

τ -metrizable spaces

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ABSTRACT

In [1], A. A. Borubaev introduced the concept of τ -metric space, where τ is an arbitrary cardinal number. The class of τ -metric spaces as τ runs through the cardinal numbers contains all ordinary metric spaces (for $\tau = 1$) and thus these spaces are a generalization of metric spaces. In this paper the notion of τ -metrizable space is considered.

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1. PRELIMINARIES AND NOTATIONS

Our notation and terminology is standard and generally follows [2]. The cardinality of a set X is denoted by $|X|$. Throughout, we denote by τ an arbitrary nonzero cardinal number. The cardinalities of the natural numbers and of the real numbers are denoted by \aleph_0 and \mathfrak{c} , respectively. The character, the weight and the density of a topological space X are denoted by $\chi(X)$, $w(X)$ and $d(X)$, respectively. As usual I denotes the closed unit interval $[0, 1]$ with the Euclidean metric topology.

By \mathbb{R}_+^τ we denote the topological product of τ copies of the space $\mathbb{R}_+ = [0, +\infty)$ (with the natural topology). On the space \mathbb{R}_+^τ , the operations of addition, multiplication, and multiplication by a scalar, as well as a partial ordering, are defined in a natural way (coordinatewise).

Now, we present the notion of τ -metric space [1]. Let X be a nonempty set. A mapping $\rho_\tau : X \times X \rightarrow \mathbb{R}_+^\tau$ is called a τ -metric on X if the following axioms hold:

- (1) $\rho_\tau(x, y) = \theta$ if and only if $x = y$, where θ is the point of the space \mathbb{R}_+^τ whose all coordinates are zeros.
- (2) $\rho_\tau(x, y) = \rho_\tau(y, x)$ for all $x, y \in X$.
- (3) $\rho_\tau(x, z) \leq \rho_\tau(x, y) + \rho_\tau(y, z)$ for all $x, y, z \in X$.

The pair (X, ρ_τ) is called a τ -metric space and the elements of X are called points.

Every τ -metric space (X, ρ_τ) generates a Tychonoff (that is, completely regular and Hausdorff) topological space (X, T_{ρ_τ}) . The topology T_{ρ_τ} on X defined by the local basis consisting of the sets of the form

$$G(x) = \{y \in X : \rho_\tau(x, y) \in O(\theta)\},$$

where $O(\theta)$ runs through all open neighbourhoods of the point θ in the space \mathbb{R}_+^τ , of each point $x \in X$ is called the topology induced by the τ -metric ρ_τ .

In this paper the notion of τ -metrizable space is introduced. The paper is organized as follows. Section 2 contains the basic concepts of τ -metrizable spaces. Generally, τ -metrizable spaces may be not metrizable. We prove that if $\tau \leq \aleph_0$, then every τ -metrizable space is metrizable. In section 3 we obtain a generalization of the classical metrization theorem of Urysohn. More precisely, we prove that every Tychonoff space of weight $\tau > \aleph_0$ is τ -metrizable. Finally, in section 4 we prove that every compact τ -metrizable space has density less than or equal to τ .

2. BASIC CONCEPTS

The notion of a τ -metric space leads to the notion of a τ -metrizable space which is inserted in the following definition.

Definition 2.1. A topological space (X, T) is called τ -metrizable if there exists a τ -metric ρ_τ on the set X such that the topology T_{ρ_τ} induced by the τ -metric ρ_τ coincides with the original topology T of X . τ -metrics on the set X which induce the original topology of X will be called τ -metrics on the space X .

Note that τ -metrizable spaces are useful because only such spaces can be presented as limits of τ -long projective systems of metric spaces [1, Theorem 3].

Proposition 2.2. *A metric space is τ -metrizable.*

Proof. Let (X, ρ) be a metric space, t_ρ be the topology induced by the metric ρ , and let τ be a cardinal number. Consider a set Λ such that $|\Lambda| = \tau$ and set $\rho_\lambda = \rho$ for each $\lambda \in \Lambda$. The mapping $\rho_\tau : X \times X \rightarrow \mathbb{R}_+^\tau$ defined by $\rho_\tau(x, y) = \{\rho_\lambda(x, y)\}_{\lambda \in \Lambda}$ for every $x, y \in X$ is a τ -metric on X . It is easy to see that $t_\rho = T_{\rho_\tau}$. □

Proposition 2.3. *A τ -metrizable space is τ' -metrizable for every cardinal number $\tau' > \tau$.*

Proof. Let X be a τ -metrizable space, ρ_τ be a τ -metric on the space X and τ' be a cardinal number such that $\tau' > \tau$. Consider two sets K and Λ such that $K \subset \Lambda$, $|K| = \tau$ and $|\Lambda| = \tau'$, and set $\rho_\tau(x, y) = \{\rho_\tau^k(x, y)\}_{k \in K}$ for every $x, y \in X$. Let k_0 be one fixed element of K . The mapping $\rho_{\tau'} : X \times X \rightarrow \mathbb{R}_+^{\tau'}$ defined by $\rho_{\tau'}(x, y) = \{\rho_{\tau'}^\lambda(x, y)\}_{\lambda \in \Lambda}$ for every $x, y \in X$, where

$$\rho_{\tau'}^\lambda(x, y) = \begin{cases} \rho_\tau^\lambda(x, y), & \text{if } \lambda \in K \\ \rho_\tau^{k_0}(x, y), & \text{if } \lambda \in \Lambda \setminus K, \end{cases}$$

is a τ' -metric on X such that $T_{\rho_{\tau'}} = T_{\rho_\tau}$. □

The following examples show that τ -metrizable spaces may be not metrizable.

Example 2.4. The product $\mathbb{R}^c = \prod_{\lambda \in \Lambda} X_\lambda$, where $X_\lambda = \mathbb{R}$ for every $\lambda \in \Lambda$ and $|\Lambda| = \mathfrak{c}$, of uncountably many copies of the real line \mathbb{R} is not metrizable, since it is not first-countable. However, the space \mathbb{R}^c is \mathfrak{c} -metrizable. Assuming each copy X_λ of \mathbb{R} has its usual metric d_λ , the mapping $\rho_c : \mathbb{R}^c \times \mathbb{R}^c \rightarrow \mathbb{R}_+^c$ defined by $\rho_c(x, y) = \{d_\lambda(x_\lambda, y_\lambda)\}_{\lambda \in \Lambda}$ for every $x = \{x_\lambda\}_{\lambda \in \Lambda} \in \mathbb{R}^c$ and $y = \{y_\lambda\}_{\lambda \in \Lambda} \in \mathbb{R}^c$ is a \mathfrak{c} -metric on \mathbb{R}^c and the topology induced by ρ_c coincides with the product topology.

Example 2.5. Let \mathbb{R} be the set of real numbers with the discrete topology \mathcal{D} and $(\mathbb{R}_\infty, \mathcal{D}_\infty)$ be the Alexandroff's one-point compactification of the space $(\mathbb{R}, \mathcal{D})$, that is $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$ and $\mathcal{D}_\infty = \mathcal{D} \cup \{\mathbb{R}_\infty \setminus K : K \text{ is a finite subset of } \mathbb{R}\}$. The space $(\mathbb{R}_\infty, \mathcal{D}_\infty)$ is not metrizable (because it is not separable). We prove that the space $(\mathbb{R}_\infty, \mathcal{D}_\infty)$ is \mathfrak{c} -metrizable. Let $Fin(\mathbb{R})$ be the collection of all the nonempty finite subsets of \mathbb{R} with $|Fin(\mathbb{R})| = \mathfrak{c}$. For every $F \in Fin(\mathbb{R})$ we define:

- (1) $\rho_F(x, x) = 0$ for each $x \in \mathbb{R}_\infty$.
- (2) $\rho_F(x, \infty) = \rho_F(\infty, x) = \begin{cases} 0, & \text{if } x \notin F \\ 1, & \text{otherwise} \end{cases}$ for each $x \in \mathbb{R}$.
- (3) $\rho_F(x, y) = \begin{cases} 0, & \text{if } x \notin F \text{ and } y \notin F \\ 1, & \text{otherwise} \end{cases}$ for each $x, y \in \mathbb{R}$ with $x \neq y$.

The mapping $\rho_c : \mathbb{R}_\infty \times \mathbb{R}_\infty \rightarrow \mathbb{R}_+^c$ defined by $\rho_c(x, y) = \{\rho_F(x, y)\}_{F \in Fin(\mathbb{R})}$ for every $x, y \in \mathbb{R}_\infty$ is a \mathfrak{c} -metric on \mathbb{R}_∞ . We prove that the topology T_{ρ_c} induced by the \mathfrak{c} -metric ρ_c coincides with the topology \mathcal{D}_∞ .

Let $x \in \mathbb{R}$. If $G(x) = \{y \in \mathbb{R}_\infty : \rho_c(x, y) \in O(\theta)\}$, where $O(\theta)$ is an open neighbourhood of the point θ in the space \mathbb{R}_+^c , then $\{x\} \in \mathcal{D}_\infty$ and $\{x\} \subseteq G(x)$. Moreover, for the open neighbourhood $\prod_{F \in Fin(\mathbb{R})} W_F$ of the point θ , where

$$W_F = \begin{cases} [0, \frac{1}{2}), & \text{if } F = \{x\} \\ \mathbb{R}_+, & \text{otherwise} \end{cases}$$

we have $G(x) = \{y \in \mathbb{R}_\infty : \rho_c(x, y) \in \prod_{F \in Fin(\mathbb{R})} W_F\} \subseteq \{x\}$.

Now, we consider the point ∞ of \mathbb{R}_∞ . If $\{\infty\} \cup (\mathbb{R} \setminus K)$, where $K \in Fin(\mathbb{R})$ is an open neighbourhood of the point ∞ in the space \mathbb{R}_∞ , then for the open neighbourhood $\prod_{F \in Fin(\mathbb{R})} W_F$ of the point θ , where

$$W_F = \begin{cases} [0, \frac{1}{2}), & \text{if } F = K \\ \mathbb{R}_+, & \text{otherwise} \end{cases}$$

we have $G(\infty) = \{y \in \mathbb{R}_\infty : \rho_c(\infty, y) \in \prod_{F \in Fin(\mathbb{R})} W_F\} \subseteq \{\infty\} \cup (\mathbb{R} \setminus K)$. Finally, let $\prod_{F \in Fin(\mathbb{R})} U_F$ be an open neighbourhood of the point θ in the space \mathbb{R}_+^c and suppose that $\{F \in Fin(\mathbb{R}) : U_F \neq \mathbb{R}_+\} = \{K_1, \dots, K_m\}$. Then,

$$\{\infty\} \cup (\mathbb{R} \setminus (K_1 \cup \dots \cup K_m)) \subseteq G(\infty) = \{y \in \mathbb{R}_\infty : \rho_c(\infty, y) \in \prod_{F \in Fin(\mathbb{R})} U_F\}.$$

However, a τ -metrizable space may be metrizable considering addition conditions as the following assertions show.

Proposition 2.6. *A n -metric space is metrizable for every finite cardinal number n .*

Proof. Let (X, ρ_n) be a n -metric space and T_{ρ_n} be the topology induced by ρ_n . Consider a vector expression of the form $\rho_n(x, y) = (\rho_n^1(x, y), \dots, \rho_n^n(x, y))$ for every $x, y \in X$. The mapping $\rho : X \times X \rightarrow \mathbb{R}_+$ defined by

$$\rho(x, y) = \max\{\rho_n^1(x, y), \dots, \rho_n^n(x, y)\}$$

for every $x, y \in X$ is a metric on X . It is easy to see that the metric topology is the same as T_{ρ_n} . □

Definition 2.7. Two τ -metrics $\rho_{1\tau}$ and $\rho_{2\tau}$ on a set X are called equivalent if they induce the same topology on X , that is $T_{\rho_{1\tau}} = T_{\rho_{2\tau}}$.

Example 2.8. Let ρ_τ be a τ -metric on X . Consider a set Λ such that $|\Lambda| = \tau$ and let us set $\rho_\tau(x, y) = \{\rho_\tau^\lambda(x, y)\}_{\lambda \in \Lambda}$ for every $x, y \in X$. The mapping $\rho_\tau^* : X \times X \rightarrow \mathbb{R}_+^\tau$ defined by $\rho_\tau^*(x, y) = \{\min\{1, \rho_\tau^\lambda(x, y)\}\}_{\lambda \in \Lambda}$ for every $x, y \in X$ is a τ -metric on X equivalent to ρ_τ .

Proposition 2.9. *An \aleph_0 -metric space is metrizable.*

Proof. Let (X, ρ_{\aleph_0}) be an \aleph_0 -metric space. Consider the equivalent \aleph_0 -metric $\rho_{\aleph_0}^*$ to ρ_{\aleph_0} of Example 2.8. Let $\rho_{\aleph_0}^*(x, y) = (\rho_{\aleph_0}^{*1}(x, y), \rho_{\aleph_0}^{*2}(x, y), \dots)$ for every $x, y \in X$. The mapping $\rho : X \times X \rightarrow \mathbb{R}_+$ defined by

$$\rho(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \rho_{\aleph_0}^{*i}(x, y)$$

for every $x, y \in X$ is a metric on X . The process of proving that the topology induced by the metric ρ coincides with the topology $T_{\rho_{\aleph_0}}$ is similar to the proof of the Theorem 4.2.2 of [2]. □

Corollary 2.10. *If $\tau \leq \aleph_0$, then every τ -metrizable space is metrizable.*

Proof. Follows directly from Propositions 2.6 and 2.9. □

Proposition 2.11. *For each $\tau > \aleph_0$ there is a τ -metrizable space X_τ with $w(X_\tau) = \tau$, which is not metrizable.*

Proof. Let X_τ be the Alexandroff's one-point compactification of a discrete space X of cardinality τ , where $\tau > \aleph_0$. The space X_τ is not metrizable (because it is not separable). It is known that $|Fin(X)| = |X| = \tau$. Therefore, in the same manner as in Example 2.5, we can prove that the space X_τ is τ -metrizable. Let us note that $w(X_\tau) = \tau$. □

Proposition 2.12. *For every $\tau \geq \aleph_0$ and every τ -metrizable space X , we have $\chi(X) \leq \tau$.*

Proof. Let X be a τ -metrizable space and ρ_τ be a τ -metric on the space X with $\tau \geq \aleph_0$. Consider a set Λ such that $|\Lambda| = \tau$. The family \mathcal{B}_θ of all products $\prod_{\lambda \in \Lambda} W_\lambda$, where finitely many W_λ are intervals of the form $[0, b)$ with rational b and the remaining $W_\lambda = \mathbb{R}_+$, form a local basis of the point θ in the space \mathbb{R}_+^τ . Hence, for every $x \in X$, the family

$$\mathcal{B}(x) = \{G(x) = \{y \in X : \rho_\tau(x, y) \in B\} : B \in \mathcal{B}_\theta\}$$

is a local basis of the point x in the space X . Since $|\mathcal{B}_\theta| = \tau$, we have $|\mathcal{B}(x)| \leq \tau$. □

3. A τ -METRIZATION THEOREM

Metrization theorems are theorems that give sufficient conditions for a topological space to be metrizable (see [2, 5]). In this section we obtain a generalization of the classical metrization theorem of Urysohn.

Lemma 3.1. *If (X, ρ_τ) is a τ -metric space and A is a subspace of X , then the topology induced by the restriction of the τ -metric ρ_τ to $A \times A$ is the same as the subspace topology of A in X .*

Theorem 3.2. *Every Tychonoff space of weight $\tau > \aleph_0$ is τ -metrizable.*

Proof. Let X be a Tychonoff space such that $w(X) = \tau > \aleph_0$. The space $I^\tau = \prod_{\lambda \in \Lambda} X_\lambda$, where $X_\lambda = I$ for every $\lambda \in \Lambda$ and $|\Lambda| = \tau$ is τ -metrizable (see Example 2.4). Assuming each copy X_λ of I has its usual metric d_λ , the mapping $d_\tau : I^\tau \times I^\tau \rightarrow \mathbb{R}_+^\tau$ defined by $d_\tau(x, y) = \{d_\lambda(x_\lambda, y_\lambda)\}_{\lambda \in \Lambda}$ for every $x = \{x_\lambda\}_{\lambda \in \Lambda} \in I^\tau$ and $y = \{y_\lambda\}_{\lambda \in \Lambda} \in I^\tau$ is a τ -metric on I^τ . We shall prove that X is τ -metrizable by imbedding X into the τ -metrizable space I^τ , i.e. by showing that X is homeomorphic with a subspace of I^τ . But this follows immediately from the fact that the Tychonoff cube I^τ is universal for all Tychonoff spaces of weight τ (see [2, Theorem 2.3.23]). By Lemma 3.1, the space X is τ -metrizable. □

As every τ -metrizable space is Tychonoff (see [1]), we get the following result.

Corollary 3.3. *A space of weight $\tau > \aleph_0$ is τ -metrizable if and only if it is Tychonoff.*

Remark 3.4. We can use Theorem 3.2 to find τ -metrizable spaces, where $\tau > \aleph_0$, that are not metrizable. Below we consider some examples. Example 3.5 is a \mathfrak{c} -metrizable space which is not second-countable, Example 3.6 is a \mathfrak{c} -metrizable space which is not normal and Example 3.7 is a 2^τ -metrizable space, where $\tau \geq \mathfrak{c}$, which is not metrizable.

Example 3.5. Let S be the Sorgenfrey line, that is the real line with the topology in which local basis of x are the sets $[x, y)$ for $y > x$. Since X is separable but not second-countable, it cannot be metrizable. Furthermore, S is Tychonoff and $w(S) = \mathfrak{c}$. From Theorem 3.2 it follows that the Sorgenfrey line is a \mathfrak{c} -metrizable space.

Example 3.6. Let $P = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta > 0\}$ be the open upper half-plane with the Euclidean topology and $L = \{(\alpha, \beta) \in \mathbb{R}^2 : \beta = 0\}$. We set $X = P \cup L$. For every $x \in P$ let $B(x)$ be the family of all open discs in P centered at x . For every $x \in L$ let $B(x)$ be the family of all sets of the form $\{x\} \cup D$, where D is an open disc in P which is tangent to L at the point x . The family T of all subsets of X that are unions of subfamilies of $\cup\{B(x) : x \in X\}$ is a topology on X and the family $\{B(x) : x \in X\}$ is a neighbourhood system for the topological space (X, T) . The space X is called the Niemytzki plane (see, for example, [2, 4]). X is a Tychonoff space with $w(X) = \mathfrak{c}$, which is not normal. Therefore, by Theorem 3.2, X is a \mathfrak{c} -metrizable space, but not metrizable.

Example 3.7. Let $\beta D(\tau)$ be the Čech-Stone compactification of the discrete space $D(\tau)$ of cardinality $\tau \geq \mathfrak{c}$. Then, $w(\beta D(\tau)) = 2^\tau$ (see [2, Theorem 3.6.11]). Since $\beta D(\tau)$ is zero-dimensional (see [2, Theorem 3.6.13]), it is Tychonoff. The space $D(\tau)$ is not compact. Therefore, $\beta D(\tau)$ is not metrizable (see [3, Exercise 9, §38, Ch.5]). From Theorem 3.2 it follows that $\beta D(\tau)$ is 2^τ -metrizable. Particularly, if one assumes the continuum hypothesis, the Čech-Stone compactification $\beta\omega$ of the discrete space of the non-negative integers $\omega = \{0, 1, 2, \dots\}$ is \mathfrak{c} -metrizable.

Remark 3.8. A space X may be τ -metrizable for some infinite cardinal number $\tau < w(X)$, as shown in the following example.

Example 3.9. Let Λ be a set of cardinality $\tau > \aleph_0$, $D(\kappa)$ the discrete space of cardinality $\kappa > \tau$, and $F = \prod_{\lambda \in \Lambda} X_\lambda$, where $X_\lambda = D(\kappa)$ for every $\lambda \in \Lambda$, with the Tychonoff product topology. We note that the points of F are functions from Λ to $D(\kappa)$. The space F is not metrizable for $\chi(F) = \tau$ (see [2, Exercise 2.3.F(b)]). Moreover, $w(F) = \kappa$ (see [2, Exercise 2.3.F(a)]). We prove that the space F is τ -metrizable. For every $\lambda \in \Lambda$ we define:

- (1) $\rho_\lambda(f, f) = 0$ for each $f \in F$.
- (2) $\rho_\lambda(f, g) = \begin{cases} 0, & \text{if } f(\lambda) = g(\lambda) \\ 1, & \text{otherwise} \end{cases}$ for each $f, g \in F$ with $f \neq g$.

The mapping $\rho_\tau : F \times F \rightarrow \mathbb{R}_+^\tau$ defined by $\rho_\tau(f, g) = \{\rho_\lambda(f, g)\}_{\lambda \in \Lambda}$ for every $f, g \in F$ is a τ -metric on F . We prove that the topology T_{ρ_τ} induced by the τ -metric ρ_τ coincides with the Tychonoff product topology.

Let $f \in F$, $\prod_{\lambda \in \Lambda} U_\lambda$ be an open neighbourhood of the point θ in the space \mathbb{R}_+^τ , and suppose that $\{\lambda \in \Lambda : U_\lambda \neq \mathbb{R}_+\} = \{\lambda_1, \dots, \lambda_m\}$. For the open neighbourhood $\prod_{\lambda \in \Lambda} W_\lambda$ of the point f , where

$$W_\lambda = \begin{cases} \{f(\lambda)\}, & \text{if } \lambda \in \{\lambda_1, \dots, \lambda_m\} \\ D(\kappa), & \text{otherwise} \end{cases}$$

we have $\prod_{\lambda \in \Lambda} W_\lambda \subseteq G(f) = \{g \in F : \rho_\tau(f, g) \in \prod_{\lambda \in \Lambda} U_\lambda\}$.

Now, let $f \in F$ and $\prod_{\lambda \in \Lambda} W_\lambda$ be an open neighbourhood of the point f in the space F , and suppose that $\{\lambda \in \Lambda : W_\lambda \neq D(\kappa)\} = \{\lambda_1, \dots, \lambda_m\}$. For the open neighbourhood $\prod_{\lambda \in \Lambda} U_\lambda$ of the point θ , where

$$U_\lambda = \begin{cases} [0, \frac{1}{2}), & \text{if } \lambda \in \{\lambda_1, \dots, \lambda_m\} \\ \mathbb{R}_+, & \text{otherwise} \end{cases}$$

we have $G(f) = \{g \in F : \rho_\tau(f, g) \in \prod_{\lambda \in \Lambda} U_\lambda\} \subseteq \prod_{\lambda \in \Lambda} W_\lambda$.

4. COMPACT τ -METRIZABLE SPACES

It is well known that every compact metrizable space is separable. An analogous result for τ -metrizable spaces is stated in this section.

Let us consider a set Λ such that $|\Lambda| = \tau \geq \aleph_0$ and let \mathcal{B}_ε be the family of all open subsets $\prod_{\lambda \in \Lambda} W_\lambda$ of the product \mathbb{R}_+^τ , where finitely many W_λ are intervals of the form $[0, \varepsilon)$ and the remaining $W_\lambda = \mathbb{R}_+$.

Definition 4.1. Let (X, ρ_τ) be a τ -metric space. A subset A of X is called O_ε -dense in (X, ρ_τ) , where $O_\varepsilon \in \mathcal{B}_\varepsilon$, if for every $x \in X$ there exists $a \in A$ such that $\rho_\tau(x, a) \in O_\varepsilon$.

Definition 4.2. A τ -metric space (X, ρ_τ) is called ε -totally bounded if for every $O_\varepsilon \in \mathcal{B}_\varepsilon$ there exists a finite subset A of X which is O_ε -dense in (X, ρ_τ) . The τ -metric space (X, ρ_τ) is called totally bounded if it is ε -totally bounded for every $\varepsilon > 0$.

Recall that the density $d(X)$ of a topological space X , is defined to be $d(X) = \min\{|D| : D \text{ is a dense subset of } X\}$.

Proposition 4.3. For every totally bounded τ -metric space X , the inequality $d(X) \leq \tau$ holds.

Proof. Let $n \in \{1, 2, \dots\}$. For each $O_{1/n} \in \mathcal{B}_{1/n}$, let $A(O_{1/n})$ be a finite $O_{1/n}$ -dense subset of X and consider the subset $A_n = \cup\{A(O_{1/n}) : O_{1/n} \in \mathcal{B}_{1/n}\}$ of X with $|A_n| \leq \tau$. The subset $A = \cup_{n=1}^\infty A_n$ of X is dense and $|A| \leq \tau$. \square

Proposition 4.4. Every compact τ -metric space X is totally bounded.

Proof. Let $\varepsilon > 0$. For every $O_\varepsilon \in \mathcal{B}_\varepsilon$ the family

$$\{G(x) = \{y \in X : \rho_\tau(x, y) \in O_\varepsilon\} : x \in X\}$$

forms an open cover of X . By compactness of X , there exists a finite subset A of X such that $\bigcup_{a \in A} G(a) = X$. For every $x \in X$ there exists $a \in A$ with

$x \in G(a)$. Therefore, $\rho_\tau(x, a) \in O_\varepsilon$ and the subset A of X is O_ε -dense in (X, ρ_τ) . \square

Theorem 4.5. *For every compact τ -metrizable space X we have $d(X) \leq \tau$.*

Proof. Let X be a compact τ -metrizable space. According to Proposition 4.4, the space X is totally bounded. Therefore, by virtue of Proposition 4.3, $d(X) \leq \tau$. \square

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