APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 20, no. 1 (2019), 19-31
doi:10.4995/agt.2019.8843

A note on measure and expansiveness on uniform spaces

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Communicated by F. Balibrea

Abstract

We prove that the set of points doubly asymptotic to a point has measure zero with respect to any expansive outer regular measure for a bi-measurable map on a separable uniform space. Consequently, we give a class of measures which cannot be expansive for Denjoy homeomorphisms on S^1. We then investigate the existence of expansive measures for maps with various dynamical notions. We further show that measure expansive (strong measure expansive) homeomorphisms with shadowing have periodic (strong periodic) shadowing. We relate local weak specification and periodic shadowing for strong measure expansive systems.

2010 MSC: *54H20; 54E15.*

Keywords: *expansiveness; measure expansiveness; expansive measures; equicontinuity; shadowing; specification.*

1. Introduction

The fact that symbolic flows are expansive has been recognized for several years and has provided the impetus for the study of expansive homeomorphisms in a more general settings beginning with the Utz's paper [17] in the middle of the twentieth century. Evidently, mathematicians used the size of some particular subsets of the phase space to unleash many interesting phenomena of such homeomorphisms. For instance, J. F. Jacobsen [10] proved that there exists no expansive self-homeomorphism of a closed 2-cell by using the fact that the set

Received 29 October 2017 – Accepted 13 January 2019

of points doubly asymptotic to any fixed point of such a homeomorphism of a compact metric space is at most countable. In recent years, A. Arbieto proved [1] that the set of sinks of any homeomorphism with canonical coordinates has measure zero with respect to any positively expansive measure by using the fact that a set on which a measurable map of a separable metric space is Lyapunov stable has measure zero with respect to any positively expansive measure. Further, he proved that the set of heteroclinic points of a homeomorphism on a compact metric space has measure zero with respect to any expansive measure with the help of the fact that the set of points with converging semiorbits under a homeomorphism of a compact metric space has measure zero with respect to any expansive measure. Many of these results have been generalized [15] for expansive measures for maps on non-compact, non-metrizable spaces.

Treading on the same path, in section 3, we conclude some results analogous to that of expansive self-homeomorphisms on compact metric spaces by using the fact that the set of points positively asymptotic to one point and negatively asymptotic to another point has measure zero with respect to any expansive outer regular measure for a bi-measurable map on a separable uniform space. For instance, we give a class of measures which cannot be expansive for Denjoy homeomorphisms on S^1. We also prove that every stable class has measure zero with respect to any positively expansive outer regular measure of a measurable map on a separable uniform space. Then, we prove that the set of equicontinuous homeomorphisms and the set of homeomorphisms admitting expansive measure on a Lindelöf uniform space are disjoint. We prove that sink does not exist for a bi-measurable map having canonical coordinates and strictly positive positively expansive measure. Finally, in section 4, we prove that measure expansive (strong measure expansive) homeomorphisms with shadowing have periodic (strong periodic) shadowing. We further relate local weak specification and periodic shadowing for strong measure expansive systems.

2. Preliminaries

We denote the set of non-negative integers by $\mathbb{N}$ and the set of all integers by $\mathbb{Z}$. A point $x \in X$ is called atom for a measure μ if $\mu(\{x\}) > 0$. A measure μ on X is said to be non-atomic if it has no atom. Let us denote the set of all Borel measures, the set of all non-atomic Borel measures and the set of all strictly positive non-atomic Borel measures by $M(X)$, $NAM(X)$ and $SPNAM(X)$ respectively.

In 2013, A. Arbieto defined [1] the sets $\Phi_\delta(x) = \{y \in X \mid d(f^i(x), f^i(y)) \leq \delta$ for all $i \in \mathbb{N}\}$ and $\Gamma_\delta(x) = \{y \in X \mid d(f^i(x), f^i(y)) \leq \delta$ for all $i \in \mathbb{Z}\}$ for a measurable and bi-measurable map respectively, on a metric space (X, d) to introduce the following concepts.

Definition 2.1. An expansive (positively expansive) measure for a bi-measurable (measurable) map $f : X \to X$ on a metric space (X, d) is a Borel measure μ on X for which there is $\delta > 0$ such that $\mu(\Gamma_\delta(x))$ $(\mu(\Phi_\delta(x))) = 0$ for all $x \in X$. In both the cases, such δ is called an expansive constant for μ.

Observe that a positively expansive measure for a bi-measurable map is also an expansive measure.

In 2015, C. A. Morales generalized [15] these concepts to uniform spaces with the sole difference being the role of spherical neighborhoods now played by uniform neighborhoods. To understand his generalization we need the following terminology.

Let X be a non-empty set. Then the diagonal of $X \times X$ is given by $\Delta(X) = \{(x,x)|x \in X\}$. For a subset R of $X \times X$, we define $R^{-1} = \{(y,x)|(x,y) \in R\}$. We say that R is symmetric if $R = R^{-1}$. For two subsets U and V of $X \times X$, we define their composition as $U \circ V = \{(x,y) \in X \times X|$ there is $z \in X$ satisfying $(x,z) \in U$ and $(z,y) \in V\}$. In this paper, we assume that the phase space of a dynamical system is a uniform space $(X,\mathcal{U})$, where $\mathcal{U}$ is a collection of subsets of $X \times X$ satisfying the following properties ([11],[12]):

(1) Every $D \in \mathcal{U}$ contains $\Delta(X)$.
(2) If $D \in \mathcal{U}$ and $E \supset D$, then $E \in \mathcal{U}$.
(3) If $D, D' \in \mathcal{U}$, then $D \cap D' \in \mathcal{U}$.
(4) If $D \in \mathcal{U}$, then $D^{-1} \in \mathcal{U}$.
(5) For every $D \in \mathcal{U}$ there is symmetric $D' \in \mathcal{U}$ such that $D' \circ D' \subset D$.

The members of $\mathcal{U}$ are called entourages of X. If $(X,\mathcal{U})$ is a uniform space, then we can generate a topology on X by characterizing that a subset $Y \subset X$ is open if and only if there exists $U \in \mathcal{U}$ such that for each $x \in Y$ the cross section $U[x] = \{y \in X \mid (x,y) \in U\}$ is contained in Y.

Morales introduced [15] $\Phi_D(x) = \{y \in X \mid (f^i(x), f^i(y)) \in D \text{ for all } i \in \mathbb{N}\}$ and $\Gamma_D(x) = \{y \in X \mid (f^i(x), f^i(y)) \in D \text{ for all } i \in \mathbb{Z}\}$ for some $D \in \mathcal{U}$ to generalize the respective concepts of expansive and positively expansive measures.

Definition 2.2. An expansive (positively expansive) measure for bi-measurable (measurable) map $f : X \to X$ on a uniform space is a Borel measure μ on X for which there exists $D \in \mathcal{U}$ such that $\mu(\Gamma_D(x))(\mu(\Phi_D(x))) = 0$ for all $x \in X$. In both the cases, such $D \in \mathcal{U}$ is referred to as expansive entourage for μ.

Remark 2.3. One can verify that Remark 3.2 [8] holds for all dynamical notions in the present paper.

Definition 2.4 ([15])**.** We say that a bi-measurable (measurable) map $f : X \to X$ on a uniform space X is expansive (positively expansive) if there exists $D \in \mathcal{U}$ such that for any distinct $x, y \in X$ there exists $n \in \mathbb{Z}$ ($n \in \mathbb{N}$) such that $(f^n(x), f^n(y)) \notin D$.

Remark 2.5. (i) Any non-atomic Borel measure on a uniform space X is expansive (positively expansive) measure for any expansive (positively expansive) map on X. Recall from [13] that if X has no perfect subset and every Borel probability measure on X is net additive, then a Borel probability measure is atomic. So, there exists no expansive (positively expansive) map with expansive (positively expansive) probability measures on a Lindelöf uniform space without any perfect subset. For instance, (a) consider $X = \mathbb{R}$ and $\mathcal{U} = \{U_t \mid 0 < t < \infty\}$, where $U_t = \{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 < t\} \cup \{(x, x) \mid x \in \mathbb{R}\}$. Then, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is expansive homeomorphism. (b) consider $X = \{1/n, 1-1/n \mid n \in \mathbb{N}\}$. Then $f : X \to X$ given by $f(0) = 0, f(1) = 1$ and $f(x) = x^+$ for all $x \neq 0, 1$ where x^+ denotes the point immediate right to x, is an expansive homeomorphism. But both of the homeomorphism have no expansive measure. Thus, expansiveness of a homeomorphism neither imply nor is implied by the existence of an expansive measure for the map.

(ii) If $f : X \to X$ is an expansive homeomorphism with expansive entourage E, then X must be T_1 space. Indeed, if $x \neq y$ in X, then there exists $n \in \mathbb{Z}$ such that $y \notin f^{-n}(E[f^n(x)])$. On the other hand, Proposition 2.18 [15] shows that there are homeomorphisms on non-T_1 uniform spaces admitting expansive measure.

(iii) In [15], authors proved that any expansive probability measure for a bi-measurable map on a Lindelöf uniform space is non-atomic. Thus, the measure of a countable set under such a measure is equal to zero.

(iv) Let $f : X \to X$ be a measurable map and μ an ergodic invariant probability measure for f. Then μ is positively expansive with expansive entourage D if and only if $\mu(\Phi_D(x)) = 0$ for all $x \in E$ with $\mu(E) > 0$. Indeed, if $E_D = \{x \in X \mid \mu(\Phi_D(x)) = 0\}$, then by Lemma 2.10 [15] it is enough to show that $\mu(E_D) = 1$. For fix $x \in X$, we have that $\Phi_D(x) \subset f^{-1}(\Phi_D(f(x)))$. Then by the fact that μ is invariant under f we have, $\mu(\Phi_D(x)) \leq \mu(\Phi_D(f(x)))$ which implies that $f^{-1}(E_D) \subset E_D$ and hence $\mu(f^{-1}(E_D) = \mu(E_D)$. Since μ is ergodic, we have $\mu(E_D) \in \{0, 1\}$. But since $E \subset E_D$ we must have $\mu(E_D) = 1$.

Example 2.6. Let Z be the Sorgenfrey line, i.e. $\mathbb{R}$ equipped with the topology based on the intervals of the form $[x, y)$, where $(x, y) \in \mathbb{R} \times \mathbb{Q}$ with $x < y$.
(i) The map $f : Z \to Z$ given by $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = 2x$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ is not expansive but the Lebesgue measure is an expansive measure for f.
(ii) The map $f : Z \to Z$ given by $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 2x$ for $x \in \mathbb{R} \setminus \mathbb{Q}$ is not positively expansive but the Lebesgue measure is a positively expansive measure for f.

3. Expansive Measures

In [10], Jacobsen and Utz stated that the set of points doubly asymptotic to a fixed point of an expansive homeomorphism on a compact metric space is at most countable. In [18], R. Williams proved a stronger result which states that the set of points doubly asymptotic to a given point of an expansive homeomorphism on a compact metric space is at most countable. The purpose of this section is to extend this result to expansive measures for bi-measurable maps on separable uniform space. We prove a stronger result and conclude our result as a direct consequence.

Definition 3.1. Let $f : X \to X$ be a bi-measurable map on a uniform space X and let $x \in X$ be given. A point $y \in X$ is said to be positively asymptotic to x if for every $E \in \mathcal{U}$ there is a positive integer N such that $(f^n(x), f^n(y)) \in E$ for all $n \geq N$. On the other hand, a point $y \in X$ is said to be negatively asymptotic to x if for every $E \in \mathcal{U}$ there is a positive integer M such that $(f^n(x), f^n(y)) \in E$ for all $n \leq -M$. A point $y \in X$ is said to be doubly asymptotic to x if for every $E \in \mathcal{U}$ there is a positive integer N such that $(f^n(x), f^n(y)) \in E$ for all $\mid n \mid \geq N$.

Recall that a Borel measure μ on a topological space is outer regular if for every measurable subset A and any $\epsilon > 0$ there is an open subset $O \supset A$ such that $\mu(O \setminus A) < \epsilon$. The following lemma was proved in [15]. For the sake of completeness we present it here.

Lemma 3.2. *Let μ be a Borel measure on a topological space. Then for every measurable Lindelöf subset K with $\mu(K) > 0$ there are $z \in K$ and an open neighborhood U of z such that $\mu(K \cap W) > 0$ for every open neighborhood $W \subset U$ of z.*

Proof. Otherwise, for every $z \in K$ there is an open neighborhood $U_z \subset U$ satisfying $\mu(K \cap U_z) = 0$. Since K is Lindelöf, the open cover $\{K \cap U_z : z \in K\}$ of K admits a countable sub-cover, i.e. there is a sequence $\{z_l\}_{l \in \mathbb{N}}$ in K satisfying $K = \bigcup_{l \in \mathbb{N}} (K \cap U_{z_l})$. So, $\mu(K) = \sum_{l \in \mathbb{N}} \mu(K \cap U_{z_l}) = 0$, a contradiction. □

Theorem 3.3. *The set of points positively asymptotic to p and negatively asymptotic to q for given $p, q \in X$ for a bi-measurable map $f : X \to X$ on a separable uniform space has measure zero with respect to any expansive outer regular measure μ of f.*

Proof. Let $p, q \in X$ be given. Let D be an expansive entourage for μ and $D' \in \mathcal{U}$ be symmetric such that $D' \circ D' \subset D$. Let A be the set of all points positively asymptotic and negatively asymptotic to p and q respectively. If $A_N = \{x \in X \mid (f^n(x), f^n(p)) \in D'$ for all $n \geq N$ and $(f^n(x), f^n(q)) \in D'$ for all $n \leq -N\}$. It is easy to verify that $A \subset \bigcup_{N \geq 0} A_N$ and each A_N is measurable. We show that $\mu(\bigcup_{N \geq 1} A_N) = 0$. If possible, suppose $\mu(\bigcup_{N \geq 1} A_N) > 0$. So, there is $M \geq 1$ such that $\mu(A_M) > 0$.

Since X is separable uniform space, it is second countable and since μ is outer regular, Lusin Theorem [9] implies that for every $\epsilon > 0$ there is a measurable subset C_ϵ with $\mu(X \setminus C_\epsilon) < \epsilon$ such that $f^i \mid_{C_\epsilon}$ is continuous for all integer i with $\mid i \mid \leq M$. Taking $\epsilon = \mu(A_M)/2$ we obtain a measurable subset $C = C_{\mu(A_M)/2}$ such that $f^i \mid_C$ continuous for all integer i with $\mid i \mid \leq M$ and $\mu(A_M \cap C) > 0$.

Since $K = A_M \cap C$ is Lindelöf, we can apply Lemma 3.2 to obtain $z \in A_M \cap C$ and $D_0 \in \mathcal{U}$ satisfying $\mu(A_M \cap C \cap D_1[z]) > 0$ for all entourage $D_1 \subset D_0$...**(*)**.

Fix such a z and D_0. Since $z \in C$ and $f^i \mid_C$ is continuous for all $\mid i \mid \leq M$, we can fix $D^* \in \mathcal{U}$ with $D^* \subset D_0$ such that $f^i(w) \in D[f^i(z)]$ whenever $\mid i \mid \leq M$ and $w \in C \cap D^*[z]$.

Now, take $w \in A_M \cap C \cap D^*[z]$. Since $w \in C \cap D^*[z]$, we already have $f^i(w) \in D[f^i(z)]$ for all $\mid i \mid \leq M$. Since $z \in A_M \cap C$, we have $w, z \in A_M$. Hence, $(f^i(w), f^i(p)) \in D'$ and $(f^i(z), f^i(q)) \in D'$ for all $i \geq M$ and $(f^i(w), f^i(p)) \in D'$ and $(f^i(z), f^i(q)) \in D'$ for all $i \leq -M$. Since D' is symmetric and $D' \circ D' \subset D$, we get $(f^i(z), f^i(w)) \in D$ for all $\mid i \mid \geq M$, i.e., $f^i(w) \in D[f^i(z)]$ for all $\mid i \mid \geq M$. Thus $f^i(w) \in D[f^i(z)]$ for all $i \in \mathbb{Z}$ which implies $A_M \cap C \cap D^*[z] \subset \phi_D(z)$. Since D is an expansive entourage for μ, we have $\mu(A_M \cap C \cap D^*[z]) = 0$. On the other hand, since $D^* \subset D_0$, we can take $D_1 = D^*$ in **(*)** to obtain $\mu(A_M \cap C \cap D^*[z]) > 0$, a contradiction. This completes the proof. □

Lemma 1 [2] shows that if $f : X \to X$ is an expansive homeomorphism on a compact metric space, then $A = \bigcup_{N \geq 0} A_N$ but it is not true in general for bi-measurable map admitting an expansive measure.

Example 3.4. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ for all $x \in \mathbb{Q}$ and $f(x) = 2x$ for all $x \in \mathbb{R} \setminus \mathbb{Q}$. The Lebesgue measure is an expansive measure for f with any $\delta > 0$ as expansive constant. Let $p, q \in \mathbb{Q}$ with $d(p, q) < \delta$ and choose $x = (p + q)/2$. Then, $x \in A_N$ for all $N \geq 0$ but $x \notin A$.

We have the following corollary to Theorem 3.3 and the well-known fact that every Borel probability measure on a metric space is outer regular.

Corollary 3.5. *The set of points positively asymptotic and negatively asymptotic to two given points under a homeomorphism $f : X \to X$ of a separable metric space has measure zero with respect to any expansive probability measure.*

Theorem 3.6 ([14])**.**

(i) *There exists no self-homeomorphism on $[0, 1]$ admitting an expansive measure.*

(ii) *A self-homeomorphism of S^1 admits expansive measure if and only if it is Denjoy.*

The following corollary significantly cuts down the collection of expansive measures for Denjoy homeomorphisms on S^1.

Corollary 3.7. *An outer regular measure $\mu \in SPNAM(X)$ cannot be expansive for a Denjoy homeomorphism on S^1.*

Proof. Suppose by contradiction that an outer regular measure $\mu \in SPNAM(X)$ is expansive for a Denjoy homeomorphism f on S^1 with an expansive constant δ. As is well known, f exhibits unique minimal set M which is infinite, totally disconnected and has no isolated point. Since $S^1 \setminus M$ is an open subset of S^1, it can be written as disjoint union of countably many open arcs, say $\{I_j\}$ so that $\mathrm{diam}(I_j) \to 0$ as $j \to \infty$ and for fixed j, $f^n(I_j) \neq I_j$ for all $n \neq 0$ because of Theorem 3.6(i). Thus, we have $\mathrm{diam}(f^n(I_j)) \to 0$ as $|\, n \,| \to \infty$. For $a, b \in I_j$, let $f^{n_0}(\overline{ab})$ be the longest length in $\{f^n(\overline{ab}) \mid n \in \mathbb{Z}\}$. Take $a, b \in I_j$ such that the length between $f^{n_0}(a)$ and $f^{n_0}(b)$ is less than δ. Then, $\mathrm{diam}(f^n(\overline{ab})) < \delta$ for all $n \in \mathbb{Z}$. Since $\mu(\overline{ab}) > 0$, it follows that δ is not expansive constant for μ, a contradiction. $\square$

As in [7], let the stable set of $x \in X$ is

$$W^s(x) = \{y \in X \mid \forall B \in \mathcal{U}, \exists n \in \mathbb{N} \text{ such that } (f^i(x), f^i(y)) \in B \text{ for all } i \geq n\}$$

and in the invertible case, the unstable set of $x \in X$ is

$$W^u(x) = \{y \in X \mid \forall B \in \mathcal{U}, \exists n \in \mathbb{N} \text{ such that } (f^i(x), f^i(y)) \in B \text{ for all } i \leq -n\}.$$

If $x \in X$, then for $E \in \mathcal{U}$, we define the local stable set

$$W^s(x, E) = \{y \in X \mid (f^n(x), f^n(y)) \in E, \forall n \in \mathbb{N}\}$$

and in the invertible case, the local unstable set

$$W^u(x, E) = \{y \in X \mid (f^{-n}(x), f^{-n}(y)) \in E, \forall n \in \mathbb{N}\}.$$

Definition 3.8. A point $x \in X$ is called heteroclinic point for a map $f : X \to X$ if $x \in W^s(O(p)) \cap W^u(O(q))$ for some periodic points $p, q \in X$.

Corollary 3.9. *The set of heteroclinic points of a bi-measurable map $f : X \to X$ of separable uniform space, with at most countably many periodic points has measure zero with respect to any expansive outer regular measure μ of f.*

Proof. From Theorem 3.3, it follows that $W^s(x) \cap W^u(y)$ for any $x, y \in X$ has measure zero and then the result follows from the fact that the countable union of measure zero set has measure zero. $\square$

Next example shows that the set of heteroclinic points of a bi-measurable map with more than countably many periodic points and an expansive measure on a separable uniform space may have measure zero.

Example 3.10. Let C be the Cantor set in $[-1, 0]$ and $X = C \cup (0, \infty)$ and let $f : X \to X$ be given by $f(x) = x$ for all $x \in C$ and $f(x) = 2x$ for all $x \in (0, \infty)$. The Lebesgue measure is expansive measure for f.

Theorem 3.11. *The set of points doubly asymptotic to a given point for a bi-measurable map $f : X \to X$ on a separable uniform space has measure zero with respect to any expansive outer regular measure μ of f.*

Proof. It follows in the same fashion if we take $p = q$ in Theorem 3.3. $\square$

Note that any non-atomic Borel probability measure is an expansive measure for an expansive homeomorphism of a compact metric space. So, Theorem 3.11 implies that the set of points doubly asymptotic to a given point under an expansive homeomorphism has measure zero with respect to any non-atomic Borel probability measure. From this and well-known measure theoretical results [16], we obtain that the set of points doubly asymptotic to a given point under an expansive homeomorphism is at most countable, which is the fact used by R. Williams [18] to prove that there exists no expansive self-homeomorphisms on a closed 2-cell.

Corollary 3.12. *Let X be a separable uniform space and $f : X \to X$ be a bi-measurable map with an expansive measure $\mu \in SPNAM(X)$. Then, for each $x \in X$ and each neighbourhood N of x there exists $y \in N$ such that x and y are not doubly asymptotic.*

One can prove the following theorem in same fashion as in the proof of Theorem 3.3.

Theorem 3.13. *Every stable class of a measurable map f on a separable uniform space has measure zero with respect to any positively expansive outer regular measure μ of f.*

Corollary 3.14. *Let X be a separable uniform space and $\mu \in SPNAM(X)$ be a positively expansive measure of a measurable map f on X. Then, for each $x \in X$ and each neighbourhood N of x there exists $y \in X$ such that x and y are not asymptotic.*

Definition 3.15. Let X be a uniform space. A homeomorphism (continuous map) $f : X \to X$ is called equicontinuous (positively equicontinuous) if for each $x \in X$ and every $E \in \mathcal{U}$ there exists $D_x \in \mathcal{U}$ (depends on x) such that $(x, y) \in D_x$ implies $(f^n(x), f^n(y)) \in E$ for all $n \in \mathbb{Z}$ ($n \in \mathbb{N}$).

Remark 3.16. Observe that Corollary 3.11 and Corollary 3.14 are obvious generalizations of Theorem 3 [2] of B. F. Bryant. From these corollaries it is clear that an equicontinuous (positively equicontinuous) map on a separable uniform space cannot have any expansive (positively expansive) measure. We now give a direct proof of this result for maps on Lindelöf uniform spaces.

Theorem 3.17. *Let $f : X \to X$ be equicontinuous (positively equicontinuous) map on a Lindelöf uniform space X. Then, f cannot have any expansive (positively expansive) measure.*

Proof. Suppose by contradiction that f has an expansive measure μ with expansive entourage D. Then, for each $x \in X$ there exists $D_x \in \mathcal{U}$ such that $(x, y) \in D_x$ implies $(f^n(x), f^n(y)) \in D$ for all $n \in \mathbb{Z}$. Thus, $D_x[x] \subset \Gamma_D(x)$. Since D is expansive entourage for μ, $\mu(D_x[x]) \leq \mu(\Gamma_D(x)) = 0$ for all $x \in X$. Now, $\{D_x[x] \mid x \in X\}$ is an open cover for X and since X is Lindelöf there is $\{x_i\}_{i\in\mathbb{N}}$ such that $\{D_{x_i}[x_i] \mid i \in \mathbb{N}\}$ is an open covering for X. Therefore, $\mu(X) \leq \Sigma_{i\in\mathbb{N}}\mu(D_{x_i}[x_i])$ which implies $\mu(X) = 0$, which is not the case. The proof for positively equicontinuous map is similar. □

Example 3.18. The map $f : (0, 1) \to (0, 1)$ given by $f(x) = x^n$ for some $n \in \mathbb{N}^+$, the rotation of the unit circle, a contraction [15] on any Lindelöf uniform space are equicontinuous and therefore, they cannot have expansive measure.

Thus, the two classes of homeomorphisms important in topological dynamics namely equicontinuous homeomorphisms and homeomorphisms admitting an expansive measure are disjoint. On the other hand, the identity map on a discrete uniform space is both expansive and equicontinuous.

Definition 3.19. Let $f : X \to X$ be a map on a uniform space X and let $D, E \in \mathcal{U}$ be given. Then, a sequence $\{x_i\}_{i\in\mathbb{N}}$ is said to be D-pseudo orbit if $(f(x_i), x_{i+1}) \in D$ for all $i \in \mathbb{N}$. A sequence $\{x_i\}_{i\in\mathbb{N}}$ is said to be E-shadowed by some point $x \in X$ if $(f^i(x), x_i) \in E$ for all $i \in \mathbb{N}$.

(i) f is said to have shadowing if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that every D-pseudo orbit is E-shadowed by some point $x \in X$.

(ii) f is said to have h-shadowing if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that if $\{x_0, x_1, ..., x_m\}$ satisfy $(f(x_i), x_{i+1}) \in D$ for $0 \leq i \leq (m-1)$, then there exists $x \in X$ such that $(f^i(x), x_i) \in E$ for $0 \leq i \leq (m-1)$ and $f^m(x) = x_m$.

Corollary 3.20. *Let X be a uniform space and let $f : X \to X$ be a bi-measurable map which satisfies either one of the following.*

(i) *f has h-shadowing.*

(ii) *f is positively expansive and has shadowing.*

Then f^{-1} cannot have positively expansive measure.

Proof. (i) Let $E \in \mathcal{U}$ be given and let $D \in \mathcal{U}$ be given for E by the h-shadowing of f. Let $x \in X$ be fixed and y be such that $(x, y) \in D$. For any fixed natural number $m > 0$ the finite sequence $\{f^{-m}(x), f^{-m+1}(x), ..., f^{-1}(x), y\}$ is a D-chain. Hence, by h-shadowing there is $z \in X$ such that $(f^i(z), f^{-m+i}(x)) \in E$ for $0 \leq i \leq (m-1)$ and $f^m(z) = y$ which implies $((f^{-1})^m(y), (f^{-1})^m(x)) = (f^{-m}(x), z) \in E$. This shows that f^{-1} is positively equicontinuous and hence, by Theorem 3.17 it cannot have positively expansive measure.

(ii) Let $E \in \mathcal{U}$ be expansive entourage for f and $D \in \mathcal{U}$ with $D \subset E$ be given for E by the shadowing of f. Fix a finite D-chain $\{x_0, x_1, x_2, x_3, ..., x_m\}$ and extend to a D-pseudo orbit $\{x_0, x_1, x_2, x_3, ..., x_m, f(x_m), f^2(x_m), f^3(x_m), ...\}$. If $z \in X$ is a point which E-shadows the above D-pseudo orbit then $(f^i(z), x_i) \in E$ for $i < m$ and $(f^{j+m}(z), f^j(x_m)) \in E$ for all $j \geq 0$ which implies $(f^i(z), x_i) \in E$ for $i < m$ and $(f^j(f^m(z)), f^j(x_m)) \in E$ for all $j \geq 0$. Hence, $(f^i(z), x_i) \in E$ for $i < m$ and $f^m(z) = x_m$. This shows that f has h-shadowing. Then by (i), f^{-1} cannot have positively expansive measure. □

Definition 3.21. Let $f : X \to X$ be a bijective map on a uniform space X. Then, a point $x \in X$ is called sink [15] (source) for f if $W^u(x, D) = \{x\}$ ($W^s(x, D) = \{x\}$) for some entourage D. We say that f has canonical coordinates if for every entourage E there is an entourage D such that $(x, y) \in D$ implies $W^s(x, E) \cap W^u(y, E) \neq \phi$.

Proposition 3.22. *Sink does not exist for a bi-measurable map $f : X \to X$ with canonical coordinates admitting a positively expansive measure $\mu \in SPNAM(X)$.*

Proof. Let $x \in X$ be a sink and let $E \in \mathcal{U}$ be an expansive entourage for the positively expansive measure μ. Then by Lemma 4.6 [15], there exists $D \in \mathcal{U}$ such that $D[x] \subset W^s(x, E)$. But $W^s(x, E)$ has measure zero and hence, $D[x]$ must have measure zero. This is a contradiction to the fact that the measure is strictly positive. Thus, there does not exist any sink for such f. □

4. Measure Expansive Systems

In this section, we study measure expansive and strong measure expansive homeomorphisms on uniform spaces. For the metric definitions of the following notions reader may see in [6].

Definition 4.1. Let X be a uniform space and $f : X \to X$ be a homeomorphism. Let $D, E \in \mathcal{U}$ be a given entourage. Then,

(i) A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be D-pseudo orbit if $(f(x_i), x_{i+1}) \in D$ for all $i \in \mathbb{Z}$. A D-pseudo orbit is said to be periodic with period $N \geq 1$ if $x_{i+N} = x_i$ for all $i \in \mathbb{Z}$. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be E-shadowed by some point $x \in X$ if $(f^i(x), x_i) \in E$ for all $i \in \mathbb{Z}$.

(ii) f is said to have shadowing if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that every D-pseudo orbit is E-shadowed by some point in X.

(iii) f is said to have periodic shadowing if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that every periodic D-pseudo orbit is E-shadowed by some periodic point in X.

(iv) f is said to have strong periodic shadowing if for every $E \in \mathcal{U}$ there exists $D \in \mathcal{U}$ such that every periodic D-pseudo orbit with period N is E-shadowed by some periodic point with period N.

(v) f is said to have local weak specification if for every $E \in \mathcal{U}$ there exists $M \geq 1$ and $D \in \mathcal{U}$ such that if $\{x_0, x_1, x_2, ..., x_k\}$ satisfy $(f^n(x_i), x_{i+1}) \in D$ for some $n \geq M$, then there exists $x \in X$ such that $(f^{j+in}(x), f^j(x_i)) \in E$ for $0 \leq i \leq (k-1)$ and $0 \leq j < n$.

Definition 4.2. Let X be a uniform space and $f : X \to X$ be a homeomorphism. Then,

(i) f is said to be measure expansive if there exists $D \in \mathcal{U}$ such that for every invariant non-atomic probability measure μ on X, we have $\mu(\Gamma_D(x)) = 0$ for all $x \in X$. Such D is called measure expansive entourage.

(ii) f is said to be strong measure expansive if there exists $D \in \mathcal{U}$ such that for every invariant probability measure μ on X, we have $\mu(\Gamma_D(x)) = \mu(\{x\})$ for all $x \in X$. Such D is called strong measure expansive entourage.

Theorem 4.3. *Let X be a locally compact, para-compact, Hausdorff uniform space and let $f : X \to X$ be an uniform equivalence. Then, f has local weak specification if and only if it has shadowing.*

Proof. Suppose that f has local weak specification. Let $E \in \mathcal{U}$ be symmetric and let $M \geq 1$, $D \in \mathcal{U}$ be given by the local weak specification of f. By para-compactness, we can assume that E be such that $E[K]$ is compact for compact subset $K \subset X$. Let $\{x_i\}_{i\in\mathbb{Z}}$ be a sequence such that $(f^n(x_i), x_{i+1}) \in D$ for some $n \geq M$ and for all $i \in \mathbb{Z}$. Then for each $k \geq 1$, there exists y_k such that $(f^{j+in}(y_k), f^j(x_i)) \in E$ for $0 \leq| i |\leq (k-1)$ and $0 \leq j \leq (n-1)$. Thus for each $k \geq 1$, $y_k \in E[x_0]$. By local compactness, there exists a compact neighborhood K of x_0 such that $y_k \in E[K]$, which is compact. Therefore, $\{y_k\}_{k\in\mathbb{N}}$ has a convergent subsequence converging to some point, say y in X. By continuity of f, we have $(f^{j+in}(y), f^j(x_i)) \in E$ for all $i \in \mathbb{Z}$ and $0 \leq j \leq (n-1)$. This means that f^n has shadowing for each $n \geq M$ and similarly as in Proposition 3.4(a) [8], we conclude that f has shadowing. On the other hand, by definition shadowing implies local weak specification with $N = 1$. □

Theorem 4.4. *Let X be a first countable, Hausdorff uniform space and $f : X \to X$ a measure expansive homeomorphism. If f has shadowing, then it has periodic shadowing.*

Proof. Let $E'' \in \mathcal{U}$ be a measure expansive entourage for f. Let $E \in \mathcal{U}$ be closed, symmetric such that $E^2 \subset E''$. Further, let $E' \in \mathcal{U}$ be symmetric such that $E'^2 \subset E$. Let D be given for E' by the shadowing of f. If $\{x_i\}_{i\in\mathbb{Z}}$ is a periodic D-pseudo orbit with period N, then there exists $x \in X$ such that $(f^i(x), x_i) \in E'$ for all $i \in \mathbb{Z}$. If x is periodic, then nothing to show. Suppose that x is not periodic.

Let us fix $0 \leq l < N$ and $k \in \mathbb{Z}$.

Since $x_{i+l+kN} = x_{i+l}$, $(f^i(f^l(x)), x_{i+l}) \in E'$ and $(x_{i+l}, f^i(f^{kN}(f^l(x))) \in E'$ for all $i \in \mathbb{Z}$
we have $(f^i(f^l(x)), f^i(f^{kN}(f^l(x)))) \in E'^2 \subset E$ for all $i \in \mathbb{Z}$.

Thus, $f^{kN}(f^l(x)) \subset \Gamma_E(f^l(x))$ for every $k \in \mathbb{Z}$.

Thus, the closure of the orbit of x is contained in $\bigcup_{l=0}^{N-1} \Gamma_E(f^l(x))$. If μ is a weak accumulation point of the uniform distribution supported on the finite orbit $\{x, f(x), f^2(x), ..., f^{N-1}(x)\}$, then it is invariant and $\mu(\bigcup_{l=0}^{N-1} \Gamma_E(f^l(x))) = 1$. Hence, $\mu(\Gamma_E(f^l(x))) > 0$ for some $0 \leq l < N$. Since f is measure expansive, μ must be atomic. Thus there exists $z \in \bigcup_{l=0}^{N-1} \Gamma_E(f^l(x))$ such that $\mu(\{z\}) > 0$. Since μ is invariant probability measure, z must be periodic.

Let $z \in \Gamma_E(f^l(x))$ for some $0 \leq l < N$. Then,

$(f^i(x), f^i(f^{-l}(z))) = (f^i(x), f^{i-l}(z)) \in E$ and also, $(f^i(x), x_i) \in E' \subset E$ for all $i \in \mathbb{Z}$.
Therefore, $(f^i(f^{-l}(z)), x_i) \in E^2$ for all $i \in \mathbb{Z}$.

This completes the proof. □

Example 4.5.

(i) Similarly as in Theorem A [5], we can produce N-expansive homeomorphisms with shadowing. Such homeomorphisms are clearly measure expansive and therefore, has periodic shadowing.

(ii) Let f be an expansive homeomorphism with shadowing. Then, one can verify that the N-expansive homeomorphisms constructed in Proposition 2.18 [15] have shadowing. Since such homeomorphisms are measure expansive, they have periodic shadowing.

Corollary 4.6. *Let X be a first countable, locally compact, para-compact, Hausdorff uniform space and $f : X \to X$ a strong measure expansive uniform equivalence with $Per(f) \neq \phi$. If f has local weak specification then it has periodic shadowing.*

Proof. It follows from Theorem 4.3 and Theorem 4.4. □

Lemma 4.7. *Let X be a uniform space and $f : X \to X$ a strong measure expansive homeomorphism such that $Per(f) \neq \phi$. Then, $f\mid_{Per(f)}$ is expansive.*

Proof. Let $D \in \mathcal{U}$ be a strong measure expansive entourage for f. If $Per(f)$ is a single point, then we have nothing to show. Therefore, assume that $Per(f)$ contains more than one point. If possible, suppose there exists $x \neq y$ in $Per(f)$ such that $y \in \Gamma_D(x)$. Let μ be an invariant probability measure such that $\mu(x) > 0$ and $\mu(y) > 0$. Thus, $\mu(\Gamma_D(x)) \geq \mu(x) + \mu(y) > \mu(x)$, a contradiction. So, $f\mid_{Per(f)}$ is expansive with expansive entourage D. □

Lemma 4.8. *Let X be a uniform space and $f : X \to X$ a homeomorphism such that $Per(f) = \phi$. Then, f is measure expansive if and only if it is strong measure expansive.*

Hint: An invariant atomic probability measure for f must be supported on periodic points.

Theorem 4.9. *Let X be a first countable, Hausdorff uniform space and $f : X \to X$ a strong measure expansive homeomorphism with $Per(f) \neq \phi$. If f has shadowing, then it has strong periodic shadowing.*

Proof. By Theorem 4.4, f has periodic shadowing. Let $E' \in \mathcal{U}$ be a strong measure expansive entourage for f. Let $E \in \mathcal{U}$ be symmetric such that $E^2 \subset E'$ and let $D \in \mathcal{U}$ be given for E by shadowing of f. Let $\{x_i\}_{i\in\mathbb{Z}}$ be a D-pseudo orbit with period $N \geq 1$. Then, by periodic shadowing there exists $x \in Per(f)$ such that

$$(f^i(x), x_i) \in E \text{ for all } i \in \mathbb{Z}.$$

Since $x_{i+N} = x_i$ for all $i \in \mathbb{Z}$, we have $(x_i, f^i(f^N(x))) = (x_{i+N}, f^{i+N}(x)) \in E$ for all $i \in \mathbb{Z}$. Therefore, $(f^i(x), f^i(f^N(x))) \in E^2 \subset E'$ for all $i \in \mathbb{Z}$. Since by Lemma 4.7, $f/Per(f)$ is expansive with expansive entourage E', we must have $f^N(x) = x$. This completes the proof. □

Example 4.10. Let Z be the Sorgenfrey line. Then, the map $f : Z \to Z$ given by $f(x) = 2x$ is strong measure expansive homeomorphism with shadowing. Thus, f has strong periodic shadowing.

Acknowledgements. *The first author is supported by the Department of Science and Technology, Government of India, under INSPIRE Fellowship (Registration No- IF150210) Program since March 2015.*

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