APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 19, no. 2 (2018), 223-237
doi:10.4995/agt.2018.7955

Completely simple endomorphism rings of modules

Victor Bovdi [a], Mohamed Salim [a] and Mihail Ursul [b]

[a] Department of Mathematical Sciences, UAE University, United Arab Emirates (vbovdi@gmail.com, msalim@uaeu.ac.ae)
[b] Department of Mathematics and Computer Science, University of Technology, Lae, Papua New Guinea (mihail.ursul@gmail.com)

Communicated by F. Lin

Abstract

It is proved that if A_p is a countable elementary abelian p-group, then: (i) The ring $\mathrm{End}\,(A_p)$ *does not admit a nondiscrete locally compact ring topology. (ii) Under (CH) the simple ring* $\mathrm{End}\,(A_p)/I$*, where I is the ideal of* $\mathrm{End}\,(A_p)$ *consisting of all endomorphisms with finite images, does not admit a nondiscrete locally compact ring topology. (iii) The finite topology on* $\mathrm{End}\,(A_p)$ *is the only second metrizable ring topology on it. Moreover, a characterization of completely simple endomorphism rings of modules over commutative rings is obtained.*

2010 MSC: *16W80; 16N20; 16S50; 16N40.*

Keywords: *topological ring; endomorphism ring; Bohr topology; finite topology; locally compact ring.*

1. Introduction

The notion of associative simple ring can be extended for associative topological rings in several ways:

(i) simple abstract ring endowed with a nondiscrete ring topology (for instance, the classification of nondiscrete locally compact division rings, see [25, Chapter IV] and [4, 15, 16]; we refer to some historical notes about locally compact division rings to [29]);

Received 24 August 2017 – Accepted 09 July 2018

(ii) topological ring without nontrivial closed ideals (see [22, 31]).
(iii) topological ring R with the property that if $f : R \to S$ is a continuous homomorphism in a topological ring S, then either $f = 0$ or f is a topological embedding of R into S (see [24]).

In all cases it is assumed that multiplication is not trivial.

I. Kaplansky has mentioned (see [20], p. 56) that the classification of locally compact simple rings in positive characteristic p is difficult. He proved that every simple nondiscrete locally compact simple torsion-free ring is a matrix ring over a locally compact division ring. However in [26] (see also [30]) has been constructed a nondiscrete locally compact simple ring of positive characteristic which is not a matrix ring over a division ring. Thereby the program of classification of nondiscrete locally compact simple rings was finished. Nevertheless it is interesting to look for new examples of locally compact simple rings.

If A_p is a countable elementary abelian p-group and I is the ideal of the ring $\mathrm{End}\,(A_p)$ consisting of endomorphisms with finite images, then the factor ring $\mathrm{End}\,(A_p)/I$ is a simple von Neumann regular ring. We prove that under (CH) this ring does not admit a nondiscrete locally compact ring topology.

S. Ulam (see [23, Problem 96, p. 181]) posed the following problem: "*Can the group S_∞ of all permutations of integers so metrized that the group operation (composition of permutations) is a continuous function and the set S_∞ becomes, under this metric, a compact space? (locally compact?)*". E.D. Gaughan (see [10]) has solved this problem in the negative.

We study in §3 an analogous problem for the endomorphism ring of a countable elementary abelian p-group, namely: "*Does the endomorphism ring $\mathrm{End}\,(A_p)$ of a countable elementary p-group A_p admit a nondiscrete locally compact ring topology?*". Similarly to the Ulam's problem we obtain a negative answer to this question. Moreover, we prove that $\mathcal{T}_{fin}$ is the only ring topology $\mathcal{T}$ on $\mathrm{End}\,(A_p)$ such that $(\mathrm{End}\,(A_p), \mathcal{T})$ is complete and second metrizable.

We classify in §4 the completely simple rings $(\mathrm{End}\,(M), \mathcal{T}_{fin})$ of vector spaces M over division rings. Corollary 4.4 gives a characterization of semisimple left linearly compact minimal rings. It should be mentioned that Corollary 4.4 is related to a result from [3] stating that any semisimple ring admits at most one linearly compact topology.

Furthermore, we obtain in §5 a description of completely simple rings of the form $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ of modules M over a commutative ring R. We extend the result of [28] to topological rings $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$.

2. Notation, Conventions and Preliminary Results

Rings are assumed to be associative, not necessarily with identity. Topological spaces are assumed to be completely regular. The *weight* (see [8], p.12) of the space X is denoted by $w(X)$. The *pseudocharacter* of a point $x \in X$ (see [8], p.135) is the smallest cardinal of the form $|\mathcal{U}|$, where $\mathcal{U}$ is a family of open subsets of X such that $\cap\mathcal{U} = \{x\}$. The closure of a subset A of the topological space X is denoted by $\overline{A}$ and the interior by $\mathrm{Int}(A)$ (see [8], p.14). A topological space X is called a *Baire space* (see [8], p.198) if for each sequence $\{X_1, X_2, \ldots\}$ of open dense subsets of X the intersection $\cap_{i=1}^{\infty} G_i$ is a dense set.

An abelian group A is called *elementary abelian p-group* (p prime) if $pa = 0$ for all $a \in A$. Such group is a direct sum of copies of the cyclic group $Z(p)$. The subring of a ring R generated by a subset S, is denoted by $\langle S\rangle$. A ring R is called *locally finite* if every its finite subset is contained in a finite subring. A topological ring $(R, \mathcal{T})$ is called *metrizable* if its underlying additive group satisfies the first axiom of countability. A ring R with 1 is called *Dedekind-finite* if each equality $xy = 1$ implies $yx = 1$. It is well-known that every finite ring with identity is Dedekind-finite. Since every compact ring with identity is a subdirect product of finite rings, it follows that every compact ring with identity is Dedekind-finite. If $A \subseteq R$, then $\mathrm{Ann}_l(A) := \{x \in R \mid xA = 0\}$. If X, Y are the subsets of R, then $X \cdot Y := \{xy \mid x \in X, y \in Y\}$. A topological ring R is called *compactly generated* (see [27, Chapter II]) if there exists a compact subset K such that $R = \langle K\rangle$. If $(R, \mathcal{T})$ is a topological ring and I is an ideal of R, then the quotient topology of the factor ring R/I is denoted by $\mathcal{T}/I$. If K is a subset of an abelian group A, then set

$$T(K) = \{\alpha \in \mathrm{End}\,(A) \mid \alpha(K) = 0\}.$$

When K runs over all finite subsets of A, the family $\{T(K)\}$ defines a ring topology $\mathcal{T}_{fin}$ on $\mathrm{End}\,(A)$. This topology is called the *finite topology*.

Lemma 2.1. *For any abelian group A the ring $(\mathrm{End}\,(A), \mathcal{T}_{fin})$ is complete.*

Proof. See [27, Theorem 19.2]. □

Lemma 2.2 (Cauchy's criterion)**.** *In a Hausdorff complete commutative group G, in order that a family $(x_\alpha)_{\alpha\in\Omega}$ should be summable it is necessary and sufficient that, for each neighborhood V of zero in G, there is a finite subset Ω_0 of Ω such that $\Sigma_{\alpha\in K} x_\alpha \in V$ for all finite subsets K of Ω which do not meet Ω.*

Proof. See [5], p.263. □

Lemma 2.3. *If $(x_\alpha)_{\alpha\in\Omega}$ is a summable subset in $(\mathrm{End}\,(A), \mathcal{T}_{fin})$ then every subset Δ of Ω the family $(x_\beta)_{\beta\in\Delta}$ is summable.*

Proof. Let V be a neighborhood of zero of $(\mathrm{End}\,(A), \mathcal{T}_{fin})$. We can consider without loss of generality that V is a left ideal of $\mathrm{End}\,(A)$. There exists a finite subset Ω_0 of Ω such that $\Sigma_{\alpha\in K} x_\alpha \in V$ for every finite subset K of Ω for which $K \cap \Omega_0 = \varnothing$. Let F be a finite subset of Δ such that $F \cap (\Omega_0 \cap \Delta) = \varnothing$. If $\alpha \in F$, then $\alpha \notin \Omega_0$, hence $\Sigma_{\alpha\in F} x_\alpha \in V$. By Cauchy's criterion the family $(x_\beta)_{\beta\in\Delta}$ is summable. □

A topological ring $(R, \mathcal{T})$ is called *minimal* (see, for instance, [7]) if there is no ring topology $\mathcal{U}$ such that $\mathcal{U} \leq \mathcal{T}$ and $\mathcal{U} \neq \mathcal{T}$. A topological ring $(R, \mathcal{T})$ is called *simple* if R is simple as a ring without topology. A topological ring $(R, \mathcal{T})$ is called *weakly simple* if $R^2 \neq 0$ and every its closed ideal is either 0 or R. A topological ring $(R, \mathcal{T})$ is called *completely simple* (see [24]) if $R^2 \neq 0$ and for every continuous homomorphism $f : (R, \mathcal{T}) \to (S, \mathcal{U})$ in a topological ring $(S, \mathcal{U})$ either $\ker(f) = R$ or f is a homeomorphism of $(R, \mathcal{T})$ on $\mathrm{Im}(f)$. Equivalently, $R^2 \neq 0$ and $(R, \mathcal{T})$ is weakly simple and minimal. Let M be a unitary right R-module over a commutative ring R with 1. The module M is called *divisible* if $Mr = M$ for every $0 \neq r \in R$. A right R-module M is called *faithful* if $Mr = 0$ implies $r = 0$ $(r \in R)$. A right R-module M is called *torsion-free* if $mr = 0$ implies that either $m = 0$ or $r = 0$, where $m \in M$ and $r \in R$. Recall that a submodule N of an R-module M is called *fully invariant* $\alpha(N) \subseteq N$ for every endomorphism α of M_R. We use in the sequel the notion and results from the books [8, 27].

Remark 2.4. If R is a von Neumann regular ring, then $R^2 = R$.

Lemma 2.5. *An ideal I of a von Neumann regular ring is von Neumann regular.*

Proof. Let $i \in I$. Thus there exists $x \in R$ such that $ixi = i$. It follows that $ixixi = i$ and $xix \in I$. □

Corollary 2.6. *If I an ideal of a von Neumann regular ring R, then any ideal H of I is an ideal of R, too.*

Proof. $RH = RH^2 \subseteq IH \subseteq H$. Similarly, $HR \subseteq H$. □

If A_p is a p-elementary countable group, then

$$I = \{\alpha \in \mathrm{End}\,(A_p) \mid |\mathrm{Im}(\alpha)| < \aleph_0\}.$$

Fix a linear basis $\{v_i \mid i \in \mathbb{N}\}$ of A_p over the field $\mathbb{F}_p$. Using this fixed basis, we define the map $e_i : A \to A$ such that

$$e_i(v_j) = \delta_{ij} v_j, \qquad (i, j \in \mathbb{N})$$

where δ_{ij} is the Kronecker delta.

Lemma 2.7. *We have for* $\mathrm{End}\,(A_p)$*:*

(i) *I is a von Neumann regular ring.*
(ii) *I is a simple ring.*
(iii) *The factor ring* $\mathrm{End}(A_p)/I$ *is simple von Neumann regular.*
(iv) *I is a locally finite ring.*

Proof. (i): The ring $\mathrm{End}\,(A_p)$ is regular (see [21, Theorem 4.27, p. 63]), so I is von Neumann regular by Lemma 2.5.

(ii), (iii): The ideal I is the only nontrivial ideal of the ring $\mathrm{End}\,(A_p)$ (see [17, §17, Theorem 1, p. 93]). This means that $\mathrm{End}\,(A)/I$ is simple. It is regular by the part (i).

(iv) Since I is simple (see [17, §12, Proposition 1]), it suffices to show that I contains a nonzero locally finite right ideal.

Let us show that the left ideal Ie_1 of I is locally finite as a ring (equivalently, as a $\mathbb{F}_p$-algebra). We have $0 \neq e_1 \in Ie_1$. If H is the left annihilator of Ie_1, then, obviously, H is a locally finite ring, hence it is locally finite as a $\mathbb{F}_p$-algebra. We claim that Ie_1/H is finite. Define $\beta_n \in H$ $(n \geq 2)$ in the following way

$$\beta_n(v_i) = \begin{cases} v_n, & \text{for} \quad i = 1; \\ 0, & \text{for} \quad i \neq 1. \end{cases}$$

Let us prove that $Ie_1 = \mathbb{F}_p e_1 + \Sigma_{n=2}^{\infty} \mathbb{F}_p \beta_n$.

If $\alpha \in I$, then $\alpha(v_1) = r_1 v_1 + \cdots + r_n v_n$, where $r_i \in \mathbb{F}_p$ and $n \in \mathbb{N}$, so

$$\begin{aligned} \alpha e_1(v_1) &= r_1 e_1(v_1) + r_2 \beta_2(v_1) + \cdots + r_n \beta_n(v_1) \\ &= (r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n)(v_1); \\ \alpha e_1(v_j) &= (r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n)(v_j) \qquad (j \neq 1). \end{aligned}$$

This yields

$$\alpha e_1 = r_1 e_1 + r_2 \beta_2 + \cdots + r_n \beta_n$$

and so $Ie_1 = \mathbb{F}_p e_1 + \Sigma_{n=2}^{\infty} \mathbb{F}_p \beta_n$.

In particular, $Ie_1 = \mathbb{F}_p e_1 + H$, and so H has a finite index in Ie_1. Clearly, Ie_1 is a locally finite $\mathbb{F}_p$-algebra (see [17, Proposition 1, p. 241]) and I is a locally finite $\mathbb{F}_p$-algebra (see [17, Proposition 2, p. 242]). □

The next result can be deduced from [27, Lemma 36.11].

Lemma 2.8. *Let A be a locally compact, compactly generated, and totally disconnected ring. If A contains a dense locally finite subring B, then A is compact.*

Proof. Let $A = \langle V \rangle$, where V is a compact symmetric neighborhood of zero. Since V is compact, the subset $V + V + V \cdot V$ also is compact. Since B is dense, $A = B + V$. By compactness of $V + V + V \cdot V$ there exists a finite subset $H \subseteq B$ such that $V + V + V \cdot V \subseteq H + V$. Since B is a locally finite ring, we can assume without loss of generality that H is a subring. Let $H \setminus \{0\} = \{h_1, \ldots, h_k\}$. The subset

$$H + h_1 V + \cdots + h_k V + V$$

is an open subgroup of $R(+)$. Indeed, this subset is symmetric and

$$\begin{aligned} (H + h_1 V + \cdots + h_k V + V) + (H + h_1 V + \cdots + h_k V + V) \\ \subseteq H + h_1(V + V) + \cdots + h_k(V + V) + V + V \\ \subseteq H + h_1 V + \cdots + h_k V + V. \end{aligned}$$

We prove by induction on m that

$$V^{[m]} \subseteq H + h_1 V + \cdots + h_k V + V, \qquad (m \in \mathbb{N})$$

where $V^{[1]} = V$ and $V^{[m]} = V^{[m-1]} \cdot V$ for all m.

The inclusion is obvious for $m = 1$.

Assume that the assertion has been proved for $m \geq 1$. Clearly,

$$\begin{aligned} V^{[m+1]} = V^{[m]} \cdot V \subseteq H \cdot V + h_1(V \cdot V) + \cdots + h_k(V \cdot V) + V \cdot V \subseteq \\ h_1 V + \cdots + h_k V + h_1(H + V) + \cdots + h_k(H + V) + H + V \subseteq \\ H + h_1 V + \cdots + h_k V + V. \end{aligned}$$

Consequently, $A = H + h_1V + \cdots + h_kV + V$, therefore A is compact. $\square$

An element x of a topological ring is called *discrete* if there exists a neighborhood V of zero such that $xV = 0$ (i.e., the right annihilator of x is open).

Lemma 2.9. *The set of all discrete elements of a topological ring is an ideal. A simple ring with identity does not contain nonzero discrete elements.*

3. Locally compact ring topologies on End (A) of a countable elementary abelian p-group A

Theorem 3.1. *Let R be a simple, nondiscrete and locally compact ring of* $\operatorname{char}(R) = p > 0$ *and* $1 \in R$. *If V is a compact open subring of R and* $\{e_\alpha \mid \alpha \in \Omega\}$ *is a set of orthogonal idempotents in R, then*

$$|\Omega| \le w(V).$$

Proof. The ring R does not contain nonzero discrete elements by Lemma 2.9. Since R is locally compact and $\operatorname{char}(R) = p$, it is totally disconnected. Additionally, R has a fundamental system of neighborhoods of zero consisting of compact open subrings by [19, Lemma 9].

If V is a compact open subring of R, then by continuity of the ring operations for each $\alpha \in \Omega$ there exists an open ideal V_α of V such that $e_\alpha V_\alpha \subseteq V$. Clearly, there exists $y_\alpha \in V_\alpha$ for which $e_\alpha y_\alpha \neq 0$ since R has no nonzero discrete elements.

We claim that hold the following two properties:

(i) $e_\alpha y_\alpha \notin \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$ for each $\alpha \in \Omega$;

(ii) the set $X = \{e_\alpha y_\alpha \mid \alpha \in \Omega\}$ is a discrete subspace of V.

Indeed, if were $e_\alpha y_\alpha \in \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$ for some $\alpha \in \Omega$, then

$$\begin{aligned} e_\alpha y_\alpha = e_\alpha e_\alpha y_\alpha &\in e_\alpha \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}} \\ &\subseteq \overline{\{e_\alpha e_\beta y_\beta \mid \beta \neq \alpha\}} \\ &= \{0\}, \end{aligned}$$

so $e_\alpha y_\alpha = 0$, a contradiction. The part (i) is proved.

(ii) Now, for each $\alpha \in \Omega$ we have $V \setminus \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}$ is open and, consequently,

$$(V \setminus \overline{\{e_\beta y_\beta \mid \beta \neq \alpha\}}) \cap X = \{e_\alpha y_\alpha\},$$

by (i). Therefore the point $e_\alpha y_\alpha (\alpha \in \Omega)$ of X is isolated. In other words, the subspace X of V is discrete.

Since X is discrete, $|\Omega| = |X| = w(X) \le w(V)$ (see [1, Exercises 98-99, p. 72]). $\square$

Theorem 3.2. *Let A_p be a countable elementary abelian p-group. Then the ring*

$$I = \{\alpha \in \operatorname{End}(A_p) \mid |\operatorname{Im}(\alpha)| < \aleph_0\}$$

does not admit a nondiscrete ring topology $\mathcal{U}$ such that $(I, \mathcal{U})$ is a Baire space.

Proof. Put $S_n = \{\alpha \in I \mid \alpha(A) \subseteq \mathbb{F}_p v_1 + \cdots + \mathbb{F}_p v_n\}$, where $n \in \mathbb{N}$. Clearly, $I = \cup_{n\in\mathbb{N}} S_n$ and

$$S_n = \{\alpha \in I \mid e_i\alpha = 0 \quad \text{for} \quad i > n\} = \mathrm{Ann}_r\big(\{e_k \mid k > n\}\big).$$

This yields that the subset S_n is closed due the continuity of the ring operations.

Since I is a Baire space, there exists $n \in \mathbb{N}$ such that $\mathrm{Int}(S_n) \neq \varnothing$, hence S_n is an open subgroup.

Set $\beta \in I$ such that

$$\beta(v_i) = \begin{cases} v_{n+i}, & \text{for} \quad i = 1, \dots, n; \\ 0 \quad , & \text{for} \quad i > n. \end{cases}$$

Let $W \subseteq S_n$ be a neighborhood of zero of $(I, \mathcal{U})$ such that $\beta W \subseteq S_n$. If $w \in W \setminus \{0\}$, then there exist $a \in A$ and $r_1, \dots, r_n \in \mathbb{F}_p$ such that

$$0 \neq w(a) = \sum_{i=1}^{n} r_i v_i \qquad \text{and} \qquad \beta(w(a)) = \sum_{i=1}^{n} r_i v_{n+i}.$$

There exists $j \in 1, \dots, n$ such that $r_j \neq 0$. Then

$$e_{n+j}\beta w(a) = r_j v_{n+j} \neq 0,$$

hence $e_{n+j}\beta w \neq 0$ and so $\beta w \notin S_n$, a contradiction. □

Corollary 3.3. *Under the notation of Theorem 3.2 the ring I does not admit a nondiscrete locally compact ring topology.*

Proof. This follows from the fact that each locally compact space is a Baire space (see [6, Theorem 1, p. 117]). □

Our main result is the following.

Theorem 3.4. *The endomorphism ring* $\mathrm{End}\,(A_p)$ *of a countable elementary abelian p-group A_p does not admit a nondiscrete locally compact ring topology.*

Proof. We use the notation and results from section 2. Denote $R = \mathrm{End}\,(A_p)$. Assume on the contrary that there exists on R a nondiscrete locally compact ring topology $\mathcal{T}$.

Fact 1. The ring $(R, \mathcal{T})$ has a fundamental system of neighborhoods of zero consisting of compact open subrings.

Since the additive group of the ring R has exponent p, it is totally disconnected (this follows from [12, Theorem 9.14, p. 95]). By I. Kaplansky's result (see [19, Lemma 9]), the ring $(R, \mathcal{T})$ has a fundamental system of neighborhoods of zero consisting of compact open subrings.

Fact 2. The group Re_i is countable for each $i \in \mathbb{N}$.

We claim that Re_i is infinite. Indeed, for each $j \in \mathbb{N}$ put $\beta_j \in R$ such that

$$\beta_j(v_k) = \begin{cases} v_j, & \text{for} \quad k = i; \\ 0, & \text{for} \quad k \neq i. \end{cases}$$

If $j \neq s$, then $\beta_j e_i(v_i) = \beta_j(v_i) = v_j$ and $\beta_s e_i(v_i) = \beta_s(v_i) = v_s$, hence $\beta_j e_i \neq \beta_s e_i$, so Re_i is infinite.

The ring Re_i is countable. Indeed, consider the mapping $\psi : Re_i \to A_p^{\mathbb{F}_p v_i}$, where

$$\psi(\alpha e_i)(rv_i) = \alpha(rv_i) \qquad \text{for all} \qquad r \in \mathbb{F}_p.$$

If $\alpha e_i \neq \beta e_i$ $(\alpha, \beta \in R)$, then there exists an element $x = \sum_j r_j v_j \in A_p$ such that $\alpha e_i(x) \neq \beta e_i(x)$, hence, $\alpha(r_i v_i) \neq \beta(r_i v_i)$. Thus

$$\psi(\alpha e_i)(r_i v_i) = \alpha(r_i v_i) \neq \beta(r_i v_i) = \psi(\beta e_i)(r_i v_i).$$

The latter means that ψ is an injective mapping of Re_i into $A^{\mathbb{F}_p v_i}$. Since $A^{\mathbb{F}_p v_i}$ is countable, Re_i is countable, too.

Fact 3. I is a closed ideal of R. We claim that I is not dense in the topological ring $(R, \mathcal{T})$. Assume the contrary. Since I is locally finite and is a maximal ideal, $(R, \mathcal{T})$ is topologically locally finite by Lemma 2.8. The ring R contains two elements x, y such that $xy = 1$ and $yx \neq 1$. The subring $\overline{\langle x, y\rangle}$ is compact, hence Dedekind-finite, a contradiction. We obtained that $(R/I, \mathcal{T}/I)$ is a nondiscrete metrizable locally compact ring.

Fact 4. I is a discrete ideal of R.

This follows from Theorem 3.2.

Fact 5. Re_i is a discrete left ideal of R for every $i \in \mathbb{N}$.

Indeed, $Re_i \subseteq I$ and I is discrete by Fact 4 for every $i \in \mathbb{N}$.

Fact 6. $\mathrm{Ann}_l(e_i)$ is open in R for every $i \in \mathbb{N}$.

Indeed, the group homomorphism $q : R \to Re_i, r \mapsto re_i$, is continuous. Since Re_i is discrete $q^{-1}(0) = \mathrm{Ann}_l(e_i)$ is open.

Fact 7. $\cap_i \mathrm{Ann}_l(e_i) = 0$.

Obvious.

Fact 8. $\mathcal{T} \geq \mathcal{T}_{fin}$.

We notice that $\mathrm{Ann}_l(e_i) = T(\{v_i\})$ for every $i \in \mathbb{N}$. For, if $\alpha e_i = 0$, then $\alpha(v_i) = \alpha e_i(v_i) = 0$, i.e., $\alpha \in T(\{v_i\})$. Conversely, if $\alpha \in T(\{v_i\})$, then $\alpha e_i(v_i) = \alpha(v_i) = 0$. If $j \neq i$ then $\alpha e_i(v_j) = 0$. Therefore $\alpha e_i = 0$. Moreover

$$T(\{v_1, \dots, v_n\}) = \cap_{i=1}^n T(\{v_i\}) = \cap_{i=1}^n \mathrm{Ann}_l(e_i) \in \mathcal{T} \qquad (\forall n \in \mathbb{N}).$$

Since the family $\{T(\{v_1, \dots, v_n\})\}$ forms a fundamental system of neighborhoods of zero of $(R, \mathcal{T}_{fin})$, we get that $\mathcal{T}_{fin} \leq \mathcal{T}$.

Fact 9. The ring $(R, \mathcal{T})$ is metrizable.

Since $\cap_{i\in\mathbb{N}} \mathrm{Ann}_l(e_i) = 0$, the pseudocharacter of $(R, \mathcal{T})$ is $\aleph_0$. If V is a compact open subring of $(R, \mathcal{T})$ (see Fact 1), then the pseudocharacter of V also is $\aleph_0$. However in every compact space the pseudocharacter of a point coincides with its character. Therefore $(R, \mathcal{T})$ is metrizable.

Fact 10. $(R/I, \mathcal{T}/I)$ has an open compact subring.

Indeed, it is well-known (see [19]) that every totally disconnected ring has a fundamental system of neighborhood of zero consisting of compact open subrings. Henceforth V is a fixed open compact subring of $(R/I, \mathcal{T}/I)$.

Fact 11. R/I contains a family of orthogonal idempotents of cardinality $2^{\aleph_0}$.

Indeed, the family $\{e_i\}_{i\in\mathbb{N}}$ of idempotents of the ring $(R, \mathcal{T}_{fin})$ is summable and $1_A = \Sigma_{n\in\mathbb{N}} e_n$, where 1_A is the identity of R.

The first ordinal number of cardinality $\mathfrak{c}$ of continuum is denoted by $\omega(\mathfrak{c})$. Let $\{\mathbb{N}(\alpha) \mid \alpha < \omega(\mathfrak{c})\}$ be a family of infinite almost disjoint subsets of $\mathbb{N}$ (see [8, Example 3.6.18, p. 175–176]). Put $f_{\mathbb{N}(\alpha)} = \Sigma_{i\in\mathbb{N}(\alpha)} e_i$ for each $\alpha < \omega(\mathfrak{c})$. The element $f_{\mathbb{N}(\alpha)}$ exists by Lemma 2.3. Then:

(i) $f_{\mathbb{N}(\alpha)} \notin I$ for every $\alpha < \omega(\mathfrak{c})$;

(ii) $f_{\mathbb{N}(\alpha)} f_{\mathbb{N}(\beta)} \in I$ for each $\alpha, \beta < \omega(\mathfrak{c})$ and $\alpha \neq \beta$.

If $g_\alpha = f_{\mathbb{N}(\alpha)} + I$ for each $\alpha < \omega(\mathfrak{c})$, then $\{g_\alpha \mid \alpha < \omega(\mathfrak{c})\}$ is the required system of orthogonal idempotents.

The subring V is metrizable (by Fact 9). Since V is compact and R/I is a simple von Neumann regular ring by Lemma 2.7 and $w(V) \leq \aleph_0$, we obtain a contradiction to Theorem 3.1. □

Theorem 3.5. *(CH) Under the notation of Theorem 3.4, the ring R/I does not admit a nondiscrete locally compact ring topology.*

Proof. Assume on the contrary that the factor ring R/I admits a nondiscrete locally compact ring topology $\mathcal{T}$, so $(R/I, \mathcal{T})$ contains an open compact subring V. Since the cardinality of R/I is continuum and V is infinite, the power of V is continuum. Since we have assumed (CH), the subring V is metrizable, hence second metrizable (see [14, 18]). However we have proved in Theorem 3.4 that the ring R/I contains a family of orthogonal idempotents of cardinality $\mathfrak{c}$, a contradiction with Theorem 3.1. □

Theorem 3.6. *The finite topology $\mathcal{T}_{fin}$ is the only second metrizable ring topology $\mathcal{T}$ on R for which $(R, \mathcal{T}_{fin})$ is complete.*

Proof. Let $K = \langle F\rangle$, where F is a finite subset of A. Clearly, there exists a subgroup A' of A such that $A = K \oplus A'$. Choose $\mathrm{e}_F \in R$ such that $\mathrm{e}_F \restriction_K = \mathrm{id}_K$ and $\mathrm{e}_F(A') = 0$. Clearly,

$$T(K) = R(1 - e_F)$$

and $\alpha K = 0$ if and only if $\alpha \in R(1 - e_F)$, so the family $\{R(1 - e_F)\}$, where F runs over all finite subset of A, forms a fundamental system of neighborhoods of zero for $(R, \mathcal{T}_{fin})$.

There exists an injective map of Re_F to $\mathrm{Hom}(K, A)$, so the left ideal Re_F is countable, due to countability $\mathrm{Hom}(K, A)$. Since $e_F^2 = e_F$, the Peirce decomposition

$$R = Re_F \oplus R(1 - e_F)$$

of R with respect to the idempotent e_F is a decomposition of the topological group $(R, +, \mathcal{T})$. It follows that Re_F is discrete, hence $R(1-e_F)$ is open (in the topology $\mathcal{T}$). Hence $\mathcal{T} \geq \mathcal{T}_{fin}$, so $\mathcal{T} = \mathcal{T}_{fin}$ (see [9, Theorem 30] or [11]). □

4. Completely simple topological endomorphism rings of vector spaces

Theorem 4.1. *Let A_F be a right vector space over a division ring F and $S = \mathrm{End}\,(A_F)$. The following conditions are equivalent:*

(i) *$(S, \mathcal{T}_{fin})$ is a completely simple topological ring.*

(ii) $\dim(A_F) = \infty$ *or* $\dim(A_F) < \infty$ *and* F *does not admit a nondiscrete ring topology.*

Proof. (i)⇒ (ii): If A_F is finite-dimensional, then S is discrete and isomorphic to the matrix ring $\mathrm{M(n, F)}$, where n is the dimension of A_F. Then, obviously, F does not admit a nondiscrete ring topology.

(ii)⇒ (i): If $\dim(A_F) = n < \infty$, then $S \cong \mathrm{M(n, F)}$. Since F does not admit nondiscrete ring topologies, the same holds for $\mathrm{M(n, F)}$.

Let A_F be infinite dimensional. Fix a basis $\{x_\alpha\}_{\alpha<\tau}$ over F, where τ is an infinite ordinal number. It is well-known that the topological ring $(S, \mathcal{T}_{fin})$ is weakly simple (see [22, Satz 12, p. 258]) and the family $\{T(x_\alpha)\}_{\alpha<\tau}$ is a prebase at zero for the finite topology $\mathcal{T}_{fin}$ of S.

Assume on the contrary that there exists a Hausdorff ring topology $\mathcal{T}$, coarser that $\mathcal{T}_{fin}$ and different from it. Let $e_\alpha \in S$ such that $e_\alpha^2 = e_\alpha$ and $e_\alpha(x_\beta) = \delta_{\alpha\beta} x_\alpha$ for each $\alpha < \tau$, where $\delta_{\alpha\beta}$ is the Kronecker delta.

Fact 1. $T(x_\alpha) = \mathrm{Ann}_l(e_\alpha)$ for each $\alpha < \tau$.

Indeed, if $p \in T(x_\alpha)$, then $pe_\alpha(x_\alpha) = p(x_\alpha) = 0$. If $\beta \neq \alpha$, then $e_\alpha(x_\beta) = 0$, hence $pe_\alpha = 0$, i.e. $p \in \mathrm{Ann}_l(e_\alpha)$. Conversely, if $pe_\alpha = 0$, then we have $p(x_\alpha) = pe_\alpha(x_\alpha) = 0$, i.e. $p \in T(x_\alpha)$.

Fact 2. There exists $\alpha_0 < \tau$ for which Se_{α_0} is nondiscrete in $(S, \mathcal{T})$.

Assume on the contrary that for every $\alpha < \tau$ there exists a neighborhood V_α of zero of $(S, \mathcal{T})$ such that $Se_\alpha \cap V_\alpha = 0$. If U_α is a neighborhood of zero of $(S, \mathcal{T})$ such that $U_\alpha e_\alpha \subseteq V_\alpha$, then $U_\alpha e_\alpha = 0$, hence $\mathrm{Ann}_l(e_\alpha) = T(x_\alpha)$ is open in $(S, \mathcal{T})$. Hence $\mathcal{T}_{fin} \leq \mathcal{T}$ and $\mathcal{T} = \mathcal{T}_{fin}$, a contradiction.

Fact 3. $(Se_{\alpha_0} \cap V)x_{\alpha_0} \nsubseteq \oplus_{\beta\in K} x_\beta F$
for any neighborhood V of of zero of $(S, \mathcal{T})$ and any finite subset K of the set $[0, \tau)$ of all ordinal numbers less than τ.

Assume on the contrary that there exists a finite subset K of $[0, \tau)$ and a neighborhood V of zero of $(S, \mathcal{T})$ such that

$$(Se_{\alpha_0} \cap V)x_{\alpha_0} \subseteq \oplus_{\beta\in K} x_\beta F. \tag{4.1}$$

Fix $\gamma \in [0, \tau) \setminus K$. For each $\beta \in K$ define $q_\beta \in S$ such that $q_\beta(x_\beta) = x_\gamma$ and $q(x_\delta) = 0$ for $\delta \neq \beta$.

Let V_0 be a neighborhood of zero of $(S, \mathcal{T})$ such that $V_0 \subseteq V$ and $q_\beta V_0 \subseteq V$ for all $\beta \in K$. There exists $0 \neq h \in Se_{\alpha_0} \cap V_0$ by Fact 2 and $hx_{\alpha_0} \neq 0$ by Fact 1. Since $Se_{\alpha_0} \cap V_0 \subseteq Se_{\alpha_0} \cap V$, we obtain that $hx_{\alpha_0} = \Sigma_{\beta\in K} x_\beta f_\beta$, $(f_\beta \in F)$ by (4.1). There exists $\beta_0 \in K$ such that $f_{\beta_0} \neq 0$ (because $hx_{\alpha_0} \neq 0$), so

$$q_{\beta_0} h = q_{\beta_0}(\Sigma_{\beta\in K} x_\beta f_\beta) = r_{\beta_0} x_\gamma \notin \oplus_{\beta\in K} x_\beta F,$$

a contradiction. Therefore Fact 3 is proved.

Now let V be a neighborhood of zero of $(S, \mathcal{T})$. Pick up a neighborhood V_0 of zero of $(S, \mathcal{T})$ such that $V_0 \cdot V_0 \subseteq V$. Since $\mathcal{T} \leq \mathcal{T}_{fin}$, there exists a finite subset K of $[0, \tau)$ such that

$$T(\{x_\beta \mid \beta \in K\}) \subseteq V_0.$$

We have $(Se_{\alpha_0} \cap V_0)x_{\alpha_0} \nsubseteq \oplus_{\beta \in K} x_\beta F$ by Fact 3. It follows that there exists $q \in Se_{\alpha_0} \cap V_0$ such that

$$q(x_{\alpha_0}) \notin \oplus_{\beta \in K} x_\beta F.$$

Clearly, $q(x_{\alpha_0}) \in A_F$, so it can be written as $q(x_{\alpha_0}) = \sum_{\alpha < \tau} x_\alpha f_\alpha$, where $f_\alpha \in F$ and there exists $\beta_0 \notin K$ such that $f_{\beta_0} \neq 0$.

Consider the element $s \in S$ such that $s(x_{\beta_0}) = x_{\alpha_0} f_{\beta_0}^{-1}$ and $s(x_\lambda) = 0$ for $\lambda \neq \beta_0$. Evidently, $s \in T(K)$, hence

$$sq \in T(K) \cdot V_0 \subseteq V_0 \cdot V_0 \subseteq V.$$

Moreover, $sq(x_{\alpha_0}) = s(x_{\beta_0} f_{\beta_0} + \cdots) = x_{\alpha_0}$. Since $q \in Se_{\alpha_0}$, we obtain that $sq(x_\beta) = 0$ for $\beta \neq \alpha_0$. Consequently, $e_{\alpha_0} = sq \in V$ for every neighborhood V of zero of $(S, \mathcal{T})$, a contradiction. □

Remark 4.2. The question of existence of a uncountable division ring which does not admit a nondiscrete Hausdorff ring topology is open. Several results on this topic can be found in Chapter 5 of [2].

Theorem 4.3. *Let $\prod_{\alpha \in \Omega} R_\alpha$ be a family of compact rings with identity. Then the product $(\prod_{\alpha \in \Omega} R_\alpha, \prod_{\alpha \in \Omega} \mathcal{T}_\alpha)$ is a minimal ring if and only if every $(R_\alpha, \mathcal{T}_\alpha)$ is a minimal topological ring. (Here $\prod_{\alpha \in \Omega} \mathcal{T}_\alpha$ is the product topology on the ring $\prod_{\alpha \in \Omega} R_\alpha$.)*

Proof. $\Rightarrow$: Assume on the contrary that there exists $\beta \in \Omega$ and a ring topology $\mathcal{T}'$ on R_β such that $\mathcal{T}' \leq \mathcal{T}_\beta$ and $\mathcal{T}' \neq \mathcal{T}_\beta$. Consider the product topology $\mathcal{U}$ on $\prod_{\alpha \in \Omega} R_\alpha$, where R_α is endowed with $\mathcal{T}_\alpha$ when $\alpha \neq \beta$ and R_β is endowed with $\mathcal{T}'$. Obviously, $\mathcal{U} \leq \prod_{\alpha \in \Omega} \mathcal{T}_\alpha$ and $\mathcal{U} \neq \prod_{\alpha \in \Omega} \mathcal{T}_\alpha$, a contradiction.
$\Leftarrow$: Denote by $\pi_\alpha (\alpha \in \Omega)$ the projection of $\prod_{\alpha \in \Omega} R_\alpha$ on R_α. By definition of the product topology, $\prod_{\alpha \in \Omega} \mathcal{T}_\alpha$ is the coarsest topology on $\prod_{\alpha \in \Omega} R_\alpha$ for which the projections $\pi_\alpha (\alpha \in \Omega)$ are continuous.

Let $\mathcal{U}$ be a ring topology on $\prod_{\alpha \in \Omega} R_\alpha$, $\mathcal{U} \leq \prod_{\alpha \in \Omega} \mathcal{T}_\alpha$ and $\beta \in \Omega$. Since

$$\mathcal{U} \restriction_{R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}} \leq \Big(\prod_{\alpha \in \Omega} \mathcal{T}_\alpha \Big) \restriction_{R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}},$$

it follows that $\mathcal{U} \restriction_{R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}} = (\prod_{\alpha \in \Omega} \mathcal{T}_\alpha) \restriction_{R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}}$ by minimality of $(R_\beta, \mathcal{T}_\beta)$.

Then the family $\{V \times \prod_{\gamma \neq \beta} \{0_\gamma\}\}$ when V runs all neighborhoods of zero of $(R_\beta, \mathcal{T}_\beta)$ is a fundamental system of neighborhoods of zero of

$$\Big(R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}, \quad \mathcal{U} \restriction_{R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}} \Big).$$

Since $R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}$ is an ideal with identity of $\prod_{\alpha \in \Omega} R_\alpha$, the topological ring $(\prod_{\alpha \in \Omega} R_\alpha, \mathcal{U})$ is a direct sum of ideals $R_\beta \times \prod_{\gamma \neq \beta} \{0_\gamma\}$ and $\{0_\beta\} \times \prod_{\gamma \neq \beta} R_\gamma$. Let V be a neighborhood of zero of $(R_\beta, \mathcal{T}_\beta)$. Then $V \times \prod_{\gamma \neq \beta} R_\gamma$ be a neighborhood of zero of $(\prod_{\alpha \in \Omega} R_\alpha, \mathcal{U})$ and $\pi_\beta(V \times \prod_{\gamma \neq \beta} R_\gamma) = V$.

We have proved that π_β is a continuous function from $(\prod_{\alpha \in \Omega} R_\alpha, \mathcal{U})$ to $(R_\beta, \mathcal{T}_\beta)$. It follows that $\prod_{\alpha \in \Omega} \mathcal{T}_\alpha \leq \mathcal{U}$ and so $\mathcal{U} = \prod_{\alpha \in \Omega} \mathcal{T}_\alpha$. □

Corollary 4.4. *A left linearly compact semisimple ring is minimal if and only if has no direct summands of the form $M(n, \Delta)$, where Δ is a division ring which does not admit a nondiscrete Hausdorff ring topology.*

Proof. This follows from Theorems 4.1, 4.3 and the Theorem of Leptin (see [22, Theorem 13, p. 258]) about the structure of left linearly compact semisimple rings. □

Corollary 4.5. *A semisimple linearly compact ring $(R, \mathcal{T})$ having no ideals isomorphic to matrix rings over infinite division rings is minimal.*

5. Completely simple endomorphism rings of modules

The endomorphism ring of a right R-module M is denoted by $\mathrm{End}\,(M_R)$.

Lemma 5.1. *Let M be a divisible, torsion-free module over a commutative domain R and K the field of fractions of R. The additive group of M has a structure of a vector K-space such that R-endomorphisms of M are exactly the K-linear transformations.*

Proof. We define a structure of a right vector K-space as follows: if $\frac{a}{b} \in K$ and $m \in M$, then there exists a unique $x \in M$ such that $ma = xb$; set $m \circ \frac{a}{b} = x$. Moreover, if $\frac{a}{b} = \frac{c}{d}$ and $0 \neq m \in M$, then $m \circ \frac{a}{b} = m \circ \frac{c}{d}$. Indeed, if $m \circ \frac{a}{b} = x$ and $m \circ \frac{c}{d} = y$, then $mad = xbd$ and $mbc = ybd$ which means that $xbd = ybd$, hence $x = y$.

Let $\alpha \in \mathrm{End}\,(M_R)$, $\frac{a}{b} \in K$, $m \in M$. By definition, $am = b(\frac{a}{b} \circ m)$, hence, $a\alpha(m) = b\alpha(\frac{a}{b} \circ m)$, which means that $\alpha(\frac{a}{b} \circ m) = \frac{a}{b} \circ \alpha(m)$, so α is a K-linear transformation. Note that, if $a \in R$ and $m \in M$, then $m \circ \frac{a}{1} = ma$.

Conversely, if α is a K-linear transformation, $a \in R$, $m \in M$, then

$$\alpha(\tfrac{a}{1} \circ m) = \tfrac{a}{1} \circ \alpha m,$$

i.e. $\alpha(am) = a\alpha(m)$. We have proved that every K-linear transformation is an right R-module homomorphism. □

Remark 5.2. The center $Z(R)$ of a weakly simple ring R is a domain.

Remark 5.3. For every right R-module M the underlying group $M(+)$ is a discrete left topological $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$-module.

Indeed, $T(m)(m) = 0$ for every $m \in M$. Moreover, $\mathrm{End}\,(M_R)\{0\} = \{0\}$, so M is a discrete left topological $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$-module.

Theorem 5.4. *Let M_R be a module over a commutative ring R.*

If the topological ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is weakly simple, then:

(i) *$P = \{r \in R \mid Mr = 0\}$ is a prime ideal of R.*
(ii) *M is a vector space over the field K of fractions of R/P and the R-endomorphisms of M are exactly the K-linear transformations.*

Conversely, if M_R is an R-module and are satisfied (i) and (ii), then the ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is a weakly simple topological ring.

Proof. $\Rightarrow$: If $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is weakly simple, then the mapping:

$$\alpha_r : M \to M, \qquad m \mapsto mr \qquad (r \in R) \tag{5.1}$$

is an R-module homomorphism and $\alpha_r \in Z(=$ the center of $\mathrm{End}\,(M_R))$.

First we show that the part (i) holds. Indeed, if $a, b \in R$ and $ab = 0$, then $\alpha_a \alpha_b = 0$ (see (5.1)). Thus $(\mathrm{End}\,(M_R)\alpha_a) \cdot (\mathrm{End}\,(M_R)\alpha_b) = 0$, so

$$(\overline{\mathrm{End}\,(M_R)\alpha_a}) \cdot (\overline{\mathrm{End}\,(M_R)\alpha_b}) = 0.$$

Since $\mathrm{End}\,(M_R)$ is weakly simple, one of them, say $\overline{\mathrm{End}\,(M_R)\alpha_a}$, is zero. This implies that $\alpha_a = 0$, hence $a \in P$.

(ii) The structure of R/P-module on M is defined as follows: if $r \in R$ and $m \in M$, then put $M(r + P) = mr$.

Note that M is a torsion-free right R/P-module. Assume that $m(r+P) = 0$, where $0 \neq r + P \in R/P$ and $0 \neq m \in M$. Then $mr = 0 = \alpha_r(m)$ (see (5.1)). Thus $\mathrm{End}\,(M_R)\alpha_r(m) = 0$. It follows that $\big(\overline{\mathrm{End}\,(M_R)\alpha_r}\big)(m) = 0$ by Remark 5.3. Since $\mathrm{End}\,(M_R)$ is weakly simple

$$\overline{\mathrm{End}\,(M_R)\alpha_r} = \mathrm{End}\,(M_R).$$

We obtained that $\mathrm{End}\,(M_R)(m) = 0$, so $m = 0$, a contradiction.

Under this convention R-submodules are exactly R/P-submodules and R-endomorphisms are exactly R/P-endomorphisms.

The module M is a divisible R/P-module. Indeed, if $0 \neq r+P \in R/P$, then $0 \neq M(r + P) = Mr$. Suppose that $Mr \neq M$. Consider

$$I = \{\alpha \in \mathrm{End}\,(M_R) \mid \alpha(M) \subseteq Mr\}.$$

Since Mr is a fully invariant submodule, I is a two-sided ideal of the ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$.

The ideal I is closed. Indeed, let $\alpha \in \overline{I}$. If $m \in M$, then there exists $\beta \in I$ such that $\alpha - \beta \in T(m)$. Clearly, $\alpha(m) = \beta(m) \in Mr$ and so $\alpha \in I$. We have proved that I is closed.

Since $1_M \notin I$, $I = 0$. It follows that $\alpha_r = 0$ (see (5.1)), a contradiction.

The module M has a structure of a right K-vector space and $\mathrm{End}\,(M_R)$ is exactly the ring of endomorphisms of M by Lemma 5.1.

The converse follows from Theorem 4.1. $\square$

A characterization of completely simple topological ring $\mathrm{End}\,(M_R)$ is given by the following.

Theorem 5.5. *Let M_R be a module over a commutative ring R. The topological ring $(\mathrm{End}\,(M_R), \mathcal{T}_{fin})$ is completely simple if and only are satisfied the conditions (i) and (ii) of Theorem 5.4 and either*

(i) *M is finite or*
(ii) *M is infinite and the dimension of M over the field K is infinite.*

Proof. $\Rightarrow$: According to Theorem 5.4, the ideal P is prime and the topology of $\mathrm{End}\,(M_R)$ coincide with the finite topology of $\mathrm{End}\,(M_K)$, where K is the field of fractions of R/P. If M is finite, we have the part (i). Assume that

M is infinite. If R/P is finite, then the dimension of M over K is infinite. Suppose that R/P is infinite and $\dim_K(M) = n < \aleph_0$. Then M is isomorphic to $M(n, K)$. Since K is an infinite field, it admits a nondiscrete ring topology (see [13]) and we obtain a contradiction because $\mathrm{End}\,(M_R)$ is a discrete ring. Consequently $\dim_K(M)$ is infinite.
$\Leftarrow$ This follows from Theorems 4.1 and 5.4. □

Corollary 5.6. *The topological ring* $(\mathrm{End}\,(A), \mathcal{T}_{fin})$ *of an abelian group* A *is completely simple if and only one of the following conditions holds:*

(i) *A is a elementary abelian p-group.*
(ii) *A is a divisible torsion-free group of infinite rank.*

Acknowledgements. *Supported by UAEU UPAR (9) 2017 Grant G00002599.*

References

[1] A. V. Arkhangelskii and V. I. Ponomarev, Osnovy obshchei topologii v zadachakh i uprazhneniyakh, Izdat. Nauka, Moscow, 1974.
[2] V. I. Arnautov, S. T. Glavatsky and A. V. Mikhalev, Introduction to the theory of topological rings and modules, vol. 197 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1996.
[3] V. I. Arnautov and M. I. Ursul, Uniqueness of a linearly compact topology in rings, Mat. Issled. 53 (1979), 6–14, 221.
[4] N. Bourbaki, Kommutativnaya algebra, Izdat. Mir, Moscow, 1971. Èlementy Matematiki, Vyp. XXVII, XXVIII, XXX, XXXI. [Foundations of Mathematics, No. XXVII, XXVIII, XXX, XXXI], Translated from the French by A. A. Belskii, Edited by E. S. Golod.
[5] N. Bourbaki, General topology. Chapters 1–4. Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998.
[6] N. Bourbaki, Obshchaya topologiya. Izdat. Nauka, Moscow, 1975. Ispolʹzovanie veshchestvennykh chisel v obshchei topologii. Funktsionalʹnye prostranstva. Svodka rezulʹtatov. Slovarʹ. [Application of real numbers in general topology. Functional spaces. Resumé of results. Vocabulary], Translated from the third French edition by S. N. Kračkovskii, Edited by D. A. Raikov.
[7] D. Dikranjan, Minimal topological rings, Serdica 8, no. 2 (1982), 149–165.
[8] R. Engelking, General topology, vol. 6 of Sigma Series in Pure Mathematics. Heldermann Verlag, Berlin, ii ed., 1989.
[9] H. Freudenthal, Einige sätze über topologische gruppen, Ann. of Math. (2) 37, no. 1 (1936), 46–56. 1936.
[10] E. D. Gaughan, Topological group structures of infinite symmetric groups, Proc. Nat. Acad. Sci. U.S.A. 58 (1967), 907–910.
[11] M. I. Graev, Theory of topological groups. I. Norms and metrics on groups. Complete groups. Free topological groups, Uspehi Matem. Nauk (N.S.) 5, no. 2 (36) (1950), 3–56.
[12] E. Hewitt and K. A. Ross, Abstract harmonic analysis. Vol. I, vol. 115 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin-New York, second ed., 1979.

[13] M. Hochster and J. O. Kiltinen, Commutative rings with identity have ring topologies, Bull. Amer. Math. Soc. 76 (1970), 419–420.

[14] A. Hulanicki, On locally compact topological groups of power of continuum, Fund. Math. 44 (1957), 156–158.

[15] N. Jacobson, Totally disconnected locally compact rings, Amer. J. Math. 58, no. 2 (1936), 433–449.

[16] N. Jacobson, A note on topological fields, Amer. J. Math. 59, no. 4 (1937), 889–894.

[17] N. Jacobson, Structure of rings, American Mathematical Society, Colloquium Publications, vol. 37, American Mathematical Society, 190 Hope Street, Prov., R. I., 1956.

[18] F. B. Jones, On the first countability axiom for locally compact Hausdorff spaces, Colloq. Math. 7 (1959), 33–34.

[19] I. Kaplansky, Topological rings, Amer. J. Math. 69 (1947), 153–183.

[20] I. Kaplansky, Selected papers and other writings, Springer Collected Works in Mathematics, Springer, New York, 2013.

[21] T. Y. Lam, A first course in noncommutative rings, vol. 131 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.

[22] H. Leptin, Linear kompakte moduln und ringe, Math. Z. 62 (1955), 241–267.

[23] R. D. Mauldin, ed., The Scottish Book, Birkhäuser/Springer, Cham, second ed., 2015.

[24] A. F. Mutylin, Completely simple commutative topological rings, Mat. Zametki 5 (1969), 161–171.

[25] L. S. Pontryagin, Continuous groups, Nauka, Moscow, fourth ed., 1984.

[26] L. Skornjakov, Einfache lokal bikompakte ringe, Math. Z. 87 (1965), 241–251.

[27] M. Ursul, Topological rings satisfying compactness conditions, vol. 549 of Mathematics and its Applications. Kluwer Academic Publishers, Dordrecht, 2002.

[28] R. Ware and J. Zelmanowitz, Simple endomorphism rings, Amer. Math. Monthly 77 (1970), 987–989.

[29] S. Warner, Topological fields, vol. 157 of North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 1989. Notas de Matemática [Mathematical Notes], 126.

[30] S. Warner, Topological rings, vol. 178 of North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 1993.

[31] D. Zelinsky, Linearly compact modules and rings, Amer. J. Math. 75 (1953), 79–90.

