APPLIED GENERAL TOPOLOGY

Fixed point theorems for nonlinear contractions with applications to iterated function systems

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ABSTRACT

We introduce a new type of nonlinear contraction and present some fixed point results without using continuity or semi-continuity. Our result complement, extend and generalize a number of fixed point theorems including the well-known Boyd and Wong theorem [On nonlinear contractions, Proc. Amer. Math. Soc. 20(1969), 458–464]. Also we discuss an application to iterated function systems.

1. INTRODUCTION AND PRELIMINARIES

In 1981, Hutchinson [12] introduced the concept of iterated function system (IFS) and self-similarity. A set is said to be self-similar if it is made up of a finite transformed copies of itself. Self-similar sets are special class of fractals and there are no objects in nature which have exact structures of self-similar sets. These sets are perhaps the simplest and the most basic structures in the theory of fractals. In recent years this area received great attention of many mathematicians, scientists and a huge developments took place (cf. [1, 2, 3, 4, 5, 7, 10, 11, 14, 18, 22, 23, 24, 25, 27, 28, 29, 31, 32, 33]).

The purpose of this paper is to introduce a new class of nonlinear contractions and present some fixed point results for this class of mapping without

using any kind of continuity. Our result complement, extend and generalize a number of fixed point theorems in the literature. We also discuss an application of our results to iterated function systems.

The following result which generalizes the classical Banach contraction principle (BCP) is due to Boyd and Wong [6]:

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ a self-mapping such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \varphi(d(x, y)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right on $[0, \infty)$, and satisfies $\varphi(t) < t$ for all $t > 0$. Then T has a unique fixed point in X.

Jachymski [13] established equivalence between various φ-contractive type conditions (see also [22]).

Definition 1.2 ([13])**.** Let (X, d) be a metric space and $\varphi : [0, \infty) \to [0, \infty)$ a function such that $\varphi(t) < t$ for $t > 0$. A self-mapping $T : X \to X$ is said to be φ-contractive if

$$d(Tx, Ty) \leq \varphi(d(x, y))$$

for all $x, y \in X$.

Theorem 1.3 ([13])**.** *Let (X, d) be a metric space and $T : X \to X$ a self-mapping. The following statements are equivalent:*

(i) *There exists an increasing and right continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that T is φ-contractive;*

(ii) *There exists a continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) > 0$ for $t > 0$, such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for all $x, y \in X$;

(iii) *There exists an upper semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ such that T is φ-contractive;*

(iv) *There exists a function $\varphi : [0, \infty) \to [0, \infty)$ with $\limsup_{s \to t} \varphi(s) < t$ for all $t > 0$ such that T is φ-contractive;*

(v) *There exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\lim_{n \to \infty} \varphi^n(t) = 0$ for all $t \in [0, \infty)$ and T is φ-contractive;*

(vi) *There exists a strictly increasing and continuous $\varphi : [0, \infty) \to [0, \infty)$ such that T is φ-contractive;*

On the other hand, Suzuki [30] obtained the following forceful generalization of the BCP:

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \to X$ a self-mapping. Define a non decreasing function $\theta : [0, 1) \to (1/2, 1]$ such that*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq (\sqrt{5} - 1)/2, \\ (1 - r)r^{-2}, & \text{if } (\sqrt{5} - 1)/2 \leq r \leq 2^{-1/2}, \\ (1 + r)^{-1}, & \text{if } 2^{-1/2} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that for all $x, y \in X$

(1.1) $\qquad \theta(r)d(x, Tx) \leq d(x, y) \ \text{ implies } d(Tx, Ty) \leq rd(x, y).$

Then T has a unique fixed point in X.

The above theorem has been extended and generalized by many authors in various ways (cf. [8, 9, 15, 16, 17, 19, 20, 26] and elsewhere).

2. Suzuki type generalized φ-contractive mappings

In this section, we present a generalization of Theorem 1.1 which also extends Theorem 1.4. We begin with the following definition:

Definition 2.1. Let (X, d) be a metric space. A self-mapping $T : X \to X$ will be called a Suzuki type generalized φ-contractive if for all $x, y \in X$,

(2.1) $\qquad \dfrac{1}{2}d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq \varphi(m(x, y)),$

where $m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\varphi : [0, \infty) \to [0, \infty)$ is a function such that $\varphi(t) < t$ for all $t > 0$ and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$.

The following theorem is the main result of this section.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ a Suzuki type generalized φ-contractive mapping. Then T has a unique fixed point in X.*

Proof. Pick $x_0 \in X$ arbitrary and define a sequence $\{x_n\}$ by $x_n = T^n x = Tx_{n-1}$ for all $n \in \mathbb{N}$.

Since $\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$ by (2.1), we have

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \leq \varphi(m(x_n, x_{n+1})) \\ (2.2) \qquad &= \varphi(\max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) \\ &= \varphi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}). \end{aligned}$$

for all $n \in \mathbb{N}$.

Set $a_n = d(x_n, x_{n+1})$ then $a_n \geq 0$. If there exists some $n_0 \in \mathbb{N}$ such that $a_{n_0} = d(x_{n_0}, x_{n_0+1}) = 0$, then since $\frac{1}{2}d(x_{n_0}, x_{n_0+1}) \leq d(x_{n_0}, x_{n_0+1})$ by (2.1), we have

$$a_{n_0+1} = d(x_{n_0+1}, x_{n_0+2}) \leq \varphi(m(x_{n_0}, x_{n_0+1})).$$

But $\varphi(0) = 0$ so, $a_{n_0+1} = 0$ and

$$0 \leq a_{n_0+1} = d(Tx_{n_0}, Tx_{n_0+1}) \leq \varphi(m(x_{n_0}, x_{n_0+1})).$$

Hence the sequence $\{a_n\}$ monotone decreasing and bounded below and $a_n = 0$ for all $n \geq n_0$. Therefore

$$\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0 \quad \text{for each} \quad x \in X.$$

If $a_n > 0$ for every $n \in \mathbb{N}$ then since $\varphi(t) < t$ for $t > 0$ by (2.2), we get

$$0 < a_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \leq \varphi(m(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

we obtain

$$0 < a_{n+2} \leq \varphi(a_{n+1}) < a_{n+1} \leq \varphi(a_n) < a_n$$

Hence $\{a_n\}$ and $\{\varphi(a_n)\}$ are strictly decreasing sequences, which are bounded below. So, $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} \varphi(a_n)$ exist. Suppose $0 < a = \lim_{n\to\infty} a_n$ and $a_n = a + \varepsilon_n$ $(\varepsilon_n > 0)$. If $\limsup_{s\to t^+} \varphi(s) < t$ for all $t > 0$, then $\forall \{t_n\}$, $t_n \downarrow a^+$ (as $n \to \infty$); $\limsup_{t_n \to a^+} \varphi(t_n) < a$. Hence

$$\begin{aligned} 0 < a &= \lim_{n\to\infty} a_{n+1} \leq \lim_{n\to\infty} \varphi(a_n) \leq \lim_{n\to\infty} \sup_{s\in(a,a_{n+1})} \varphi(s) \\ &= \lim_{\varepsilon_{n+1}\to 0^+} \sup_{s\in(a,a+\varepsilon_{n+1})} \varphi(s) \leq \lim_{\varepsilon\to 0^+} \sup_{s\in(a,a+\varepsilon)} \varphi(s) < a, \end{aligned}$$

a contradiction and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0$ for each $x \in X$.

Now, we show that the sequence $\{x_n\}$ is a Cauchy. Suppose $\{x_n\}$ is not Cauchy. Then there exists an $\varepsilon > 0$ and integers $m_k, n_k \in \mathbb{N}$ such that $m_k > n_k > k$ and

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \quad \text{and} \quad d(x_{n_k}, x_{m_k-1}) < \varepsilon.$$

Hence for each $k \in \mathbb{N}$, we have

$$\begin{aligned} \varepsilon &\leq d(x_{n_k}, x_{m_k}) \\ &\leq d(x_{n_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{m_k}) \\ &< \varepsilon + d(x_{m_k-1}, x_{m_k}). \end{aligned}$$

Since $\lim_{k\to\infty} d(x_{m_k-1}, x_{m_k}) = 0$, we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon.$$

Note that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. So, there exists some $k \in \mathbb{N}$ such that $\frac{1}{2} d(x_{n_k}, x_{n_k+1}) \leq d(x_{n_k}, x_{m_k})$ for $m_k > n_k \geq k$. Now by (2.1), we have

$$d(Tx_{n_k}, Tx_{m_k}) \leq \varphi(m(x_{n_k}, x_{m_k})).$$

By the triangle inequality

$$\begin{aligned} d(x_{n_k}, x_{m_k}) &\leq d(x_{n_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{m_k+1}) + d(x_{m_k}, x_{m_k+1}) \\ &= d(x_{n_k}, x_{n_k+1}) + d(Tx_{n_k}, Tx_{m_k}) + d(x_{m_k}, x_{m_k+1}) \\ &\leq a_{n_k} + \varphi(m(x_{n_k}, x_{m_k})) + a_{m_k} \\ &= a_{n_k} + \varphi(\max\{d(x_{n_k}, x_{m_k}), a_{n_k}, a_{m_k}\}) + a_{m_k} \end{aligned}$$

Letting $k \to \infty$ and using $\varphi(t) < t$ and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$, we obtain

$$\varepsilon = \lim_{k \to \infty} d(x_{n_k}, x_{m_k}) \leq \lim_{k \to \infty} \varphi(d(x_{n_k}, x_{m_k})) \leq \lim_{\varepsilon_1 \to 0^+} \sup_{s \in (\varepsilon, \varepsilon + \varepsilon_1)} \varphi(s) < \varepsilon,$$

a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\{x_n\}$ has a limit in X. Call it z. Now for all $n \in \mathbb{N}$, we show that

(2.3) either $\dfrac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, z)$ or $\dfrac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, z)$.

Arguing by contradiction, we suppose that for some $n \in \mathbb{N}$

$$d(x_n, z) < \frac{1}{2}d(x_n, x_{n+1}) \text{ and } d(x_{n+1}, z) < \frac{1}{2}d(x_{n+1}, x_{n+2}).$$

By the triangle inequality and (2.2)

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq d(x_n, z) + d(x_{n+1}, z) \\
&< \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}) \\
&< \frac{1}{2}[d(x_n, x_{n+1}) + d(x_n, x_{n+1})] = d(x_n, x_{n+1}),
\end{aligned}$$

a contradiction. Thus for all $n \in \mathbb{N}$ (2.3) holds.

In the first case, since $\dfrac{1}{2}d(x_n, x_{n+1}) = \dfrac{1}{2}d(x_n, Tx_n) \leq d(x_n, z)$, by (2.1), we have

$$d(x_{n+1}, Tz) = d(Tx_n, Tz) \leq \varphi(d(x_n, z)).$$

Letting $n \to \infty$ and using $\varphi(t) < t$ and $\limsup_{s \to t^+} \varphi(s) < t$ for all $t > 0$, we obtain

$$d(z, Tz) = \lim_{n \to \infty} d(x_{n+1}, Tz) \leq \lim_{n \to \infty} \varphi(d(x_{n+1}, Tz)) < d(z, Tz),$$

a contradiction unless $Tz = z$. Similarly, in the other case we can deduce that $Tz = z$. Uniqueness of fixed point follows easily. $\qquad\square$

Corollary 2.3. *Let (X, d) be a complete metric space, $\varphi : [0, \infty) \to [0, \infty)$ an increasing and right continuous function such that $\varphi(t) < t$ for all $t > 0$ and $T : X \to X$ a Suzuki type generalized φ-contractive mapping. Then T has a unique fixed point in X.*

Proof. It may be completed by following the proof of Theorem 2.2. $\qquad\square$

Corollary 2.4. *Let (X, d) be a complete metric space and $T : X \to X$ a self-mapping such that for all $x, y \in X$*

$$\frac{1}{2}d(x, Tx) \text{ implies } d(Tx, Ty) \leq \varphi(m(x, y)),$$

where φ is as in Theorem 1.1. Then T has a unique fixed point in X.

Proof. It comes from Theorem 2.2 when $\varphi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right on $[0, \infty)$, and $\varphi(t) < t$ for all $t > 0$. $\qquad\square$

Corollary 2.5. *Theorem 1.1.*

Example 2.6. Let $X = \{0, 2\} \cup \{1, 3, 5, ...\}$ be endowed with the usual metric d. Then (X, d) is a complete metric space. Define $T : X \to X$ and $\varphi : [0, \infty) \to [0, \infty)$ by

$$Tx = \begin{cases} 0, & \text{if } x = 5, \\ 2, & \text{if } x = 7, \\ 1, & \text{otherwise}. \end{cases} \quad \text{and} \quad \varphi(t) = \begin{cases} \dfrac{t^2}{2}, & \text{if } t \leq 1, \\ t - \dfrac{1}{3}, & \text{if } t > 1. \end{cases}$$

For $x = 5$ and $y = 7$, we have

$$d(Tx, Ty) = d(0, 2) = 2 > \frac{5}{3} = \varphi(2) = \varphi(d(5, 7)) = \varphi(d(x, y)).$$

Therefore T does not satisfy Theorem 1.1.

On the other hand

$$\frac{1}{2}d(5, T5) = \frac{1}{2}d(5, 0) = \frac{5}{2} > 2 = d(5, 7) \text{ and}$$

$$\frac{1}{2}d(7, T7) = \frac{1}{2}d(7, 2) = \frac{5}{2} > 2 = d(5, 7).$$

Therefore Theorem 2.2 is applicable and $z = 1$ is the unique fixed point of T.

Further, we get the same conclusion when

$$\varphi(t) = \begin{cases} \dfrac{t^2}{2}, & \text{if } t \leq 1, \\ t - \frac{1}{4}, & \text{if } t > 1. \end{cases}$$

We note that in this case φ is not upper semicontinuous on $[0, \infty)$.

3. Applications to Fractal spaces

Let (X, d) be an metric space and $C(X)$, the collection of all nonempty compact subsets of X. Define

(a) $d(A, B) := \inf\{d(a, b) : a \in A, \ b \in B\}$ (the distance between two sets).
(b) $\delta(A, B) := \sup\{d(x, B) : x \in A\}$.

The Hausdorff metric induced by d is defined by

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \ \sup_{y \in B} d(y, A)\right\} = \max\{\delta(A, B), \delta(B, A)\}$$

for all $A, B \in C(X)$, where $d(x, B) = \inf_{y \in B} d(x, y)$.

Hutchinson [12] and Barnsley [1] initiated an ingenius way to define and construct fractals as compact invariant subsets of an abstract complete metric space with respect to the union of contractions f_i, $i = 1, 2, 3, \ldots n$. Hutchinson showed that the operator

$$F(A) = f_1(A) \cup f_2(A) \cup ... \cup f_n(A), \quad A \subset X,$$

is a contraction with respect to the Hausdorff distance. Thus, the contraction mapping principle can be applied to the iteration of Hutchinson operator F.

Consequently, whatever the initial image is chosen to start the iteration under the IFS, for example A_0, the generated sequence

$$A_{k+1} = F(A_k), \quad k = 0, 1, ...$$

will tend towards a distinguished image, the attractor A_∞ of the IFS. Moreover, this image is invariant, i.e., $F(A_\infty) = A_\infty$.

Now onwards $\varphi : [0, \infty) \to [0, \infty)$ a non-decreasing continuous function. The following lemma which is modeled on the pattern of [22, Lem. 3.2] is a crucial result of this section.

Lemma 3.1. *Let (X, d) be a metric space and $T : X \to X$ a continuous Suzuki type generalized φ-contractive mapping. Then*

$$\frac{1}{2} H(A, T(A)) \leq H(A, B) \text{ implies } \quad H(F_T(A), F_T(B)) \leq \varphi(M_T(A, B))$$

for all $A, B \in C(X)$, where $M_T(A, B) = \max\{H(A, B), H(A, T(A)), H(B, T(B))\}$. That is, $F_T : C(X) \to C(X)$ is also a Suzuki type generalized φ-contractive (with the same φ), where

$$\forall \ D \in C(X), \quad F_T(D) := T(D).$$

Proof. Following the proof of Lemma 3.2 [22], for all $x \in A$ and $y \in B$, we have

$$\sup_{x \in A} \inf_{y \in B} \varphi(d(x, y)) \leq \sup_{x \in A} \varphi(\inf_{y \in B} d(x, y)) \leq \varphi(M_T(A, B)); \text{ and}$$

$$\sup_{y \in B} \inf_{x \in A} \varphi(d(x, y)) \leq \sup_{y \in B} \varphi(\inf_{x \in A} d(x, y)) \leq \varphi(M_T(A, B)).$$

Further, for all $x \in A, \quad y \in B$

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \text{ implies } \frac{1}{2} \delta(A, T(A)) \leq \frac{1}{2} H(A, T(A)) \leq H(A, B) \text{ and}$$

$$\frac{1}{2} d(y, Ty) \leq d(x, y) \text{ implies } \frac{1}{2} \delta(B, T(B)) \leq \frac{1}{2} H(B, T(B)) \leq H(A, B).$$

Now

$$\delta(F_T(A), F_T(B)) = \max_{Tx \in F(A)} \min_{Ty \in F(B)} d(Tx, Ty) = \max_{x \in A} \min_{y \in B} d(Tx, Ty).$$

Since T is Suzuki type generalized φ-contractive mapping, $\frac{1}{2}\delta(A, T(A)) \leq \frac{1}{2} H(A, T(A))$ implies

$$\delta(F_T(A), F_T(B)) \leq \sup_{x \in A} \inf_{y \in B} \varphi(d(x, y)) \leq \varphi(M_T(A, B)).$$

Similarly, $\frac{1}{2}\delta(B, T(B)) \leq \frac{1}{2} H(B, T(B))$ implies

$$\delta(F_T(B), F_T(A)) \leq \sup_{y \in B} \inf_{x \in A} \varphi(d(x, y)) \leq \varphi(M_T(A, B)).$$

Since $H(A, B) = H(B, A)$ (symmetric), and

$$H(F_T(A), F_T(B)) = \max\{\delta(F_T(A), F_T(B)), \delta(F_T(B), F_T(A))\},$$

we conclude that

$$\frac{1}{2}H(A, T(A)) \leq H(A, B) \text{ implies } H(F_T(A), F_T(B)) \leq \varphi(M_T(A, B)),$$

for all $A, B \in C(X)$. Therefore F_T is Suzuki type generalized φ-contractive mapping. $\qquad\square$

Lemma 3.2 ([1])**.** *Let (X, d) be a complete metric space. Then $(C(X), H)$ is a complete metric space.*

Lemma 3.3. *Let (X, d) be a metric space and $T_n : C(X) \rightarrow C(X)$ $(n = 1, 2, 3, ..., p)$ continuous Suzuki type generalized φ-contractive mappings, i.e., for all $A, B \in C(X)$.*

$$\frac{1}{2}H(A, T_n(A)) \leq H(A, B) \text{ implies } H(T_n(A), T_n(B)) \leq \varphi_n(M_{T_n}(A, B)).$$

Define $T : C(X) \rightarrow C(X)$ by $T(A) = T_1(A) \cup T_2(A) \cup ... \cup T_p(A) = \bigcup\limits_{n=1}^{p} T_n(A)$ for each $A \in C(X)$. Then T also satisfies

$$\frac{1}{2}H(A, T(A)) \leq H(A, B) \text{ implies } H(T(A), T(B)) \leq \eta(M_T(A, B)).$$

for all $A, B \in C(X)$, where $\eta = \max\{\varphi_n : n = 1, 2, 3..., p\}$.

Proof. We shall prove this by induction. For $n = 1$, the statement is obviously true. For $n = 2$, we have

$$
\begin{aligned}
H(T(A), T(B)) &= H(T_1(A) \cup T_2(A), T_1(B) \cup T_2(B)) \\
&\leq \max\{H(T_1(A), T_1(B)), H(T_2(A), T_2(B))\}.
\end{aligned}
$$

Since each T_1 and T_2 are Suzuki type generalized φ-contractive, that is

$$\frac{1}{2}H(A, T_1(A)) \leq H(A, B) \text{ implies } H(T_1(A), T_1(B)) \leq \varphi_1(M_{T_1}(A, B))$$

$$\frac{1}{2}H(A, T_2(A)) \leq H(A, B) \text{ implies } H(T_2(A), T_2(B)) \leq \varphi_2(M_{T_2}(A, B)),$$

we get

$$
\begin{aligned}
H(T(A), T(B)) &\leq \max\{\varphi_1(M_{T_1}(A, B)), \varphi_2(M_{T_2}(A, B))\} \\
&= \eta(\max\{H(A, B), H(A, T_1(A) \cup T_2(A)), H(B, T_1(B) \cup T_2(B))\}) \\
&= \eta(\max\{H(A, B), H(A, T(A)), H(B, T(B))\}) \\
&= \eta(M_T(A, B)),
\end{aligned}
$$

where $\eta = \max\{\varphi_1, \varphi_2\}$. $\qquad\square$

As a consequence of Theorem 2.2, and Lemmas 3.1 and 3.3, we get following result in fractal spaces.

Theorem 3.4. *Let (X, d) be a complete metric space and $T_n : C(X) \to C(X)$ continuous Suzuki type generalized φ-contractive mappings. Then the transformation $T : C(X) \to C(X)$ defined by $T(A) = \bigcup_{n=1}^{p} T_n(A)$ for each $A \in C(X)$ satisfies the following condition*

$$\frac{1}{2} H(A, T(A)) \leq H(A, B) \text{ implies } H(T(A), T(B)) \leq \eta(M(A, B)).$$

for all $A, B \in C(X)$, where $\eta = \max\{\varphi_n : n = 1, 2, 3..., p\}$.

Moreover,

(A): *T has a unique fixed point A in $C(X)$; and*
(B): $\lim_{n \to \infty} T^n(B) = A$ *for all $B \in C(X)$.*

Remark 3.5. In view of Rhoades [21], Theorems 2.2 and 3.4, generalizes certain results of [12, 23, 22, 32] and others.

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