APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 18, no. 2 (2017), 377-390
doi:10.4995/agt.2017.7452

Some fixed point theorems on non-convex sets

M. Radhakrishnan[a], S. Rajesh[b] and Sushama Agrawal[a]

[a] Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai 600 005, India. (radhariasm@gmail.com,sushamamdu@gmail.com)
[b] Department of Mathematics, Indian Institute of Technology, Tirupati 517 506, India. (srajeshiitmdt@gmail.com)

Communicated by E. A. Sánchez-Pérez

Abstract

In this paper, we prove that if K is a nonempty weakly compact set in a Banach space X, $T : K \to K$ is a nonexpansive map satisfying $\frac{x+Tx}{2} \in K$ for all $x \in K$ and if X is $3-$uniformly convex or X has the Opial property, then T has a fixed point in K.

2010 MSC: *47H09; 47H10.*

Keywords: *fixed points; nonexpansive mappings; $T-$regular sets; $k-$uniform convex Banach spaces; Opial property.*

1. Introduction

Let K be a nonempty subset of a Banach space X. A mapping $T : K \to K$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.

The following theorem was proved independently by Browder [2] and Göhde [8] in the setting of uniformly convex Banach spaces.

Theorem 1.1 ([2])**.** *Let K be a nonempty weakly compact convex subset of a uniformly convex Banach space X and $T : K \to K$ be a nonexpansive map. Then T has a fixed point in K.*

Using the notion of normal structure, Kirk [10] proved the following theorem which is more general than Theorem 1.1.

Received 28 March 2017 – Accepted 08 June 2017

Theorem 1.2 ([10]). *Let K be a nonempty weakly compact convex subset having normal structure in a Banach space X and $T : K \to K$ be a nonexpansive map. Then T has a fixed point in K.*

The convexity assumption cannot be dispense in the above theorems as can be seen from the following simple example.

Let $K = [-2, -1] \cup [1, 2] \subseteq \mathbb{R}$ and T is a self map on K defined by $Tx = -x$ for all $x \in K$. Then T is nonexpansive, but T has no fixed points in K. This implies that nonexpansive map on a non-convex set in a Banach space need not have a fixed point.

Motivated by Theorem 1.1 and Theorem 1.2, Veeramani [20] introduced the notion of $T-$regular set as follows:

Let T be a self map on a nonempty subset K of a Banach space X. Then K is said to be a $T-$regular set if $\frac{x+Tx}{2} \in K$ for all $x \in K$.

Clearly, if K is a convex set and $T : K \to K$, then K is $T-$regular. But a $T-$regular set need not be a convex set(see Example 3.2). Further, Veeramani [20] proved the following fixed point theorem.

Theorem 1.3 ([20]). *Let K be a nonempty weakly compact subset of a uniformly convex Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Then T has a fixed point in K.*

Khan and Hussain [9] used the notion of $T-$regular sets to prove the existence of fixed points for nonexpansive mappings in the setting of metrizable topological vector space. Also, Goebel and Schöneberg [6] proved the existence of fixed point for a nonexpansive map on certain nonconvex sets in a Hilbert space.

Sullivan [18] introduced the concept of $k-$uniform convexity, $k-$UC in short, where k is any positive integer and proved that every $k-$uniformly convex Banach space has normal structure. Note that for $k = 1$, it is uniformly convex.

Sullivan [18] observed that every $k-$UC Banach space is a $(k + 1)-$UC. But the converse is not true. For example, the Banach space $l^{p,1}(\mathbb{N})$ [1] for $1 < p < \infty$ is 2−UC but not 1−UC where $l^{p,1}(\mathbb{N})$ is the $l^p(\mathbb{N})$ space with suitable renorm.

Motivated by Theorem 1.2, Theorem 1.3 and the fact that $k-$UC Banach spaces have normal structure [18], we raise the following question:

Does a nonexpansive map T on a nonempty weakly compact set K in a $k-$UC Banach space have a fixed point if $\frac{x+Tx}{2} \in K$ for all $x \in K$?

In this paper, we give an affirmative answer to the above question, if X is a 3−UC Banach space. For the proof of this result, Lemma 3.3 and Lemma 3.4 (the geometric inequality on $k-$UC Banach space) are crucial.

In another direction, Opial [16] introduced a class of spaces for which the asymptotic center of a weakly convergent sequence coincides with the weak limit point of the sequence. Gossez and Lami Dozo [7] have observed that all such spaces have normal structure. Hence, in view of Kirk's theorem, every nonempty weakly compact convex set in a Banach space which satisfy

Opial's condition has fixed point property for a nonexpansive mapping. Recently, Suzuki [19] introduced a new class of mappings which also includes nonexpansive maps and proved that every nonempty weakly compact convex set in a Banach space which satisfy Opial's condition also has fixed point property for all such maps.

In this paper, we prove that if K is a nonempty weakly compact set in a Banach space X having the Opial property, $T : K \to K$ is a nonexpansive map and if K is $T-$regular set, then T has a fixed point point in K. Moreover, the Krasnoseleskii's [12] iterated sequence $\{x_n\}$ where $x_{n+1} = \frac{x_n+Tx_n}{2}$ for all $n \in \mathbb{N}$ and $x_1 \in K$ weakly converges to a fixed point.

2. Preliminaries

Now, we give some basic definitions and results which are used in this paper. Let X be a Banach space. For a nonempty subset A of X, let

$$\mathrm{co}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \geq 0, \text{ for } i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

$$\mathrm{aff}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in A, \lambda_i \in \mathbb{R}, \text{ for } i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \right\}$$

The sets $\mathrm{co}(A)$ and $\mathrm{aff}(A)$ are called the convex hull and the affine hull of A respectively.

A set A is affine if $A = \mathrm{aff}(A)$. Every affine set is a translation of a subspace and the subspace is uniquely defined by the affine set. The dimension of an affine set is the dimension of the corresponding subspace. Further, the dimension of a convex set A is defined as the dimension of the smallest affine set which contains A. This shows that the dimension of $\mathrm{co}(A)$ is the dimension of $\mathrm{aff}(A)$.

Sliverman [17] introduced the notion of volume of $k+1$ vectors, denoted by $V(x_1, x_2, \dots, x_{k+1})$, as follows:

Given $x_1, x_2, \dots, x_{k+1} \in X$,

$$V(x_1, x_2, \dots, x_{k+1}) = \frac{1}{k!} \sup \left\{ \begin{vmatrix} f_1(x_2 - x_1) & \dots & f_1(x_{k+1} - x_1) \\ f_2(x_2 - x_1) & \dots & f_2(x_{k+1} - x_1) \\ \vdots & \vdots & \vdots \\ f_k(x_2 - x_1) & \dots & f_k(x_{k+1} - x_1) \end{vmatrix} : f_1, \dots, f_k \in B_{X^*} \right\}$$

By the consequences of Hahn-Banach theorem, $V(x_1, x_2) = \|x_1 - x_2\|$ for any $x_1, x_2 \in X$. Note that $V(x_1, x_2, \dots, x_{k+1}) = 0$ iff the dimension of the convex hull of $\{x_1, x_2, \dots, x_{k+1}\}$ does not exceed $k - 1$.

Using the notion of volume of $k+1$ vectors, Sullivan [18] defined the concept of $k-$uniform convexity.

We put $\mu_X^{(k)} = \sup\{V(x_1, \dots, x_{k+1}) : x_1, \dots, x_{k+1} \in B_X\}$.

Definition 2.1 ([18]). The modulus of $k-$convexity is defined as

$$\delta_X^{(k)}(\epsilon) = \inf\left\{1 - \frac{1}{k+1}\left\|\sum_{i=1}^{k+1} x_i\right\| : x_1, \ldots, x_{k+1} \in B_X \text{ and } V(x_1, \ldots, x_{k+1}) \geq \epsilon\right\}$$

where $\epsilon \in [0, \mu_X^{(k)})$.

A Banach space X is said to be $k-$uniformly convex if $\delta_X^{(k)}(\epsilon) > 0$ for every $0 < \epsilon < \mu_X^{(k)}$.

Note that all Banach spaces of dimension less than $k+1$ are $k-$UC. For more information on $k-$UC, one can refer to [11, 14, 15].

Lim [13] proved the continuity of modulus $\delta_X^{(k)}$ of $k-$convexity using the following inequality.

Theorem 2.2 ([13]). *Let X be a Banach space and k be any positive integer. For every $0 < \epsilon_1 < c < \epsilon_2 < \mu_X^{(k)}$,*

$$\frac{\delta_X^{(k)}(c) - \delta_X^{(k)}(\epsilon_1)}{c - \epsilon_1} \leq \frac{1}{k(\epsilon_2^{1/k} - \epsilon_1^{1/k})\epsilon_1^{1-1/k}}$$

Corollary 2.3 ([13]). *Let X be a Banach space. Then $\delta_X^{(k)}(\epsilon)$ is continuous on $[0, \mu_X^{(k)})$.*

Definition 2.4 ([16]). A Banach space X is said to have the Opial property if $\{x_n\}$ is a weakly convergent sequence in X with limit z, then

$$\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq z$.

It is known that [5] Hilbert spaces, finite dimensional Banach spaces and $l^p(\mathbb{N})$ $(1 < p < \infty)$ have the Opial property.

Edelstein [3] introduced the notion of asymptotic center as follows:

Definition 2.5 ([3]). Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a bounded sequence in X. For each $x \in X$, define $r(x) = \limsup_{n\to\infty} \|x - x_n\|$. The number $r = \inf_{x\in K} r(x)$ and the set $A(K, \{x_n\}) = \{x \in K : r(x) = r\}$ are called the asymptotic radius and asymptotic center of $\{x_n\}$ with respect to K respectively.

We use the next lemma in the sequel, which is proved by Goebel and Kirk [4].

Lemma 2.6 ([4]). *Let $\{z_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space X and let $\lambda \in (0, 1)$. Suppose that $z_{n+1} = \lambda w_n + (1 - \lambda)z_n$ and $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \|w_n - z_n\| = 0$.*

3. Main results

3.1. **3−UC Banach spaces.** In this section, we first give the convergence theorem for a nonexpansive map T defined on a compact $T-$regular set in a Banach space X. Also, we prove the existence of fixed points for a nonexpansive map T defined on a weakly compact $T-$regular set in a 3−UC Banach space X.

Theorem 3.1. *Let K be a nonempty compact subset of a Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Define a sequence $\{x_n\}$ in K by $x_{n+1} = \frac{x_n+Tx_n}{2}$ for $n \in \mathbb{N}$ and $x_1 \in K$. Then T has a fixed point in K and $\{x_n\}$ strongly converges to a fixed point of T.*

Proof. Since $x_{n+1} = \frac{x_n+Tx_n}{2}$ for $n \in \mathbb{N}$, by Lemma 2.6, we have $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$.

Since K is compact and $\{x_n\} \subseteq K$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_k}\}$ converges to z. Now, by the continuity of T, $\{Tx_{n_k}\}$ converges to Tz.

But, note that $\lim\limits_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$. Hence $\{x_{n_k}\}$ also converges to Tz. This implies that $Tz = z$.

Also, note that $\{\|x_n - z\|\}$ is a decreasing sequence. For,

$$\|x_{n+1} - z\| \leq \frac{1}{2}\|x_n - z\| + \frac{1}{2}\|Tx_n - z\| \leq \|x_n - z\|, \text{ for all } n \in \mathbb{N}$$

Therefore $\{x_n\}$ converges to z, as $\{x_{n_k}\}$ converges to z in norm. □

Example 3.2. Let $K = \{(x, 0, \frac{1}{2^n}), (0, y, \frac{1}{2^n}), (x, x, \frac{1}{2^n}), (x, 0, 0), (0, y, 0), (x, x, 0) : 0 \leq x, y \leq 1 \text{ and } n \in \mathbb{N}\}$ be a subset of $(\mathbb{R}^3, \|.\|_2)$. Define a map $T : K \to K$ by $T(x, y, z) = (y, x, 0)$ for all $(x, y, z) \in K$.

It is easy to see that K is $T-$regular. Also, note that T is nonexpansive. For, let $x = (x_1, y_1, z_1), y = (x_2, y_2, z_2) \in K$.

$$\begin{aligned} \text{Then } \|Tx - Ty\|_2 &= \|(y_1 - y_2, x_1 - x_2, 0)\|_2 \\ &\leq \|(x_1 - x_2, y_1 - y_2, z_1 - z_2)\|_2 = \|x - y\|_2 \end{aligned}$$

By Theorem 3.1, T has a fixed point in K, since K is compact and $T-$regular. Also, note that $\mathrm{Fix}(T) = \{(x, x, 0) : 0 \leq x \leq 1\}$.

Lemma 3.3. *Let K be a nonempty weakly compact subset of a Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Define a sequence $\{x_n\}$ in K by $x_{n+1} = \frac{x_n+Tx_n}{2}$ for $n \in \mathbb{N}$ and $x_1 \in K$. Then the asymptotic center $A(K, \{x_n\})$ of $\{x_n\}$ with respect to K is also a nonempty weakly compact $T-$regular subset of K. Moreover, if K is a minimal weakly compact $T-$regular set, then $A(K, \{x_n\}) = K$.*

Proof. Since $r(x) = \limsup\limits_{n\to\infty} \|x - x_n\|$ is a weakly lower semicontinuous function on X and K is weakly compact, $A(K, \{x_n\}) = \{x \in K : r(x) = \inf\limits_{y\in K} r(y) = r\}$ is nonempty.

Also $\{x \in X : r(x) \leq \inf_{y\in K} r(y)\}$ is a weakly closed set, this implies that $A(K, \{x_n\}) = \{x \in X : r(x) \leq \inf_{y\in K} r(y)\} \cap K$ is a weakly closed set.

Moreover, since T is nonexpansive and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $A(K, \{x_n\})$ is $T-$invariant.

Now, it is claimed that $A(K, \{x_n\})$ is a $T-$regular set.

Let $x \in A(K, \{x_n\})$. Then $Tx \in A(K, \{x_n\})$ and

$$\left\| \frac{x + Tx}{2} - x_n \right\| \leq \frac{1}{2}\|x - x_n\| + \frac{1}{2}\|Tx - x_n\|.$$

This implies that

$$\limsup_{n\to\infty} \left\| \frac{x + Tx}{2} - x_n \right\| = r.$$

Therefore $\frac{x+Tx}{2} \in A(K, \{x_n\})$. Hence $A(K, \{x_n\})$ is a nonempty weakly compact $T-$regular subset of K.

Suppose that K is a nonempty minimal weakly compact $T-$regular set. Then $A(K, \{x_n\}) = K$, as $A(K, \{x_n\}) \subseteq K$ is also a nonempty weakly compact $T-$regular set. □

Lemma 3.4. *Let X be a $k-UC$ Banach space, for some $k \in \mathbb{N}$ and $x_1, x_2, \ldots, x_{k+1} \in B_X$ such that $V(x_1, x_2, \ldots, x_{k+1}) = \epsilon > 0$.*

Then $\|t_1x_1 + t_2x_2 + \cdots + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, \ldots, t_{k+1}\}\delta_X^{(k)}(\epsilon)$, where $\sum_{i=1}^{k+1} t_i = 1,\ t_i \geq 0$ for $i = 1, 2, \ldots, k+1$.

Proof. Without loss of generality, we can assume that $t_1 = \min\{t_1, t_2, \ldots, t_{k+1}\}$.

$$\begin{aligned}
\|t_1x_1 + t_2x_2 + \cdots + t_{k+1}x_{k+1}\| &= \|t_1(x_1 + \cdots + x_{k+1}) + (t_2 - t_1)x_2 + (t_3 - t_1)x_3 \\
&\quad + \cdots + (t_{k+1} - t_1)x_{k+1}\| \\
&\leq (k+1)t_1 \left\| \frac{x_1 + x_2 + \cdots + x_{k+1}}{k+1} \right\| + (t_2 - t_1)\|x_2\| \\
&\quad + (t_3 - t_1)\|x_3\| + \cdots + (t_{k+1} - t_1)\|x_{k+1}\| \\
&\leq (k+1)t_1(1 - \delta_X^{(k)}(\epsilon)) + t_2 + t_3 + \cdots + t_{k+1} - kt_1 \\
&= (k+1)t_1 - (k+1)t_1\delta_X^{(k)}(\epsilon) + 1 - (k+1)t_1 \\
&= 1 - (k+1)t_1\delta_X^{(k)}(\epsilon)
\end{aligned}$$

Hence $\|t_1x_1 + t_2x_2 + \cdots + t_{k+1}x_{k+1}\| \leq 1 - (k+1)\min\{t_1, t_2, \ldots, t_{k+1}\}\delta_X^{(k)}(\epsilon)$. □

Remark 3.5. Now from Lemma 3.4, we have:

(1) If $k = 2$ and $t_1 = t_2 = \frac{1}{4}$, then

$$\left\| \frac{x_1}{4} + \frac{x_2}{4} + \frac{x_3}{2} \right\| \leq 1 - \frac{3}{4}\delta_X^{(2)}(\epsilon).$$

(2) If $k = 3$ and $t_1 = t_2 = \frac{1}{8}, t_3 = \frac{1}{4}$ then

$$\left\| \frac{x_1}{8} + \frac{x_2}{8} + \frac{x_3}{4} + \frac{x_4}{2} \right\| \leq 1 - \frac{1}{2}\delta_X^{(3)}(\epsilon).$$

(3) If $k = 3$ and $t_1 + t_2 + t_3 = \frac{1}{2}$, then

$$\left\| t_1x_1 + t_2x_2 + t_3x_3 + \frac{1}{2}x_4 \right\| \leq 1 - 4\min\{t_1, t_2, t_3\}\delta_X^{(3)}(\epsilon).$$

We obtain the intuitive and geometric idea for the proof of our main result Theorem 3.7 from the proof technique of the following theorem.

Theorem 3.6. *Let K be a nonempty weakly compact subset of a $2-$uniformly convex Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Then T has a fixed point in K.*

Proof. Define $\mathcal{F} = \{F \subseteq K : F \text{ is nonempty weakly compact } T-\text{regular set}\}$.

It is easy to see that the set inclusion $\subseteq$, defines a partial order relation on $\mathcal{F}$. By Zorn's lemma, we get a minimal element in $\mathcal{F}$.

Without loss of generality, we can assume that K is minimal in $\mathcal{F}$.

Let $x_1 \in K$ and define $x_{k+1} = \frac{x_k+Tx_k}{2} \in K$, for $k \in \mathbb{N}$.

By Lemma 3.3, we have $A(K, \{x_k\}) = K$ i.e., $r(x) = \limsup_{k\to\infty} \|x - x_k\| = r$, for all $x \in K$.

Note that $r = 0$ if and only if K is singleton.

For, if $r = 0$, then $\limsup_{k\to\infty} \|x - x_k\| = 0$, for all $x \in K$. This gives $\{x_k\}$ converges to every point in K. Hence K is singleton.

Conversely, suppose that K is singleton. Then it is easy to see that $r = 0$, as $\{x_k\} \subseteq K$.

We claim that $r = 0$. Suppose that $r > 0$. This implies that $x \neq Tx$, for all $x \in K$.

It is claimed that $Tx_n \in \text{aff}\{x_1, Tx_1\}$ for all $n \in \mathbb{N}$.

Suppose that there exists $n \in \mathbb{N}$ such that $Tx_n \notin \text{aff}\{x_1, Tx_1\}$.

Without loss of generality, we can assume that $Tx_2 \notin \text{aff}\{x_1, Tx_1\}$.

This gives $\{x_1, Tx_1, Tx_2\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_2\} = 2$. Hence $V(x_1, Tx_1, Tx_2) = \epsilon$ for some $\epsilon > 0$.

Since X is $2-$UC and $\delta_X^{(2)}$ is continuous, we have

$$\lim_{\rho\to 0}(r + \rho)\left(1 - \frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r + \rho)^2}\right)\right) = r\left(1 - \frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{r^2}\right)\right) < r$$

This implies that there is a $\rho_0 > 0$ such that

$$(r + \rho_0)\left(1 - \frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r + \rho_0)^2}\right)\right) < r.$$

Since $A(K, \{x_k\}) = K$ and for this $\rho_0 > 0$, there exists $N \in \mathbb{N}$ such that for $k \geq N$, we have

$$\begin{aligned} \|x_1 - x_k\| &\leq r + \rho_0 \\ \|Tx_1 - x_k\| &\leq r + \rho_0 \\ \|Tx_2 - x_k\| &\leq r + \rho_0 \end{aligned}$$

As X is $2-$UC, we have

$$\left\|\frac{x_1 + Tx_1 + Tx_2}{3} - x_k\right\| \leq (r + \rho_0)\left(1 - \delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho_0)^2}\right)\right), \text{ for } k \geq N.$$

Note that $x_3 = \frac{x_1}{4} + \frac{Tx_1}{4} + \frac{Tx_2}{2} \in \text{co}\{x_1, Tx_1, Tx_2\}$ and by Lemma 3.4, we get

$$\begin{aligned} \|x_3 - x_k\| &= \left\|\frac{x_1}{4} + \frac{Tx_1}{4} + \frac{Tx_2}{2} - x_k\right\| \\ &\leq (r + \rho_0)\left(1 - \frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho_0)^2}\right)\right), \text{ for } k \geq N. \end{aligned}$$

This implies that

$$\begin{aligned} r(x_3) &= \limsup_{k\to\infty} \|x_3 - x_k\| \\ &\leq (r + \rho_0)\left(1 - \frac{3}{4}\delta_X^{(2)}\left(\frac{\epsilon}{(r+\rho_0)^2}\right)\right) < r. \end{aligned}$$

This gives a contradiction to $A(K, \{x_k\}) = K$.

Therefore $Tx_n \in \text{aff}\{x_1, Tx_1\}$, for all $n \in \mathbb{N}$. This implies that $\{x_n\} \subseteq \text{aff}\{x_1, Tx_1\}$.

Since $\{x_n\}$ is a bounded sequence and $dim(\text{aff}\{x_1, Tx_1\}) = 1$, so it has a convergent subsequence say $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in K$ such that $x_{n_j} \to z$ as $j \to \infty$. Since $\lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = 0$ and T is nonexpansive, $Tz = z$. Hence $r = 0$.

This implies that K is singleton and T has a fixed point in K. □

Next we prove the main result of this paper.

Theorem 3.7. *Let K be a nonempty weakly compact subset of a $3-$uniformly convex Banach space X and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Then T has a fixed point in K.*

Proof. Note that by using Zorn's lemma, we get a nonempty minimal weakly compact $T-$regular subset of K.

Without loss of generality, we can assume that K is a nonempty minimal weakly compact $T-$regular set.

Let $x_1 \in K$ and define $x_{k+1} = \frac{x_k + Tx_k}{2} \in K$, for $k \in \mathbb{N}$.

By Lemma 3.3, we have $A(K, \{x_k\}) = K$ i.e., $r(x) = \limsup_{k\to\infty} \|x - x_k\| = r$, for all $x \in K$.

We claim that $r = 0$. Suppose that $r > 0$. This implies that $x \neq Tx$, for all $x \in K$.

Suppose that for every $n \in \mathbb{N}$, $Tx_n \in \text{aff}\{x_1, Tx_1\}$. Then $\{x_n\}$ is a bounded sequence in $\text{aff}\{x_1, Tx_1\}$, as K is bounded.

Hence $\{x_n\}$ has a convergent subsequence. This implies that T has a fixed point in K.

Suppose that there exists $n \in \mathbb{N}$ such that $Tx_n \notin \text{aff}\{x_1, Tx_1\}$.

Without loss of generality, we can assume that $Tx_2 \notin \text{aff}\{x_1, Tx_1\}$.

It is claimed that $Tx_n \in \text{aff}\{x_1, Tx_1, Tx_2\}$, for all $n \in \mathbb{N}$.

We use mathematical induction to prove our claim.

Case 1. It is claimed that $Tx_3 \in \text{aff}\{x_1, Tx_1, Tx_2\}$. Suppose that $Tx_3 \notin \text{aff}\{x_1, Tx_1, Tx_2\}$.

This gives $\{x_1, Tx_1, Tx_2, Tx_3\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_2, Tx_3\}) = 3$. Hence $V(x_1, Tx_1, Tx_2, Tx_3) = \epsilon$, for some $\epsilon > 0$.

Since X is $3-$UC and $\delta_X^{(3)}$ is continuous, there is a $\rho_0 > 0$ such that

$$(r+\rho_0)\left(1-\frac{1}{2}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r.$$

Since $A(K, \{x_k\}) = K$, there exists $N \in \mathbb{N}$ such that for $k \geq N$, we have

$$\begin{aligned} \|x_1 - x_k\| &\leq r+\rho_0 \\ \|Tx_1 - x_k\| &\leq r+\rho_0 \\ \|Tx_2 - x_k\| &\leq r+\rho_0 \\ \|Tx_3 - x_k\| &\leq r+\rho_0 \end{aligned}$$

As X is $3-$UC, we have for $k \geq N$

$$\left\|\frac{x_1+Tx_1+Tx_2+Tx_3}{4} - x_k\right\| \leq (r+\rho_0)\left(1-\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right).$$

Note that $x_4 = \frac{x_3+Tx_3}{2} = \frac{x_2+Tx_2}{4} + \frac{Tx_3}{2} = \frac{x_1}{8} + \frac{Tx_1}{8} + \frac{Tx_2}{4} + \frac{Tx_3}{2} \in \text{co}\{x_1, Tx_1, Tx_2, Tx_3\}$. Now, by Lemma 3.4, we get

$$\begin{aligned} \|x_4 - x_k\| &= \left\|\frac{x_1}{8} + \frac{Tx_1}{8} + \frac{Tx_2}{4} + \frac{Tx_3}{2} - x_k\right\| \\ &\leq (r+\rho_0)\left(1-\frac{1}{2}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right), \text{for } k \geq N. \end{aligned}$$

This implies that

$$\begin{aligned} r(x_4) &= \limsup_{k\to\infty} \|x_4 - x_k\| \\ &\leq (r+\rho_0)\left(1-\frac{1}{2}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r. \end{aligned}$$

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_3 \in \text{aff}\{x_1, Tx_1, Tx_2\}$.

Case 2. It is claimed that $Tx_4 \in \text{aff}\{x_1, Tx_1, Tx_2\}$. Suppose that $Tx_4 \notin \text{aff}\{x_1, Tx_1, Tx_2\}$.

This gives $\{x_1, Tx_1, Tx_2, Tx_4\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_2, Tx_4\}) = 3$.

Since $Tx_3 \in \text{aff}\{x_1, Tx_1, Tx_2\}$, we have the following cases:

(a). $Tx_3 \in \text{aff}\{x_2, Tx_2\}$
(b). $Tx_3 \notin \text{aff}\{x_2, Tx_2\}$.

Subcase 2(a). Suppose that $Tx_3 \in \text{aff}\{x_2, Tx_2\}$. Then $Tx_3 = (1-\mu_3)x_2 + \mu_3 Tx_2$, for some $\mu_3 \in \mathbb{R}$. By the nonexpansiveness of T, we have
$\frac{1}{2}\|Tx_2 - x_2\| = \|x_3 - x_2\| \geq \|Tx_3 - Tx_2\| = |1-\mu_3|\|Tx_2 - x_2\|$.

This gives $\frac{1}{2} \leq \mu_3 \leq \dfrac{3}{2}$. Note that $\mu_3 \neq \frac{1}{2}$. For, if $\mu_3 = \frac{1}{2}$, then $Tx_3 = x_3$.

$$\begin{aligned}
\text{Now } x_4 = \frac{x_3 + Tx_3}{2} &= \frac{1}{2}\left(\frac{x_2 + Tx_2}{2} + Tx_3\right) \\
&= \frac{x_2}{4} + \frac{1}{4}\left(\frac{Tx_3 - (1-\mu_3)x_2}{\mu_3}\right) + \frac{Tx_3}{2} \\
&= \left(\frac{2\mu_3 - 1}{4\mu_3}\right)x_2 + \left(\frac{2\mu_3 + 1}{4\mu_3}\right)Tx_3 \\
&= \left(\frac{2\mu_3 - 1}{8\mu_3}\right)x_1 + \left(\frac{2\mu_3 - 1}{8\mu_3}\right)Tx_1 + \left(\frac{2\mu_3 + 1}{4\mu_3}\right)Tx_3 \\
&= t_1 x_1 + t_1 Tx_1 + (1-2t_1)Tx_3 \text{ where } t_1 = \frac{2\mu_3 - 1}{8\mu_3}.
\end{aligned}$$

Since $\mu_3 > \frac{1}{2}$, we have $t_1 > 0$ and $1 - 2t_1 > 0$. This gives x_4 lies in the interior of $\text{co}\{x_1, Tx_1, Tx_3\}$.

Since $\{x_1, Tx_1, Tx_2, Tx_4\}$ is affinely independent and $Tx_3 \in \text{aff}\{x_2, Tx_2\}$, we have $\{x_1, Tx_1, Tx_3, Tx_4\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_3, Tx_4\}) = 3$. Hence $V(x_1, Tx_1, Tx_3, Tx_4) = \epsilon$ for some $\epsilon > 0$.

Since $\delta_X^{(3)}$ is continuous and X is $3-$UC, there is a $\rho_0 > 0$ such that

$$(r+\rho_0)\left(1 - 2\min\{t_1, 1-2t_1\}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r$$

As $A(K, \{x_k\}) = K$, there exist $N \in \mathbb{N}$ such that for $k \geq N$, we have

$$\left\|\frac{x_1 + Tx_1 + Tx_3 + Tx_4}{4} - x_k\right\| \leq (r+\rho_0)\left(1 - \delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right).$$

Note that $x_5 = \frac{x_4 + Tx_4}{2} = \frac{1}{2}\left(t_1x_1 + t_1Tx_1 + (1-2t_1)Tx_3 + Tx_4\right)$.

This implies that x_5 lies in the interior of $\text{co}\{x_1, Tx_1, Tx_3, Tx_4\}$. Now, by Lemma 3.4, for $k \geq N$ we have

$$\begin{aligned}
\|x_5 - x_k\| &= \left\|\frac{1}{2}(t_1x_1 + t_1Tx_1 + (1-2t_1)Tx_3 + Tx_4) - x_k\right\| \\
&\leq (r+\rho_0)\left(1 - 2\min\{t_1, 1-2t_1\}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
r(x_5) &= \limsup_{k\to\infty}\|x_5 - x_k\| \\
&\leq (r+\rho_0)\left(1 - 2\min\{t_1, 1-2t_1\}\delta_X^{(3)}\left(\frac{\epsilon}{(r+\rho_0)^3}\right)\right) < r.
\end{aligned}$$

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_4 \in \text{aff}\{x_1, Tx_1, Tx_2\}$.
Subcase 2(b). Suppose that $Tx_3 \notin \text{aff}\{x_2, Tx_2\}$. Then $\{x_2, Tx_2, Tx_3\}$ is affinely independent and $dim(\text{co}\{x_2, Tx_2, Tx_3\}) = 2$.

Since $Tx_3 \in \text{aff}\{x_1, Tx_1, Tx_2\}$ and $Tx_3 \notin \text{aff}\{x_2, Tx_2\}$, we have $Tx_3 = ax_1 + bTx_1 + (1 - (a + b))Tx_2$, for $a, b \in \mathbb{R}$ with $a \neq b$.

Since $\{x_1, Tx_1, Tx_2, Tx_4\}$ is affinely independent and $Tx_3 = ax_1 + bTx_1 + (1 - (a + b))Tx_2$, we have $\{x_2, Tx_2, Tx_3, Tx_4\}$ is affinely independent and $dim(\text{co}\{x_2, Tx_2, Tx_3, Tx_4\}) = 3$. This implies that $V(x_2, Tx_2, Tx_3, Tx_4) = \epsilon$, for some $\epsilon > 0$.

Therefore by case 1, we get $r(x_5) < r$.

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_4 \in \text{aff}\{x_1, Tx_1, Tx_2\}$.
Case 3. Now, we assume that $Tx_n \in \text{aff}\{x_1, Tx_1, Tx_2\}$, for $1 \leq n \leq m - 1$.

To prove that $Tx_m \in \text{aff}\{x_1, Tx_1, Tx_2\}$.

Suppose not. Then $\{x_1, Tx_1, Tx_2, Tx_m\}$ is affinely independent.

Since $Tx_k \in \text{aff}\{x_1, Tx_1, Tx_2\}$ for $3 \leq k \leq m - 1$, we have the following cases:

(a). $Tx_k \in \text{aff}\{x_2, Tx_2\}$ for $k = 3, 4, \ldots, m - 1$
(b). $Tx_k \notin \text{aff}\{x_2, Tx_2\}$ for some $k \in \{3, 4, \ldots, m - 1\}$.

Subcase 3(a). Suppose that $Tx_k \in \text{aff}\{x_2, Tx_2\}$ for $3 \leq k \leq m - 1$. Then $x_k \in \text{aff}\{x_2, Tx_2\}$ for $3 \leq k \leq m$, as $x_k = \frac{x_{k-1}+Tx_{k-1}}{2}$.

Let $x_k = (1 - \lambda_k)x_2 + \lambda_k Tx_2$ for some $\lambda_k \in \mathbb{R}$, $2 \leq k \leq m$ and $Tx_k = (1 - \mu_k)x_2 + \mu_k Tx_2$ for some $\mu_k \in \mathbb{R}$, $2 \leq k \leq m - 1$. Note that $\lambda_{k+1} = \frac{\lambda_k+\mu_k}{2}$, for $2 \leq k \leq m - 1$, as $x_{k+1} = \frac{x_k+Tx_k}{2}$. Hence $\lambda_3 = \frac{1}{2}$, as $\lambda_2 = 0$, $\mu_2 = 1$.

Since we work with the $\text{aff}\{x_2, Tx_2\}$, we can identify the $\text{aff}\{x_2, Tx_2\}$ with the real line $\mathbb{R}$ by assuming $x_2 = 0$ and $Tx_2 = 1$. In this way, we get that $x_k = \lambda_k$ and $Tx_k = \mu_k$ for $2 \leq k \leq m - 1$.

As $Tx_k \neq x_k$, we have $\lambda_k \neq \mu_k$ and $\lambda_k \neq \lambda_{k+1}$ for $2 \leq k \leq m - 1$.

Note that, from case 2(a), we have $\lambda_3 < \mu_3$. This implies that $\lambda_3 < \lambda_4 < \mu_3$, as $\lambda_{k+1} = \frac{\lambda_k+\mu_k}{2}$.

It is claimed that $\lambda_k < \lambda_{k+1}$ and $\lambda_k < \mu_k$, for $4 \leq k \leq m - 1$.

Since T is nonexpansive, we have

$|\mu_4 - \mu_3| \|x_2 - Tx_2\| = \|Tx_3 - Tx_4\| \leq \|x_3 - x_4\| = (\lambda_4 - \lambda_3)\|x_2 - Tx_2\|$.

This implies that $-\lambda_4 + \lambda_3 \leq \mu_4 - \mu_3 \leq \lambda_4 - \lambda_3$. Now, since $\lambda_4 = \frac{\lambda_3+\mu_3}{2}$, we have $\lambda_4 < \mu_4$. This gives $\lambda_4 < \lambda_5 < \mu_4$.

Continuing in this way, we get $\lambda_k < \lambda_{k+1} < \mu_k$ for $3 \leq k \leq m - 1$.

Hence $0 = \lambda_2 < \lambda_3 < \lambda_4 < \cdots < \lambda_{m-1} < \lambda_m < \mu_{m-1}$.

This implies that λ_k lies in the interior of $\text{co}\{\lambda_2, \mu_{m-1}\}$ for $3 \leq k \leq m$.

Hence x_k lies in the interior of $\text{co}\{x_2, Tx_{m-1}\}$ for $3 \leq k \leq m$.

This implies that x_m lies in the interior of $\text{co}\{x_1, Tx_1, Tx_{m-1}\}$, as $x_2 = \frac{x_1+Tx_1}{2}$.

Now, since $\text{aff}\{x_1, Tx_1, Tx_2\} = \text{aff}\{x_1, Tx_1, Tx_{m-1}\}$ and $Tx_m \notin \text{aff}\{x_1, Tx_1, Tx_2\}$, we have $\{x_1, Tx_1, Tx_{m-1}, Tx_m\}$ is affinely independent and $dim(\text{co}\{x_1, Tx_1, Tx_{m-1}, Tx_m\}) = 3$.

Hence x_{m+1} lies in the interior of $\text{co}\{x_1, Tx_1, Tx_{m-1}, Tx_m\}$, as $x_{m+1} = \frac{x_m+Tx_m}{2}$.

Now, by using the arguments as in case 2(a), it is easy to see that $r(x_{m+1}) = \limsup_{k\to\infty} \|x_{m+1} - x_k\| < r$.

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_m \in \text{aff}\{x_1, Tx_1, Tx_2\}$.

Subcase 3(b). Suppose that there exists $k \in \mathbb{N}$ such that $3 \leq k \leq m-1$ and $Tx_k \notin \text{aff}\{x_2, Tx_2\}$.

Let k_0 be the least integer satisfying $Tx_{k_0} \notin \text{aff}\{x_2, Tx_2\}$. This implies $Tx_3, Tx_4, \ldots, Tx_{k_0-1} \in \text{aff}\{x_2, Tx_2\}$.

Then $\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\}$ is affinely independent and $\text{aff}\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\} = \text{aff}\{x_1, Tx_1, Tx_2\}$.

Now, we consider the set $\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\}$.

Suppose that $Tx_k \in \text{aff}\{x_{k_0}, Tx_{k_0}\}$ for $k_0 + 1 \leq k \leq m-1$.

Then using the arguments as in case 3(a), it is easy to see that x_{m+1} lies in the interior of $\text{co}\{x_{k_0-1}, Tx_{k_0-1}, Tx_{m-1}, Tx_m\}$ and $\{x_{k_0-1}, Tx_{k_0-1}, Tx_{m-1}, Tx_m\}$ is affinely independent. Now, it is apparent that $r(x_{m+1}) < r$, as X is 3$-$UC.

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_m \in \text{aff}\{x_1, Tx_1, Tx_2\}$.

Suppose that there exists $k \in \mathbb{N}$ such that $k_0 + 1 \leq k \leq m-1$ and $Tx_k \notin \text{aff}\{x_{k_0}, Tx_{k_0}\}$.

Let k_1 be the least integer satisfying $Tx_{k_1} \notin \text{aff}\{x_{k_0}, Tx_{k_0}\}$. This implies that $Tx_{k_0+1}, Tx_{k_0+2}, \ldots, Tx_{k_1-1} \in \text{aff}\{x_{k_0}, Tx_{k_0}\}$.

Then $\{x_{k_1-1}, Tx_{k_1-1}, Tx_{k_1}\}$ is affinely independent and $\text{aff}\{x_{k_1-1}, Tx_{k_1-1}, Tx_{k_1}\} = \text{aff}\{x_{k_0-1}, Tx_{k_0-1}, Tx_{k_0}\}$.

Now, we consider the set $\{x_{k_1-1}, Tx_{k_1-1}, Tx_{k_1}\}$.

Continuing in this way, we can find n_0 is the largest integer such that $k_1 \leq n_0 \leq m-1$ and $Tx_{n_0} \notin \text{aff}\{x_{n_0-1}, Tx_{n_0-1}\}$. This implies that $Tx_n \in \text{aff}\{x_{n_0}, Tx_{n_0}\}$ for $n_0 \leq n \leq m-1$.

Then using the arguments as in case 3(a), it is easy to see that x_{m+1} lies in the interior of $\text{co}\{x_{n_0-1}, Tx_{n_0-1}, Tx_{m-1}, Tx_m\}$ and $\{x_{n_0-1}, Tx_{n_0-1}, Tx_{m-1}, Tx_m\}$ is affinely independent. Now, it is apparent that $r(x_{m+1}) < r$, as X is 3$-$UC.

This gives a contradiction to $A(K, \{x_k\}) = K$. Hence $Tx_m \in \text{aff}\{x_1, Tx_1, Tx_2\}$.

Hence, by mathematical indution $Tx_n \in \text{aff}\{x_1, Tx_1, Tx_2\}$, for all $n \in \mathbb{N}$. This implies that $\{x_n\} \subseteq \text{aff}\{x_1, Tx_1, Tx_2\}$.

Since $\{x_n\}$ is a bounded sequence and $dim(\text{aff}\{x_1, Tx_1, Tx_2\}) = 2$, so it has a convergent subsequence i.e., there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and $z \in K$ such that $x_{n_j} \to z$ as $j \to \infty$.

Since $\lim_{j\to\infty} \|x_{n_j} - Tx_{n_j}\| = 0$ and T is nonexpansive, we have $Tz = z$. Hence $r = 0$. This implies that K is singleton and T has a fixed point in K. □

Remark 3.8. In the light of Theorem 3.6 and Theorem 3.7, it is natural to expect that if K is a nonempty weakly compact subset of a $k-$UC Banach space X, for $k > 3$ and if $T : K \to K$ is a nonexpansive map satisfying $\frac{x+Tx}{2} \in K$ for all $x \in K$, then T has a fixed point in K.

3.2. **Banach space with Opial property.**

Theorem 3.9. *Let K be a nonempty weakly compact subset of a Banach space X having the Opial property and $T : K \to K$ be a nonexpansive map. Further, assume that K is $T-$regular. Define a sequence $\{x_n\}$ in K by $x_{n+1} = \frac{x_n + Tx_n}{2}$ for $n \in \mathbb{N}$ and $x_1 \in K$. Then T has a fixed point in K and $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. By Lemma 2.6, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since K is weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in K$ such that $\{x_{n_k}\}$ converges weakly to z. Also, we have

$$\|x_{n_k} - Tz\| \leq \|x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - Tz\|, \text{ for all } k \in \mathbb{N}.$$

Hence

$$\liminf_{k\to\infty} \|x_{n_k} - Tz\| \leq \liminf_{k\to\infty} \|x_{n_k} - z\|.$$

Since X has the Opial property, we obtain $Tz = z$. Also note that, $\{\|x_n - z\|\}$ is a decreasing sequence.

It is claimed that $\{x_n\}$ converges weakly to z. Suppose that $\{x_n\}$ does not converge weakly to z.

Then there exists a subsequence $\{x_{\widehat{n}_j}\}$ of $\{x_n\}$ which does not converge weakly to z. Since K is weakly compact and $\{x_{\widehat{n}_j}\} \subseteq K$, there exists a subsequence of $\{x_{\widehat{n}_j}\}$ whose weak limit is $w \in K$ and $z \neq w$.

Without loss of generality, we can assume that $\{x_{\widehat{n}_j}\}$ converges weakly to w. It is easy to see that $Tw = w$, as $\lim_{j\to\infty} \|x_{\widehat{n}_j} - Tx_{\widehat{n}_j}\| = 0$. Also, it is apparent that $\{\|x_n - w\|\}$ is a decreasing sequence, as $Tw = w$.

Since X has the Opial property, $\{x_{\widehat{n}_j}\}$ converges weakly to w and $\{x_{n_k}\}$ converges weakly to z, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - z\| &= \lim_{k\to\infty} \|x_{n_k} - z\| < \lim_{k\to\infty} \|x_{n_k} - w\| = \lim_{n\to\infty} \|x_n - w\| \\
&= \lim_{j\to\infty} \|x_{\widehat{n}_j} - w\| < \lim_{j\to\infty} \|x_{\widehat{n}_j} - z\| = \lim_{n\to\infty} \|x_n - z\|.
\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ weakly converges to z. □

Acknowledgements. *The authors thank the anonymous reviewer for the comments and suggestions. Also, the authors thank Prof. P. Veeramani, Department of Mathematics, IIT Madras (India) for the fruitful discussions regarding the subject matter of this paper. The first author thank the University Grants Commission (India), for providing financial support to carry out this research work in the form of Project fellow through Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai.*

References

[1] J. M. Ayerbe Toledano, T. Domínguez Benavides and G. López Acedo, Measures of Noncompactness in Metric Fixed Point Theory, in Operator Theory: Advances and Applications, Birkhõuser Verlag, Basel, 1997.

[2] F. E. Browder, Nonexpansive nonlinear operations in a Banach space, Proc. Nat. Acad. Sci. U.S.A. 54 (1965), 1041–1043.

[3] M. Edelstein, The construction of an asymptotic center with a fixed point property, Bull. Amer. Math. Soc. 78 (1972), 206–208.

[4] K. Goebel and W. A. Kirk, Iteration processes for nonexpansive mappings, Contemp. Math. 21 (1983), 115–123.

[5] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math., Cambridge Univ. Press, 1990.

[6] K. Goebel and R. Schöneberg, Moons, bridges, birds and nonexpansive mappings in Hilbert space, Bull. Austral. Math. Soc. 17 (1977), 463–466.

[7] J. P. Gossez and L. Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, Pacific J. Math. 40 (1972), 563–573.

[8] D. Göhde, Zum Prinzip der knontraktiven Abbildung, Math. Nachr. 30 (1965), 251–258.

[9] A. R. Khan and N. Hussain, Iterative approximation of fixed points of nonexpansive maps, Sci. Math. Jpn. 4 (2001), 749–757.

[10] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72(1965), 1004–1006.

[11] W. A. Kirk, Nonexpansive mappings in product spaces, set-valued mappings and k-uniform rotundity, Proc. Sympos. Pure Math. 45 (1986), 51–64.

[12] M. A. Krasnoselskii, Two remarks on the method of successive approximations, Uspehi Mat. Nauk (N.S.) 10 (1955), 123-û127.

[13] T. C. Lim, On moduli of $k-$convexity, Abstr. Appl. Anal. 4 (1999), 243–247.

[14] P. K. Lin, $k-$uniform rotundity is equivalent to $k-$uniform convexity, J. Math. Anal. Appl. 132 (1988), 349–355.

[15] P. K. Lin, K. K. Tan and H. K. Xu, Demiclosedness principle and asymptotic behavior for asymptotically nonexpansive mappings, Nonlinear Analysis, Theory, Methods and Applications 24 (1995), 929–946.

[16] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73 (1967), 591–597.

[17] E. Sliverman, Definitions of Lebesgue area for surfaces in metric spaces, Rivista Mat. Univ. Parma 2 (1951), 47–76.

[18] F. Sullivan, A generalization of uniformly rotund Banach spaces, Canad. J. Math. 31 (1979), 628–636.

[19] T. Suzuki, Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. 340 (2008), 1088–1095.

[20] P. Veeramani, On some fixed point theorems on uniformly convex Banach spaces, J. Math. Anal. Appl. 167 (1992), 160–166.