infty < t < +\infty)$.

Recall [1] that every such linearly ordered set $\mathbb{T} = (\mathbf{T}, \leq)$ can be equipped by the natural "internal" order topology $\mathfrak{Tpi}[\mathbb{T}]$, generated by the base consisting of the open sets of kind:

$$(\tau_1, \tau_2) = \{t \in \mathbf{T} \mid \tau_1 < t < \tau_2\}, \quad \text{where} \quad \tau_1, \tau_2 \in \mathbf{T} \cup \{-\infty, +\infty\}, \ \tau_1 < \tau_2. \tag{2.1}$$

Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be topological spaces. By $\mathbf{C}(X, Y)$ we denote the collection of all continuous mappings from X to Y. For a mapping $\boldsymbol{f} : X \times Y \to Z$ and a point $(x, y) \in X \times Y$ we write

$$\boldsymbol{f}^x(y) := \boldsymbol{f}_y(x) := \boldsymbol{f}(x, y).$$

Recall [3] that the mapping $\boldsymbol{f} : X \times Y \to Z$ is refereed to as ***separately continuous*** if and only if $\boldsymbol{f}^x \in \mathbf{C}(Y, Z)$ and $\boldsymbol{f}_y \in \mathbf{C}(X, Z)$ for every point $(x, y) \in X \times Y$ (see also [6–8]). The set of all separately continuous mappings $\boldsymbol{f} : X \times Y \to Z$ is denoted by $\mathbf{CC}(X \times Y, Z)$ [3, 6–8].

Let $\mathbb{T} = (\mathbf{T}, \leq)$ and $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$ be linearly ordered sets. We say that a function $f : \mathbf{T} \to \mathbf{T}_1$ is ***non-decreasing*** (***non-increasing***) on $\mathbf{T}$ if and only if for every $t, \tau \in \mathbf{T}$ the inequality $t \leq \tau$ leads to the inequality $f(t) \leq_1 f(\tau)$

$(f(\tau) \leq_1 f(t))$ correspondingly. Function $f : \mathbf{T} \to \mathbf{T}_1$, which is non-decreasing or non-increasing on $\mathbf{T}$ is called by ***monotonous***.

3. Main Results

Let $(X_1, \tau_{X_1}),\ \ldots,\ (X_d, \tau_{X_d})\ \ (d \in \mathbb{N})$ be topological spaces. Further we consider $X_1 \times \cdots \times X_d$ as a topological space under the Tychonoff topology $\tau_{X_1 \times \cdots \times X_d}$ on the Cartesian product of topological spaces $X_1, \ldots, X_d$. Recall [4, Chapter 3] that topology $\tau_{X_1 \times \cdots \times X_d}$ is generated by the base of kind:

$$\left\{ U_1 \times \cdots \times U_d \,|\, (\forall j \in \{1, \ldots, d\}) \left(U_j \in \tau_{X_j} \right) \right\}.$$

Theorem 3.1. *Let $\mathbb{T} = (\mathbf{T}, \leq)$ and $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$ be linearly ordered sets and (X, τ_X) be a topological space.*

If $\boldsymbol{f} \in \mathbf{CC}(\mathbf{T} \times X, \mathbf{T}_1)$ and function $\boldsymbol{f}_x(t) = \boldsymbol{f}(t, x)$ is monotonous on $\mathbf{T}$ for every $x \in X$, then $\boldsymbol{f}$ is continuous mapping from the topological space $(\mathbf{T} \times X, \tau_{\mathbf{T} \times X})$ to the topological space $(\mathbf{T}_1, \mathfrak{Tpi}[\mathbb{T}_1])$.

Proof. Consider any ordered pair $(t_0, x_0) \in \mathbf{T} \times X$. Take any open set $V \subseteq \mathbf{T}_1$ such that $\boldsymbol{f}(t_0, x_0) \in V$. Since the sets of kind (2.1) form the base of topology $\mathfrak{Tpi}[\mathbb{T}_1]$, there exist $\tau_1, \tau_2 \in \mathbf{T}_1 \cup \{-\infty, +\infty\}$ such that $\tau_1 <_1 \boldsymbol{f}(t_0, x_0) <_1 \tau_2$ and $(\tau_1, \tau_2) \subseteq V$, where $<_1$ is the strict linear order, generated by (non-strict) order $\leq_1$ (on $\mathbf{T}_1 \cup \{-\infty, +\infty\}$). The function $\boldsymbol{f}$ is separately continuous. So, since the sets of kind (2.1) form the base of topology $\mathfrak{Tpi}[\mathbb{T}]$, there exist $t_1, t_2 \in \mathbf{T} \cup \{-\infty, +\infty\}$ such that

$$t_1 < t_0 < t_2 \quad \text{and} \tag{3.1}$$

$$\boldsymbol{f}\left[(t_1, t_2) \times \{x_0\}\right] \subseteq (\tau_1, \tau_2). \tag{3.2}$$

Further we need the some additional denotations.

- In the case, where $(t_1, t_0) \neq \varnothing$ we choose any element $\alpha_1 \in \mathbf{T}$ such that $t_1 < \alpha_1 < t_0$ and denote $\widetilde{\alpha}_1 := \alpha_1$. In the opposite case we denote $\alpha_1 := t_0$, $\widetilde{\alpha}_1 := t_1$.
- In the case $(t_0, t_2) \neq \varnothing$ we choose any element $\alpha_2 \in \mathbf{T}$ such that $t_0 < \alpha_2 < t_2$ and denote $\widetilde{\alpha}_2 := \alpha_2$. In the opposite case we denote $\alpha_2 := t_0$, $\widetilde{\alpha}_2 := t_2$.

It is not hard to verify, that in the all cases the following conditions are performed:

$$\alpha_1, \alpha_2 \in \mathbf{T}, \quad \widetilde{\alpha}_1, \widetilde{\alpha}_2 \in \mathbf{T} \cup \{-\infty, +\infty\};$$

$$\alpha_1 \leq \alpha_2;$$

$$\widetilde{\alpha}_1 < \widetilde{\alpha}_2;$$

$$[\alpha_1, \alpha_2] \subseteq (t_1, t_2), \quad \text{where } [\alpha_1, \alpha_2] = \{t \in \mathbf{T} \,|\, \alpha_1 \leq t \leq \alpha_2\}; \tag{3.3}$$

$$t_0 \in (\widetilde{\alpha}_1, \widetilde{\alpha}_2) \subseteq [\alpha_1, \alpha_2]. \tag{3.4}$$

According to (3.3), $\alpha_1, \alpha_2 \in (t_1, t_2)$. Hence, according to (3.2), interval (τ_1, τ_2) is an open neighborhood of the both points $\boldsymbol{f}(\alpha_1, x_0)$ and $\boldsymbol{f}(\alpha_2, x_0)$.

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Since the function $\boldsymbol{f}$ is separately continuous on $\mathbf{T} \times X$, then there exist an open neighborhood $U \in \tau_X$ of the point x_0 (in the space X) such that:

$$\boldsymbol{f}\left[\{\alpha_1\} \times U\right] \subseteq (\tau_1, \tau_2)\,; \tag{3.5}$$

$$\boldsymbol{f}\left[\{\alpha_2\} \times U\right] \subseteq (\tau_1, \tau_2)\,. \tag{3.6}$$

The set $(\widetilde{\alpha}_1, \widetilde{\alpha}_2) \times U$ is an open neighborhood of the point (t_0, x_0) in the topology $\tau_{\mathbf{T}\times X}$ of the space $\mathbf{T} \times X$. Now our aim is to prove that

$$\forall\ (t, x) \in (\widetilde{\alpha}_1, \widetilde{\alpha}_2) \times U\ \ (\boldsymbol{f}\,(t, x) \in (\tau_1, \tau_2) \subseteq V)\,. \tag{3.7}$$

So, chose any point $(t, x) \in (\widetilde{\alpha}_1, \widetilde{\alpha}_2) \times U$. According to the condition (3.4), we have $(t, x) \in [\alpha_1, \alpha_2] \times U$, that is $\alpha_1 \leq t \leq \alpha_2$ and $x \in U$. In accordance with (3.5), (3.6), we have $\boldsymbol{f}\,(\alpha_1, x) \in (\tau_1, \tau_2)$ and $\boldsymbol{f}\,(\alpha_2, x) \in (\tau_1, \tau_2)$. Hence, since the function $\boldsymbol{f}_x(\cdot) = \boldsymbol{f}(\cdot, x)$ is monotonous on $\mathbf{T}$ and $\alpha_1 \leq t \leq \alpha_2$, we deduce $\boldsymbol{f}\,(t, x) \in (\tau_1, \tau_2) \subseteq V$. Thus, the correlation (3.7) is proven. Hence, the function $\boldsymbol{f}$ is continuous in (every) point $(t_0, x_0) \in \mathbf{T} \times X$. □

Theorem A is a consequence of Theorem 3.1 in the case $\mathbf{T} = X = \mathbb{R}$, where $\mathbb{R}$ is considered together with the usual linear order on the field of real numbers and usual topology.

Corollary 3.2. *Let $\mathbb{T}_0 = (\mathbf{T}_0, \leq_0)$, $\mathbb{T}_1 = (\mathbf{T}_1, \leq_1)$, …, $\mathbb{T}_d = (\mathbf{T}_d, \leq_d)$ ($d \in \mathbb{N}$) be linearly ordered sets, and (X, τ_X) be a topological space.*

If the function $\boldsymbol{f} : \mathbf{T}_1 \times \cdots \times \mathbf{T}_d \times X \to \mathbf{T}_0$ is continuous in each variable separately and $f\,(t_1, \ldots, t_d, \tau)$ is monotonous in each t_i separately ($1 \leq i \leq d$) then $\boldsymbol{f}$ is a continuous mapping from the topological space $(\mathbf{T}_1 \times \cdots \times \mathbf{T}_d \times X,$ $\tau_{\mathbf{T}_1\times\cdots\times\mathbf{T}_d\times X})$ to the topological space $(\mathbf{T}_0, \mathfrak{Tpi}\,[\mathbb{T}_0])$.

Proof. We will prove this corollary by induction. For $d = 1$ the corollary is true by Theorem 3.1. Assume, that the corollary is true for the number $d - 1$, where $d \in \mathbb{N}$, $d \geq 2$. Suppose, that function $\boldsymbol{f} : \mathbf{T}_1 \times \cdots \times \mathbf{T}_d \times X \to \mathbf{T}_0$ is satisfying the conditions of the corollary. Then we may consider this function as a mapping from $\mathbf{T}_1 \times X_{(d)}$ to $\mathbf{T}_0$, where $X_{(d)} = \mathbf{T}_2 \times \cdots \times \mathbf{T}_d \times X$. According to inductive hypothesis, function $\boldsymbol{f}\,(t_1, \cdot)$ is continuous on $X_{(d)}$ for every fixed $t_1 \in \mathbf{T}_1$. So $\boldsymbol{f}$ is a separately continuous mapping from $\mathbf{T}_1 \times X_{(d)}$ to $\mathbf{T}_0$. Moreover, $\boldsymbol{f}$ is monotonous relatively to the first variable (by conditions of the corollary). Hence, by Theorem 3.1, $\boldsymbol{f}$ is continuous on $\mathbf{T}_1 \times X_{(d)}$. □

Theorem B is a consequence of Corollary 3.2 in the case $\mathbf{T}_0 = \mathbf{T}_1 = \cdots = \mathbf{T}_d = X = \mathbb{R}$, where $\mathbb{R}$ is considered together with the usual linear order on the field of real numbers and usual topology. In the case $\mathbf{T}_0 = \mathbb{R}$, $\mathbf{T}_j = (a_j, b_j)$, $X = (a_{d+1}, b_{d+1})$ where $a_j, b_j \in \mathbb{R}$ and $a_j < b_j$ ($j \in \{1, \ldots, d + 1\}$) and intervals (a_j, b_j) are considered together with the usual linear order and topology, induced from the field of real numbers, we obtain the following corollary.

Corollary 3.3. *If the function $f : (a_1, b_1) \times \cdots \times (a_d, b_d) \times (a_{d+1}, b_{d+1}) \to \mathbb{R}$ ($d \in \mathbb{N}$) is continuous in each variable separately and $f\,(t_1, \ldots, t_d, \tau)$ is monotonous in each t_i separately ($1 \leq i \leq d$) then f is a continuous mapping from $(a_1, b_1) \times \cdots \times (a_{d+1}, b_{d+1})$ to $\mathbb{R}$.*

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Remark 3.4. In fact in the paper [5] the more general result was formulated, in comparison with Theorem B. Namely the author of [5] had considered the real valued function $f(t_1, \dots, t_d, \tau)$ defined on an open set $G \subseteq \mathbb{R}^{d+1}$, $d \in \mathbb{N}$ such that f is continuous in each variable separately and monotonous in each t_i separately ($1 \le i \le d$). But this result of [5] can be delivered from Corollary 3.3, because for each point $\mathbf{t} = (t_1, \dots, t_d, \tau) \in G$ in the open set G there exists the set of intervals $(a_1, b_1), \dots, (a_{d+1}, b_{d+1})$ such that $\mathbf{t} \in (a_1, b_1) \times \dots \times (a_{d+1}, b_{d+1}) \subseteq G$.

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