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k-semistratifiable spaces and expansions of set-valued mappings

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Abstract

In this paper, the concept of k-upper semi-continuous set-valued mappings is introduced. Using this concept, we give characterizations of k-semistratifiable and k-MCM spaces, which answers a question posed by Xie and Yan [9].

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1. Introduction

Before stating the paper, we give some definitions and notations.

For a mapping $\phi : X \to 2^Y$ and $W \subseteq Y$, the symbols $\phi^{-1}[W]$ and $\phi^{\sharp}[W]$ stand for $\{x \in X : \phi(x) \bigcap W \neq \varnothing\}$ and $\{x \in X : \phi(x) \subseteq W\}$, respectively. A set-valued mapping $\phi : X \to 2^Y$ is *lower semi-continuous* (l.s.c) if $\phi^{-1}[W]$ is open in X for every open subset W of Y. Also, a set-valued mapping $\phi : X \to 2^Y$ is *upper semi-continuous* (u.s.c) if $\phi^{\sharp}[W]$ is open in X for every open subset W of Y. For mappings ϕ, $\phi^{'} : X \to 2^Y$, we express by $\phi \subseteq \phi^{'}$ if $\phi(x) \subseteq \phi^{'}(x)$ for each $x \in X$. An operator Φ assigning to each set-valued

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mapping $\phi : X \to 2^Y$, $\Phi(\phi) : X \to 2^Y$, Φ is called as a *preserved order operator* if $\Phi(\phi) \subseteq \Phi(\phi')$ whenever $\phi \subseteq \phi'$.

For a space Y, define

$$\mathcal{F}(Y) = \{F \subseteq Y : F \text{ is a nonempty closed set in } Y\}.$$

For a metric space (Y, ρ), a subset B of Y is called *bounded* if the diameter of B (with respect to ρ) is finite, and we define

$$\mathcal{B}(Y) = \{F \subseteq Y : F \neq \varnothing, F \text{ is closed and bounded in } Y\}.$$

A sequence $\{B_n\}_{n\in\mathbb{N}}$ of closed subsets of a space Y is called a *strictly increasing closed cover* [10] if $\bigcup_{n\in\mathbb{N}} B_n = Y$ and $B_n \subsetneq B_{n+1}$ for each $n \in \mathbb{N}$. For a space Y having a strictly increasing closed cover $\{B_n\}$, a subset B of Y is said to be *bounded* [10] (with respect to $\{B_n\}$) if $B \subseteq B_n$ for some $n \in \mathbb{N}$. Define

$$\mathcal{B}(Y; \{B_n\}) = \{F \subseteq Y : F \neq \varnothing, F \text{ is closed and bounded in } Y\}.$$

For a space Y with a strictly increasing closed cover $\{B_n\}$, a mapping $\phi : X \to \mathcal{B}(Y; \{B_n\})$ is called *locally bounded at* x if there exist a bounded set V of $(Y; \{B_n\})$ and a neighborhood O of x such that $O \subseteq \phi^\sharp[V]$; if ϕ is locally bounded at each $x \in X$, then ϕ is called *locally bounded* [10] on X. Let (Y, ρ) be a metric space. For a mapping $\phi : X \to \mathcal{F}(Y)$, define

$$U_\phi = \{x \in X : \phi \text{ is locally bounded at } \ x \ \text{ with respect to } \rho\}.$$

Similarly, Let Y has a strictly increasing closed cover $\{B_n\}$. We also define

$$U_\phi = \{x \in X : \phi \text{ is locally bounded at } \ x \text{ with respect to } \{B_n\})\}$$

for a mapping $\phi : X \to \mathcal{F}(Y)$.

Clearly, U_ϕ is an open set in X.

The insertions of functions are one of the most interesting problems in general topology and have been applied to characterize some classical cover properties. For example, J. Mack characterized in [5] countably paracompact spaces with locally bounded real-valued functions as follows:

Theorem 1.1 (J. Mack [5])**.** *A space X is countably paracompact if and only if for each locally bounded function $h : X \to \mathbb{R}$ there exists a locally bounded l.s.c. function $g : X \to \mathbb{R}$ such that $|h| \leq g$.*

C. Good, R. Knight and I. Stares [3] and C. Pan [6] introduced a monotone version of countably paracompact spaces, called monotonically countably paracompact spaces (MCP) and monotonically cp-spaces, respectively, and it was proved in [3, Proposition 14] that both these notions are equivalent. Also, C. Good, R. Knight and I. Stares [3] characterized monotonically countably paracompact spaces by the insertions of semi-continuous functions. Inspired by those results, K. Yamazaki [10] characterized MCP spaces by expansions of locally bounded set-valued mappings as follows:

Theorem 1.2 (K. Yamazaki [10]). *For a space X, the following statements are equivalent:*

(1) *X is MCP;*
(2) *for every space Y having a strictly increasing closed cover $\{B_n\}$, there exists a preserved order operator Φ assigning to each locally bounded mapping $\varphi : X \to \mathcal{B}(Y; \{B_n\})$, a locally bounded l.s.c. mapping $\Phi(\varphi) : X \to \mathcal{B}(Y; \{B_n\})$ with $\varphi \subseteq \Phi(\varphi)$;*
(3) *for every metric space Y, there exists a preserved order operator Φ assigning to each locally bounded set-valued mapping $\varphi : X \to \mathcal{B}(Y)$, a locally bounded l.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{B}(Y)$ such that $\varphi \subseteq \Phi(\varphi)$;*
(4) *there exists a preserved order operator Φ assigning to each locally bounded mapping $\varphi : X \to \mathcal{B}(\mathbb{R})$, a locally bounded l.s.c. mapping $\Phi(\varphi) : X \to \mathcal{B}(\mathbb{R})$ such that $\varphi \subseteq \Phi(\varphi)$;*
(5) *there exists a space Y having a strictly increasing closed cover $\{B_n\}$, there exists a preserved order operator Φ assigning to each each locally bounded mapping $\varphi : X \to \mathcal{B}(Y; \{B_n\})$, a locally bounded l.s.c. mapping $\Phi(\varphi) : X \to \mathcal{B}(Y; \{B_n\})$ such that $\varphi \subseteq \Phi(\varphi)$.*

Recently, Xie and Yan [9] gave the following characterizations of stratifiable and semistratifiable spaces by expansions of set-valued mappings along same lines, and asked whether there are similar characterizations for k-MCM and k-semistratifiable spaces.

Theorem 1.3 (Xie and Yan [9]). *For a space X, the following statements are equivalent:*

(1) *X is stratifiable(resp. semi-stratifiable);*
(2) *for every space Y having a strictly increasing closed cover $\{B_n\}$, there exists a preserved order operator Φ assigning to each set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\Phi(\varphi)$ is locally bounded(resp. bounded) at each $x \in U_\varphi$ and that $\varphi \subseteq \Phi(\varphi)$;*
(3) *for every metric space Y, there exists a preserved order operator Φ assigning to each set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\Phi(\varphi)$ is locally bounded (resp. bounded) at each $x \in U_\varphi$ and that $\varphi \subseteq \Phi(\varphi)$;*
(4) *there exists a preserved order operator Φ assigning to each set-valued mapping $\varphi : X \to \mathcal{F}(\mathbb{R})$, an l.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(\mathbb{R})$ such that $\Phi(\varphi)$ is locally bounded (resp. bounded) at each $x \in U_\varphi$ and that $\varphi \subseteq \Phi(\varphi)$;*
(5) *there exist a space Y having a strictly increasing closed cover $\{B_n\}$ and a preserved order operator Φ assigning to each set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\Phi(\varphi)$ is locally bounded (resp. bounded) at each $x \in U_\varphi$ and that $\varphi \subseteq \Phi(\varphi)$.*

Recently, Xie and Yan posed the following question:

Question 1.4 ([9, Question 3.3])**.** *Are there monotone set-valued expansions for k-stratifiable spaces and k-MCM along the same lines?*

The purposes of this paper is to attempt to answer this question by the concept of k-u.s.c set-valued mappings.

Throughout this paper, all spaces are assumed to be regular, and all undefined topological concepts are taken in the sense given Engelking [2].

2. Main results

In this section we shall give characterization of k-MCM and k-semi stratifiable spaces. The following concept plays an important role in this paper.

Definition 2.1. For a space Y with a strictly increasing closed cover $\{B_n\}$, a mapping $\phi : X \to \mathcal{B}(Y; \{B_n\})$ is called *k-upper semi-continuous* (k-u.s.c.) if for every compact subset K of X, $\phi(K)$ is bounded.

Obviously, for every space Y with a strictly increasing closed cover $\{B_n\}$ satisfying $B_n \subset \text{Int } B_{n+1}$ and mapping $\phi : X \to \mathcal{B}(Y; \{B_n\})$:

ϕ is u.s.c $\Rightarrow$ ϕ is locally bounded $\Rightarrow$ ϕ is k-u.s.c..

Firstly, we shall give the characterization of k-MCM by expansion of set-valued mappings. Peng and Lin gave the $k\beta$ characterization as following. They renamed the $k\beta$ as k-MCM in [7].

Proposition 2.2 ([7])**.** *For a space X, the following statements are equivalent:*

(1) *X is k-MCM;*
(2) *there is an operator U assigning to a decreasing sequence of closed sets $(F_j)_{j\in\mathbb{N}}$ with $\bigcap_{j\in\mathbb{N}} F_j = \varnothing$, a decreasing sequence of open sets $(U(n,(F_j)))_{n\in\mathbb{N}}$ such that*
  (i) *$F_n \subseteq U(n,(F_j))$ for each $n \in \mathbb{N}$;*
  (ii) *for any compact subset K in X, there is $n_0 \in \mathbb{N}$ such that $U(n_0,(F_j)) \bigcap K = \varnothing$;*
  (iii) *given two decreasing sequences of closed sets $(F_j)_{j\in\mathbb{N}}$ and $(E_j)_{j\in\mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$ and that $\bigcap_{j\in\mathbb{N}} F_j = \bigcap_{j\in\mathbb{N}} E_j = \varnothing$, then $U(n,(F_j)) \subseteq U(n,(E_j))$, for each $n \in \mathbb{N}$.*

Theorem 2.3. *For a space X, the following statements are equivalent:*

(1) *X is k-MCM;*
(2) *for every space Y having a strictly increasing closed cover $\{B_n\}$, there exists a preserved order operator Φ assigning to each locally bounded set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c. and k-u.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\varphi \subseteq \Phi(\varphi)$;*
(3) *for every metric space Y, there exists a preserved order operator Φ assigning to each locally bounded set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c and k-u.s.c set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\varphi \subseteq \Phi(\varphi)$;*

(4) *there exists a preserved order operator* Φ *assigning to each locally bounded set-valued mapping* $\varphi : X \to \mathcal{F}(\mathbb{R})$, *an l.s.c. and k-u.s.c. set-valued mapping* $\Phi(\varphi) : X \to \mathcal{F}(\mathbb{R})$ *such that* $\varphi \subseteq \Phi(\varphi)$;

(5) *there exists a space* Y *having a strictly increasing closed cover* $\{B_n\}$, *there exists a preserved order operator* Φ *assigning to each locally bounded set-valued mapping* $\varphi : X \to \mathcal{F}(Y)$, *an l.s.c. and k-u.s.c. set-valued mapping* $\Phi(\varphi) : X \to \mathcal{F}(Y)$ *such that* $\varphi \subseteq \Phi(\varphi)$.

Proof. The implications of (2)⇒(3)⇒(4)⇒ (5) are trivial.

(1)⇒ (2). Assume that X is a k-MCM space. Then there exists an operator U satisfying (i), (ii) and (iii) in Proposition 2.2.

Let Y be a space having a strictly increasing closed cover $\{B_n\}$. For each locally bounded set-valued mapping $\varphi : X \to \mathcal{F}(Y)$ and each $n \in \mathbb{N}$, define $F_{n,\varphi} = \overline{\{x \in X : \varphi(x) \nsubseteq B_n\}}$. Then we have that $\bigcap_{n\in\mathbb{N}} F_{n,\varphi} = \varnothing$. Indeed, since φ is locally bounded, for each $x \in X$ there exist an open neighborhood V of x and some $i \in \mathbb{N}$ such that $\varphi(y) \subseteq B_i$ for each $y \in V$, which implies that $V \cap F_{i,\varphi} = \varnothing$. It implies that $x \notin F_{i,\varphi}$ and $\bigcap_{n\in\mathbb{N}} F_{n,\varphi} = \varnothing$. Define $\Phi(\varphi) : X \to \mathcal{F}(Y)$ as follows: $\Phi(\varphi)(x) = B_1$ whenever $x \in X - U(1,(F_{n,\varphi}))$, $\Phi(\varphi)(x) = B_{i+1}$ whenever $x \in U(i,(F_{n,\varphi})) - U(i+1,(F_{n,\varphi}))$.

Then, $\Phi(\varphi)$ is lower semi-continuous. To see this, let W be an open subset of Y and put $k = \min\{i \in \mathbb{N} : W \cap B_i \neq \varnothing\}$. Then, one can easily check that $(\Phi(\varphi))^{-1}[W] = U(k-1,(F_{n,\varphi}))$ (we set $U(0,(F_{n,\varphi})) = X$). This implies that $\Phi(\varphi)$ is lower semi-continuous.

Let K be a compact subset of X, then there exists $k \in \mathbb{N}$ such that $K \bigcap U(k+1,(F_{n,\varphi})) = \varnothing$. It implies that $\Phi(\varphi)(K) \subset B_{k+1}$. Hence $\Phi(\varphi)$ is k-upper semi-continuous.

To show that $\varphi \subseteq \Phi(\varphi)$. For each $x \in X$, there exists some $i \in \mathbb{N}$ such that $x \in U(i-1,(F_{n,\varphi})) \setminus U(i,(F_{n,\varphi}))$(we set $U(0,(F_{n,\varphi})) = X$). Since $x \notin U(i,(F_{n,\varphi}))$, we have $x \notin F_{i,\varphi}$. Hence, $\varphi(x) \subseteq B_i = \Phi(\varphi)(x)$. This completes the proof of $\varphi \subseteq \Phi(\varphi)$.

Finally, to show that Φ is order-preserving, let $\varphi, \varphi' : X \to F(Y)$ be set-valued mappings such that $\varphi \subseteq \varphi'$. Then, $F_{i,\varphi} \subseteq F_{i,\varphi'}$ for each $i \in \mathbb{N}$, and therefore, by (iii) of Proposition 2.2, we have $U(i,(F_{n,\varphi})) \subseteq U(i,(F_{n,\varphi'}))$ for each $i \in \mathbb{N}$. For each $x \in X$. Then, $\Phi(\varphi')(x) = B_{k'}$ for some $k' \in \mathbb{N}$. This implies that $x \in U(k'-1,(F_{n,\varphi'})) \setminus U(k',(F_{n,\varphi'}))$. Similarly, $\Phi(\varphi)(x) = B_k$ for some $k \in \mathbb{N}$ and $x \in U(k-1,(F_{n,\varphi})) \setminus U(k,(F_{n,\varphi}))$. Clearly, $k \leq k'$. Hence, $\Phi(\varphi)(x) = B_k \subseteq B_{k'} = \Phi(\varphi')(x)$. This completes the proof of $\Phi(\varphi) \subseteq \Phi(\varphi')$ whenever $\varphi \subseteq \varphi'$.

(5) ⇒ (1). Let Y be a space having a strictly increasing closed cover $\{B_n\}$ possessing the property in (5). Let $(F_j)_{j\in\mathbb{N}}$ be a sequence of decreasing closed subsets of X with $\bigcap_{j\in\mathbb{N}} F_j = \varnothing$. Define a set-valued mapping $\varphi_{(F_j)} : X \to \mathcal{F}(Y)$ as follows: $\varphi_{(F_j)}(x) = B_0$ whenever $x \in X - F_1$, $\varphi_{(F_j)}(x) = B_{i+1}$ whenever $x \in F_i - F_{i+1}$. Then, $\varphi_{(F_j)}$ is locally bounded. By the assumptions, there exists a preserved operator Φ assigning to each $\varphi_{(F_j)}$, an l.s.c. and k-u.s.c set-valued mapping $\Phi(\varphi_{(F_j)}) : X \to \mathcal{F}(Y)$ such that $\varphi_{(F_j)} \subseteq \Phi(\varphi_{(F_j)})$.

For every $n \in \mathbb{N}$, define

$$U(n,(F_j)) = X - (\Phi(\varphi_{(F_j)}))^\sharp[B_n]$$

It suffices to show the operator U satisfies (i), (ii) and (iii) of Proposition 2.2

Since $\varphi_{(F_j)} \subseteq \Phi(\varphi_{(F_j)})$, for each $n \in \mathbb{N}$ we have

$$F_n \subseteq X \setminus (\varphi_{(F_j)})^\sharp[B_n] \subseteq X \setminus (\Phi(\varphi_{(F_j)}))^\sharp[B_n] = U(n,(F_j)).$$

In addition, $\Phi(\varphi_{(F_j)})$ is lower semi-continuous, so $U(n,(F_j))$ is an open set of X for each $n \in \mathbb{N}$. This shows that the condition (i) is satisfied.

For each $x \in X$, $\Phi(\varphi_{(F_j)})(x)$ is bounded, so there exists some $n_0 \in \mathbb{N}$ such that $x \in (\Phi(\varphi_{(F_j)}))^\sharp[B_{n_0}]$. It implies that $x \notin U(n_0,(F_j))$. Hence, $\bigcap_{n\in\mathbb{N}} U(n,(F_j)) = \varnothing$.

Let K be a compact subset of X, then $\Phi(\varphi_{(F_j)})(K)$ is bounded. There exists some $k_0 \in \mathbb{N}$ such that $K \subset (\Phi(\varphi_{(F_j)}))^\sharp[B_{k_0}]$. It implies that $K \bigcap U(k_0,(F_j)) = \varnothing$.

Finally, we show the operator satisfies (iii). Let $(F_j)_{j\in\mathbb{N}}$ and $(F'_j)_{j\in\mathbb{N}}$ be sequences of decreasing closed subsets of X such that $F_j \subseteq F'_j$ for each $j \in \mathbb{N}$. Then one can easily show that $\varphi_{(F_j)} \subseteq \varphi_{(F'_j)}$, hence by the assumption, we have $\Phi(\varphi_{(F_j)}) \subseteq \Phi(\varphi_{(F'_j)})$. Therefore,

$$U(n,(F_j)) = X \setminus (\Phi(\varphi_{(F_j)}))^\sharp[B_n] \subseteq X \setminus (\Phi(\varphi_{(F'_j)}))^\sharp[B_n] = U(n,(F'_j))$$

holds for each $n \in \mathbb{N}$. Thus, X is a k-MCM space. □

Next, we consider the k-semi-stratifiable space.

Definition 2.4. A space X is said to be *semi-stratifiable* [1], if there is an operator U assigning to each closed set F, a sequence of open sets $U(F) = (U(n,F))_{n\in N}$ such that

(1) $F \subseteq U(n,F)$ for each $n \in \mathbb{N}$;
(2) if $D \subseteq F$, then $U(n,D) \subseteq U(n,F)$ for each $n \in \mathbb{N}$;
(3) $\bigcap_{n\in\mathbb{N}} U(n,F) = F$.

X is said to be *k-semi-stratifiable* [4], if, in addition, $(3')$ obtained from (3) by requiring (3) a further condition 'if a compact set K such that $K \bigcap F = \varnothing$, there is some $n_0 \in \mathbb{N}$ such that $K \bigcap U(n_0,F) = \varnothing$'.

The following result was proved in [8]. For the completeness, we give its proof.

Proposition 2.5. *For any topological space X, the following statements are equivalent:*

(1) *space X is k-semistratifiable;*
(2) *there is an operator U assigning to a decreasing sequence of closed sets $(F_j)_{j\in\mathbb{N}}$, a decreasing sequence of open sets $(U(n,(F_j)))_{n\in\mathbb{N}}$ such that*
    (i) *$F_n \subseteq U(n,(F_j))$ for each $n \in \mathbb{N}$;*

(ii) *for any compact subset K in X, if $\bigcap_{n\in\mathbb{N}} F_n \cap K = \varnothing$, there is $n_0 \in \mathbb{N}$ such that $U(n_0, (F_j)) \cap K = \varnothing$;*

(iii) *Given two decreasing sequences of closed sets $(F_j)_{j\in\mathbb{N}}$ and $(E_j)_{j\in\mathbb{N}}$ such that $F_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $U(n, (F_j)) \subseteq U(n, (E_j))$ for each $n \in \mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2) Let U_0 be an operator having the properties: $(1), (2)$ and $(3^{'})$ in Definition 2.4. Given any decreasing sequences of closed sets $(F_j)_{j\in\mathbb{N}}$, we can define an operator U by

$$U((F_j)) = (U(n, (F_j)))_{n\in\mathbb{N}}, \quad \text{where} \quad U(n, (F_j)) = U_0(n, F_n) \quad \text{for each} \quad n \in \mathbb{N}.$$

We shall prove that the operator U has the properties (i)-(iii) in (2). Because of U_0 having properties (i) and (ii) in Definition 2.4, one can easily verify that U has the properties (i) and (iii) in (2). We show that the property (ii) in (2) holds for U. Take any decreasing sequences of closed sets $(F_n)_{n\in\mathbb{N}}$ and any compact subset K in X such that $\bigcap_{n\in\mathbb{N}} F_n \cap K = \varnothing$. Then, there exists $n_0 \in \mathbb{N}$ such that $F_{n_0} \cap K = \varnothing$. Since X is k-semi-stratifiable, there is $i \in \mathbb{N}$ such that $U_0(i, F_{n_0}) \cap K = \varnothing$. If $i < n_0$, we have $U(n_0, (F_n)) \cap K = U_0(n_0, F_{n_0}) \cap K = \varnothing$; If $i \geq n_0$, we also have $U(i, (F_n)) \cap K = U_0(i, F_i) \cap K = \varnothing$. Hence the operator U holds for (ii).

(2) $\Rightarrow$ (1) Let U_0 be an operator having the properties (i)-(iii) in (2). Given any closed set F in X by letting $F_n = F$ for each $n \in \mathbb{N}$, we can define an operator U by

$$U(j, F) = U_0(j, (F_n)) \quad \text{where} \quad (U_0(j, (F_n)))_{j\in\omega} = U_0((F_n)).$$

One can easily verify that the operator U has the properties in Definition 2.4. □

Theorem 2.6. *For a space X, the following statements are equivalent:*

(1) *X is k-semistratifiable;*

(2) *for every space Y having a strictly increasing closed cover $\{B_n\}$, there exists a preserved order operator Φ assigning to each set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\Phi(\varphi)_{|U_\varphi}$ is k-u.s.c. and $\varphi \subseteq \Phi(\varphi)$;*

(3) *for every metric space Y, there exists a preserved order operator Φ assigning to each set-valued set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c set-valued set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\Phi(\varphi)_{|U_\varphi}$ is k-u.s.c. and $\varphi \subseteq \Phi(\varphi)$;*

(4) *there exists an order-preserving operator Φ assigning to each set-valued set-valued mapping $\varphi : X \to \mathcal{F}(\mathbb{R})$, an l.s.c. set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(\mathbb{R})$ such that $\Phi(\varphi)_{|U_\varphi}$ is k-u.s.c and $\varphi \subseteq \Phi(\varphi)$;*

(5) *there exists a space Y having a strictly increasing closed cover $\{B_n\}$, there exists a preserved order operator Φ assigning to each set-valued set-valued mapping $\varphi : X \to \mathcal{F}(Y)$, an l.s.c set-valued mapping $\Phi(\varphi) : X \to \mathcal{F}(Y)$ such that $\Phi(\varphi)_{|U_\varphi}$ is k-u.s.c. and $\varphi \subseteq \Phi(\varphi)$.*

Proof. The implications of (2)⇒(3)⇒(4)⇒ (5) are trivial.

(1) ⇒ (2). Assume that X is a k-semistratifiable space. Then there exists an operator U satisfying (i), (ii) and (iii) in Proposition 2.5. Let Y be a space having a strictly increasing closed cover $\{B_n\}$. For each set-valued mapping $\varphi : X \to \mathcal{F}(Y)$ and each $n \in \mathbb{N}$, define $F_{n,\varphi} = \overline{\{x \in X : \varphi(x) \notin B_n\}}$.

Then we have $U_\varphi = X \setminus \bigcap_{n\in\mathbb{N}} F_{n,\varphi}$. Indeed, for each $x \in U_\varphi$, then there exists an open neighborhood V of x and some $i \in \mathbb{N}$ such that $\varphi(y) \subseteq B_i$ for each $y \in V$, which implies that $V \bigcap F_{i,\varphi} = \varnothing$. It implies that $U_\varphi \subseteq X - \bigcap_{n\in\mathbb{N}} F_{n,\varphi}$. On the other hand, take any $y \in X - \bigcap_{n\in\mathbb{N}} F_{n,\varphi}$. Then there is $F_{j,\varphi}$ such that $y \notin F_{j,\varphi}$, and therefore, there exists an open neighborhood V of y such that $V \cap \{x \in X : \varphi(x) \nsubseteq B_j\} = \varnothing$. It implies that $y \in V \subseteq U_\varphi$.

Define $\Phi(\varphi) : X \to \mathcal{F}(Y)$ as follows: $\Phi(\varphi)(x) = B_0$ whenever $x \in X - U(0,(F_{n,\varphi}))$, $\Phi(\varphi)(x) = B_{i+1}$ whenever $x \in U(i,(F_{n,\varphi})) - U(i+1,(F_{n,\varphi}))$, $\Phi(\varphi)(x) = Y$ if $x \in X - U_\varphi$.

Then, $\Phi(\varphi)$ is lower semi-continuous and $\varphi \subseteq \Phi(\varphi)$. We only need to show that $\Phi(\varphi)_{|U_\varphi}$ is k-u.s.c.

Let K be a compact subset of U_φ. By Proposition 2.5, there exists $k \in \mathbb{N}$ such that $K \bigcap U(k+1,(F_{n,\varphi})) = \varnothing$. It implies that $\Phi(\varphi)(K) \subseteq B_{k+1}$.

(5) ⇒ (1). Let Y be a space having a strictly increasing closed cover $\{B_n\}$ possessing the property in (5). Let $(F_j)_{j\in\mathbb{N}}$ be a sequence of decreasing closed subsets of X. Define a set-valued mapping $\varphi_{(F_j)} : X \to \mathcal{F}(Y)$ as follows: $\varphi_{(F_j)}(x) = B_1$ whenever $x \in X - F_1$, $\varphi_{(F_j)}(x) = B_{i+1}$ whenever $x \in F_i - F_{i+1}$, $\varphi_{(F_j)}(x) = Y$ if $x \in X - \bigcap_{i\in\mathbb{N}} F_i$. By the assumptions, there exists a preserved operator Φ assigning to each $\varphi_{(F_j)}$, an l.s.c set-valued mapping $\Phi(\varphi_{(F_j)}) : X \to \mathcal{F}(Y)$ such that $\Phi(\varphi)_{|U_{\varphi(F_j)}}$ is k-u.s.c. and $\varphi_{(F_j)} \subseteq \Phi(\varphi_{(F_j)})$. For every $n \in \mathbb{N}$, define

$$U(n,(F_j)) = X - (\Phi(\varphi_{(F_j)}))^\sharp[B_n].$$

It suffices to show the operator U satisfies (i), (ii) and (iii) of Proposition 2.5.

The proof that the operator U satisfies (i) and (iii) of Proposition 2.5 is as same as Theorem 2.3, so we only shows that the operator U satisfies (ii) of Proposition 2.5.

Let K be a compact subset of X satisfying $K \cap (\bigcap_{n\in\mathbb{N}} F_n) = \varnothing$, then $K \subseteq U_\varphi$. There exists $k \in \mathbb{N}$ such that $\Phi(\varphi(F_j))(K) \subseteq B_k$. Hence $K \cap U(k,(F_j)) = \varnothing$.

Thus, X is a k-semistratifiable space. □

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