

Counting coarse subsets of a countable group

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ABSTRACT

For every countable group G , there are 2^ω distinct classes of coarsely equivalent subsets of G .

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1. INTRODUCTION AND RESULTS

Following [5], [6], we say that a *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius α* around x . It is supposed that $x \in B(x, \alpha)$ for all $x \in X, \alpha \in P$. The set X is called the *support* of \mathcal{B} , P is called the *set of radii*.

Given any $x \in X, A \subseteq X, \alpha \in P$, we set

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \quad B(A, \alpha) = \bigcup_{a \in A} B(a, \alpha),$$

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A ball structure $\mathcal{B} = (X, P, B)$ is called a *balleans* if

- for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,
 $B(x, \alpha) \subseteq B^*(x, \alpha'), \quad B^*(x, \beta) \subseteq B(x, \beta');$

- for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma);$$

- for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$.

We note that a ballean can be considered as an asymptotic counterpart of a uniform space, and could be defined [7] in terms of entourages of the diagonal Δ_X in $X \times X$. In this case a ballean is called a *coarse structure*. For categorical look at the ballians and coarse structures as "two faces of the same coin" see [2].

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be ballians. A mapping $f : X \rightarrow X'$ is called *coarse* if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that, for every $x \in X$, $f(B(x, \alpha)) \subseteq B'(f(x), \alpha')$.

A bijection $f : X \rightarrow X'$ is called an *asymorphism* between \mathcal{B} and \mathcal{B}' if f and f^{-1} are coarse. In this case \mathcal{B} and \mathcal{B}' are called *asymorphic*.

Let $\mathcal{B} = (X, P, B)$ be a ballean. Each subset Y of X defines a *subballean* $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = Y \cap B(y, \alpha)$. A subset Y of X is called *large* if $X = B(Y, \alpha)$, for $\alpha \in P$. Two ballians \mathcal{B} and \mathcal{B}' with supports X and X' are called *coarsely equivalent* if there exist large subsets $Y \subseteq X$ and $Y' \subseteq X'$ such that the subballians \mathcal{B}_Y and $\mathcal{B}'_{Y'}$ are asymorphic.

Every infinite group G can be considered as the ballean (G, \mathfrak{F}_G, B) , where \mathfrak{F}_G is the family of all finite subsets of G , $B(g, F) = Fg \cup \{g\}$.

We note that finitely generated groups are finitary coarsely equivalent if and only if G and H are quasi-isometric [3, Chapter 4].

A classification of countable locally finite groups (each finite subset generates finite subgroup) up to asymorphisms is obtained in [4] (see also [5, p. 103]).

Two countable locally finite groups G_1 and G_2 are asymorphic if and only if the following conditions hold:

- (i) *for every finite subgroup $F \subset G_1$, there exists a finite subgroup H of G_2 such that $|F|$ is a divisor of $|H|$;*
- (ii) *for every finite subgroup H of G_2 , there exists a finite subgroup F of G_1 such that $|F|$ is a divisor of $|H|$.*

It follows that there are continuum many distinct types of countable locally finite groups and each group is asymorphic to some direct sum of finite cyclic groups.

The following coarse classification of countable Abelian groups is obtained in [1].

Two countable Abelian groups are coarsely equivalent if and only if the torsion-free ranks of G and H coincide and G and H are either both finitely generated or infinitely generated.

In particular, any two countable torsion Abelian groups are coarsely equivalent.

Given a group G , we consider each non-empty subsets as a subballean of G and say that a class of all pairwise coarsely equivalent subsets is a *coarse subset* of G .

For a countable group G , we prove that there as many coarse subsets of G as possible by the cardinal arithmetic.

Theorem 1.1. *For a countable group G , there are 2^ω coarse subsets of G .*

Every countable group G contains either countable finitely generated subgroup or countable locally finite subgroup, so we split the proof into corresponding cases.

2. PROOF: FINITELY GENERATED CASE

2.1. We take a finite system S , $S = S^{-1}$ of generators of G and consider the Cayley graph Γ with the set of vertices G and the set of edges $\{\{g, h\} : gh^{-1} \in S, g \neq h\}$. We denote by ρ the path metric on Γ and choose a geodesic ray $V = \{v_n : n \in \omega\}$, v_0 is the identity of G , $\rho(v_n, v_m) = |n - m|$.

Then the subballean of G with the support V is asyomorphic to the metric ballean $(\mathbb{N}, \mathbb{N} \cup \{0\}, B)$, where $B(x, r) = \{y \in \mathbb{N} : d(x, y) \leq r\}$, $d(x, y) = |x - y|$. Thus, it suffices to find a family \mathfrak{F} , $|\mathfrak{F}| = 2^\omega$ of pairwise coarsely non-equivalent subsets of \mathbb{N} .

2.2. We choose a sequence $(I_n)_{n \in \omega}$ of intervals of \mathbb{N} , $I_n = [a_n, b_n]$, $b_n < a_{n+1}$ such that

$$(1) \quad b_n - a_n > n a_n.$$

Then we take an almost disjoint family \mathcal{A} of infinite subsets of ω such that $|\mathcal{A}| = 2^\omega$. Recall that \mathcal{A} is almost disjoint if $|W \cap W'| < \omega$ for all distinct $W, W' \in \mathcal{A}$.

For each $W \in \mathcal{A}$, we denote $I_W = \bigcup \{I_n : n \in W\}$. To show that $\mathfrak{F} = \{I_W : W \in \mathcal{A}\}$ is the desired family of subsets of \mathbb{N} , we take distinct $W, W' \in \mathcal{A}$ and assume that $I_W, I_{W'}$ are coarsely equivalent. Then there exist large subsets X, X' of $I_W, I_{W'}$, and an asyomorphism $f : X \rightarrow X'$. We choose $r \in \mathbb{N}$ such that

$$I_W \subseteq B(X, r), \quad I_{W'} \subseteq B(X', r)$$

and note that if an interval I of length $2r$ is contained in I_W then I must contain at least one point of X , and the same holds for the pair $I_{W'}, X'$.

Since f is an asyomorphism, we can take $t \in \mathbb{N}$ such that, for all $x \in X$, $x' \in X'$,

$$(2) \quad f(B_X(x, 2r + 2)) \subseteq B_{X'}(f(x), t);$$

$$(3) \quad f^{-1}(B_{X'}(x', 2r + 2)) \subseteq B_X(f^{-1}(x'), t).$$

We use (1) to choose $m \in W \setminus W'$, $m > \max(W \cap W')$ such that

$$(4) \quad b_m - a_m > 2ra_m;$$

$$(5) \quad b_m - a_m > 2t.$$

2.3. We denote $Z = X \cap [a_m, b_m]$ and enumerate Z in increasing order $Z = \{z_0, \dots, z_k\}$. Then $d(z_i, z_{i+1}) \leq 2r + 2$ because otherwise the interval $[z_i + 1, z_{i+1} - 1]$ of length $2r$ has no points of X .

If $f(z_0) < a_m$ then, by (2) and (5), $f(Z) \subseteq [1, a_m - 1]$. On the other hand, $k \geq (b_m - a_m)/2r - 1$ and, by (4), $(b_m - a_m)/2r > a_m$. Hence, $k > a_m - 1$ contradicting $f(Z) \subseteq [1, a_m - 1]$ because f is a bijection.

If $f(z_0) > b_m$ then we take $s \in W'$ such that $f(z_0) \in I_s$. Since $m > \max(W \wedge W')$ and $s > m$, we have $s \in W' \setminus W$, so we can repeat above argument for f^{-1} and I_s in place of f and I_m with usage (3) instead of (2).

3. PROOF: LOCALLY FINITE CASE

3.1. Let G be an arbitrary countable group and let X, A be infinite subsets of G . Suppose that there exist an infinite subset Y of X , a partition $A = B \cup C$ and $k, l \in \mathbb{N}$, $k < l$ such that

$$(6) \quad \text{there exists } H \in \mathfrak{F}_G \text{ such that, for every } y \in Y,$$

$$|B_X(y, H)| \geq k;$$

$$(7) \quad \text{for every } F \in \mathfrak{F}_G, \text{ there exists } Y' \in \mathfrak{F}_G \text{ such that, for every } y \in Y \setminus Y',$$

$$|B_X(y, H)| \geq l;$$

$$(8) \quad \text{there exists } K \in \mathfrak{F}_G \text{ such that, for every } b \in B,$$

$$|B_A(b, K)| > l;$$

$$(9) \quad \text{for every } F \in \mathfrak{F}_G, \text{ there exists } C' \in \mathfrak{F}_G \text{ such that, for every } c \in C \setminus C',$$

$$|B_A(c, F)| < k.$$

Then X and A are not asymorphic.

We suppose the contrary and let $f : X \rightarrow A$ be an asymorphism. We take an infinite subset I of Y such that either $f(I) \subset C$ or $f(I) \subset B$.

Assume that $f(I) \subset C$ and choose $F \in \mathfrak{F}_G$ such that, for every $x \in X$,

$$f(B_X(x, H)) \subseteq B_A(f(x), F).$$

For this F , we use (9) to choose corresponding C' . We take $y \in I$ such that $f(y) \in C \setminus C'$. By (6), $f(B(y, H)) \geq k$. By (9), $B_A(f(y), F) < k$ and we get a contradiction because f is a bijection.

If $f(I) \subset B$ then, by (8), $B_A(b, K) > l$ for every $b \in f(I)$. Since f^{-1} is coarse, there is $F \in \mathfrak{F}_G$ such that, for every $a \in A$

$$f^{-1}(B_A(a), K) \subseteq B_X(f^{-1}(a), F).$$

For this F , we choose Y' satisfying (7) and get a contradiction.

3.2. Now we assume that G is locally finite and show a plan how to choose the desired family \mathfrak{F} , $|\mathfrak{F}| = 2^\omega$ of pairwise coarsely non-equivalent subsets of G .

We construct some special sequence $(Y_n)_{n \in \omega}$ of pairwise disjoint subsets of G . Then we take a family \mathcal{A} of almost disjoint infinite subset of ω , $|\mathfrak{F}| = 2^\omega$, denote

$$(10) \quad X_W = \bigcup \{Y_n : n \in W\}, \quad W \in \mathcal{A},$$

and get \mathfrak{F} as $\{X_W : W \in \mathcal{A}\}$.

3.3. We represent G as the union of an increasing chain $\{F_n : n \in \omega\}$ of finite subgroups such that

$$(11) \quad |F_{n+1}| > |F_n|^2.$$

Then we choose a double sequence $(g_{nm})_{n,m \in \omega}$ of elements of G such that

$$(12) \quad F_n F_m g_{nm} \cap F_i F_j g_{ij} = \emptyset \text{ for all distinct } (n, m), (i, j) \text{ from } \omega \times \omega, \text{ and put}$$

$$Y_n = \bigcup \{F_m g_{nm} : m \in \omega\}.$$

3.4. We take distinct $W, W' \in \mathcal{A}$ and prove that X_W and $X_{W'}$ (see (10)) are not coarsely equivalent. We suppose the contrary and choose large asymorphic subsets Z_W and $Z_{W'}$ of X_W and $X_{W'}$. Then we take $t \in \omega$ such that

$$X_W \subseteq F_t Z_W, \quad X_{W'} \subseteq F_t Z_{W'}.$$

If $n > t$ and either $F_n g_{nm} \subset X_W$ or $F_n g_{nm} \subset X_{W'}$ then

$$(13) \quad |F_n g_{nm} \cap Z_W| \geq \frac{|F_n|}{|F_t|}, \quad |F_n g_{nm} \cap Z_{W'}| \geq \frac{|F_n|}{|F_t|}.$$

To apply 3.1, we choose $s \in W \setminus W'$, $s > t$ and denote

$$X = Z_W, \quad Y = Y_s \cap Z_W, \quad A = Z_{W'}, \quad B = \bigcup \{Y_i : i \in W', \quad i > s\},$$

$$C = \bigcup \{Y_i : i \in W', \quad i < s\}, \quad k = \frac{|F_s|}{|F_t|}, \quad l = |F_s|.$$

By (13) with $s = n$, we get (6). By (12) with $s = n$, we get (7).

If $n > s$ then $|F_n|/|F_t| > |F_n|/|F_s|$. By (11), $|F_n|/|F_s| > |F_s|$, so $|F_n|/|F_t| > |F_s|$ and, by (13), we have (8).

If $n < s$ then $|F_n| < |F_s|/|F_t|$ and, by (12), we get (9).

4. COMMENTS

A subset A of an infinite group G is called

- *thick* if, for every $F \in \mathfrak{F}_G$, there exists $g \in A$ such that $Fg \subset A$;
- *small* if $L \setminus A$ is large for every large subset L of G ;

• *thin* if, for every $F \in \mathfrak{F}_G$, there exists $H \in \mathfrak{F}_G$ such that $B_A(g, F) = \{g\}$ for each $g \in A \setminus H$.

A subset A is thick if and only if $L \cap A \neq \emptyset$ for every large subset L of G . For a countable group G , in the proof of Theorem 1.1, we construct 2^ω pairwise coarsely non-equivalent thick subsets of G .

Every large subset L of G is coarsely equivalent to G , so any two large subsets of G are coarsely equivalent. If G is countable then any two thin subset S, T of G are asymorphic: any bijection $f : S \rightarrow T$ is an asymorphism. Every thin subset is small. But a small subset S of G could be asymorphic to G : we take a group G containing a subgroup S isomorphic to G such that the index of S in G is infinite.

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