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Alternate product adjacencies in digital topology

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Abstract

We study properties of Cartesian products of digital images, using a variety of adjacencies that have appeared in the literature.

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1. Introduction

We study various adjacency relations for Cartesian products of multiple digital images. We are particularly interested in "product properties" - properties that are preserved by taking Cartesian products - and "factor properties" for which possession by a Cartesian product of digital images implies possession of the property by the factors. Many of the properties examined in this paper were considered in [9] for adjacencies based on the normal product adjacency. We consider other adjacencies in this paper, including the tensor product adjacency, the Cartesian product adjacency, and the composition or lexicographic adjacency.

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2. Preliminaries

Much of the material that appears in this section is quoted or paraphrased from [9, 12], and other papers cited in this section.

We use $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ to represent the sets of natural numbers, integers, and real numbers, respectively,

A digital image is a graph. Usually, we consider the vertex set of a digital image to be a subset of $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. Further, we often, although not always, restrict our study of digital images to finite graphs. We will assume familiarity with the topological theory of digital images. See, e.g., [3] for many of the standard definitions. All digital images X are assumed to carry their own adjacency relations (which may differ from one image to another). When we wish to emphasize the particular adjacency relation we write the image as (X, κ), where κ represents the adjacency relation.

2.1. **Common adjacencies.** To denote that x and y are κ-adjacent points of some digital image, we use the notation $x \leftrightarrow_\kappa y$, or $x \leftrightarrow y$ when κ can be understood.

The c_u-adjacencies are commonly used. Let $x, y \in \mathbb{Z}^n$, $x \neq y$. Let u be an integer, $1 \leq u \leq n$. We say x and y are c_u-adjacent, $x \leftrightarrow_{c_u} y$, if

- there are at most u indices i for which $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

A c_u-adjacency is often denoted by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

For Cartesian products of digital images, the normal product adjacency (see Definitions 2.1 and 2.2) has been used in papers including [22, 6, 11, 9] (errors in [22] are corrected in [6]). The tensor product adjacency (see Definition 2.3), Cartesian product adjacency (see Definition 2.4), and the lexicographic adjacency (see Definition 2.6) have not to our knowledge been studied in digital topology, so their respective roles in digital topology remain to be determined.

Given digital images or graphs (X, κ) and (Y, λ), the *normal product adjacency* $NP(\kappa, \lambda)$, also called the *strong product adjacency* (denoted $\kappa_*(\kappa, \lambda)$ in [11]) generated by κ and λ on the Cartesian product $X \times Y$ is defined as follows.

Definition 2.1 ([1, 28])**.** Let $x, x' \in X$, $y, y' \in Y$. Then (x, y) and (x', y') are $NP(\kappa, \lambda)$-adjacent in $X \times Y$ if and only if

- $x = x'$ and $y \leftrightarrow_\lambda y'$; or
- $x \leftrightarrow_\kappa x'$ and $y = y'$; or
- $x \leftrightarrow_\kappa x'$ and $y \leftrightarrow_\lambda y'$.

As a generalization of Definition 2.1, we have the following.

Definition 2.2 ([9])**.** Let u and v be positive integers, $1 \leq u \leq v$. Let $\{(X_i, \kappa_i)\}_{i=1}^{v}$ be digital images. Let $NP_u(\kappa_1, \ldots, \kappa_v)$ be the adjacency defined on the Cartesian product $\Pi_{i=1}^{v} X_i$ as follows. For $x_i, x_i' \in X_i$, $p = (x_1, \ldots, x_v)$ and $q = (x_1', \ldots, x_v')$ are $NP_u(\kappa_1, \ldots, \kappa_v)$-adjacent if and only if

- for at least 1 and at most u indices i, $x_i \leftrightarrow_{\kappa_i} x_i'$, and
- for all other indices i, $x_i = x_i'$.

Definition 2.3 ([20])**.** The *tensor product adjacency* on the Cartesian product $\Pi_{i=1}^{v} X_i$ of (X_i, κ_i), denoted $T(\kappa_1, \ldots, \kappa_v)$, is as follows. Given $x_i, x_i' \in X_i$, we have $(x_1, \ldots, x_v)$ and $(x_1', \ldots, x_v')$ are $T(\kappa_1, \ldots, \kappa_v)$-adjacent in $\Pi_{i=1}^{v} X_i$ if and only if for all i, $x_i \leftrightarrow_{\kappa_i} x_i'$.

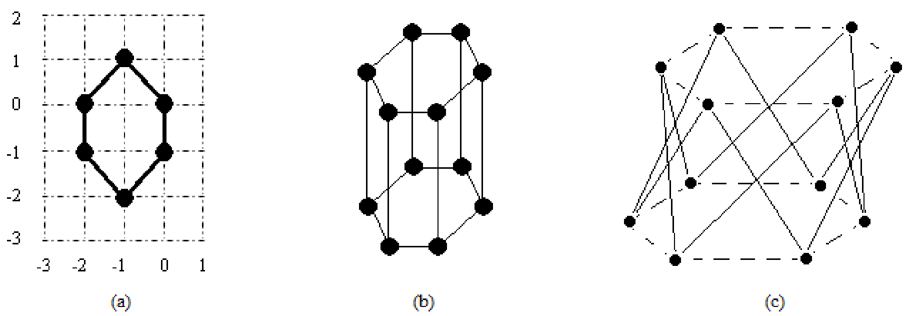

FIGURE 1. A digital simple closed curve and its Cartesian product with $[0,1]_{\mathbb{Z}}$. (a) shows the simple closed curve $MSC_8 \subset (\mathbb{Z}^2, c_2)$ [21]. (b) shows the set $MSC_8 \times [0,1]_{\mathbb{Z}} \subset \mathbb{Z}^3$ with either the $c_2 \times c_1$- or the $NP_1(c_2, c_1)$-adjacency. (c) shows the set $MSC_8 \times [0,1]_{\mathbb{Z}} \subset \mathbb{Z}^3$ with the $T(c_2, c_1)$-adjacency, where adjacencies are shown by the solid lines. If the points of MSC_8 are circularly labeled $p_0, \ldots, p_5$, then the $T(c_2, c_1)$-neighbors of (p_i, t) are $(p_{(i-1) \bmod 6}, 1-t)$ and $(p_{(i+1) \bmod 6}, 1-t)$, $t \in \{0, 1\}$.

Definition 2.4 ([26])**.** The *Cartesian product adjacency* on the Cartesian product $\Pi_{i=1}^{v} X_i$ of (X_i, κ_i), denoted $\times_{i=1}^{v} \kappa_i$ or $\kappa_1 \times \ldots \times \kappa_v$, is as follows. Given $x_i, x_i' \in X_i$, we have $(x_1, \ldots, x_v)$ and $(x_1', \ldots, x_v')$ are $\times_{i=1}^{v} \kappa_i$-adjacent in $\Pi_{i=1}^{v} X_i$ if and only if for some i, $x_i \leftrightarrow_{\kappa_i} x_i'$, and for all indices $j \neq i$, $x_j = x_j'$.

The following has an elementary proof.

Proposition 2.5. *For* $\Pi_{i=1}^{v}(X_i, \kappa_i)$, $\times_{i=1}^{v} \kappa_i = NP_1(\kappa_1, \ldots, \kappa_v)$.

Definition 2.6 ([19])**.** Let (X_i, κ_i) be digital images, $1 \leq i \leq v$. Let $x_i, x_i' \in X_i$. Let $p = (x_1, \ldots, x_v)$, $p' = (x_1', \ldots, x_v')$. We say p and p' are adjacent in the *composition* or *lexicographic* adjacency on $\Pi_{i=1}^{v} X_i$ if $x_1 \leftrightarrow_{\kappa_1} x_1'$, or if for some index j, $1 \leq j < v$, we have $(x_1, \ldots, x_j) = (x'_1, \ldots, x'_j)$ and $x_{j+1} \leftrightarrow_{\kappa_{j+1}} x'_{j+1}$. The adjacency is denoted $L(\kappa_1, \ldots, \kappa_v)$.

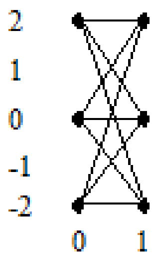

FIGURE 2. An illustration of lexicographic adjacency. This is $[0,1]_{\mathbb{Z}} \times \{-2, 0, 2\}$, with both factors regarded as subsets of $(\mathbb{Z}, c_1)$, and the $L(c_1, c_1)$ adjacency.

Remark 2.7. Notice that for p and p' to be $L(\kappa_1, \ldots, \kappa_v)$-adjacent with x_k and x'_k κ_k-adjacent, for indices $m > k$ we do not require that x_m and x'_m be either equal or adjacent. See, e.g., Figure 2, where $(0,0)$ and $(1,2)$ are $L(c_1, c_1)$-adjacent. This is unlike other adjacencies discussed above.

2.2. **Connectedness.** A subset Y of a digital image (X, κ) is κ-*connected* [25], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a sequence $\{y_i\}_{i=0}^m \subset Y$ such that $a = y_0$, $b = y_m$, and $y_i \leftrightarrow_\kappa y_{i+1}$ for $0 \leq i < m$.

For two subsets $A, B \subset X$, we will say that A and B are *adjacent* when there exist points $a \in A$ and $b \in B$ such that a and b are equal or adjacent. Thus sets with nonempty intersection are automatically adjacent, while disjoint sets may or may not be adjacent. It is easy to see that a finite union of connected adjacent sets is connected.

2.3. **Continuous functions.** The following generalizes a definition of [25].

Definition 2.8 ([4])**.** Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)-continuous if for every κ-connected A of X we have that $f(A)$ is a λ-connected subset of Y.

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be reformulated in terms of adjacency of points:

Theorem 2.9 ([25, 4])**.** *A function $f : X \to Y$ is continuous if and only if, for any adjacent points $x, x' \in X$, the points $f(x)$ and $f(x')$ are equal or adjacent.*

Note that similar notions appear in [14, 15] under the names *immersion*, *gradually varied operator*, and *gradually varied mapping*.

Theorem 2.10 ([3, 4]). *If $f : (A, \kappa) \to (B, \lambda)$ and $g : (B, \lambda) \to (C, \mu)$ are continuous, then $g \circ f : (A, \kappa) \to (C, \mu)$ is continuous.*

Example 2.11 ([25]). A constant function between digital images is continuous.

Example 2.12. The identity function $1_X : (X, \kappa) \to (X, \kappa)$ is continuous.

Definition 2.13. Let (X, κ) be a digital image in $\mathbb{Z}^n$. Let $x, y \in X$. A *κ-path of length m from x to y* is a set $\{x_i\}_{i=0}^m \subset X$ such that $x = x_0$, $x_m = y$, and x_{i-1} and x_i are equal or κ-adjacent for $1 \leq i \leq m$. If $x = y$, we say $\{x\}$ is a *path of length 0 from x to x.*

Notice that for a path from x to y as described above, the function $f : [0, m]_{\mathbb{Z}} \to X$ defined by $f(i) = x_i$ is (c_1, κ)-continuous. Such a function is also called a *κ-path of length m from x to y.*

2.4. **Digital homotopy.** A homotopy between continuous functions may be thought of as a continuous deformation of one of the functions into the other over a finite time period.

Definition 2.14 ([4]; see also [23]). Let (X, κ) and (Y, κ') be digital images. Let $f, g : X \to Y$ be (κ, κ')-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by

  $$F_x(t) \;=\; F(x, t) \text{ for all } t \in [0, m]_{\mathbb{Z}}$$

  is $(2, \kappa')-$continuous. That is, $F_x(t)$ is a path in Y.
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by

  $$F_t(x) \;=\; F(x, t) \text{ for all } x \in X$$

  is $(\kappa, \kappa')-$continuous.

Then F is a digital $(\kappa, \kappa')-$homotopy between f and g, and f and g are digitally $(\kappa, \kappa')-$homotopic in Y. If for some $x_0 \in X$ we have $F(x_0, t) = F(x_0, 0)$ for all $t \in [0, m]_{\mathbb{Z}}$, we say F holds x_0 fixed, and F is a pointed homotopy.

We denote a pair of homotopic functions as described above by $f \simeq_{\kappa,\kappa'} g$. When the adjacency relations κ and κ' are understood in context, we say f and g are *digitally homotopic* (or just *homotopic*) to abbreviate "digitally $(\kappa, \kappa')-$homotopic in Y," and write $f \simeq g$.

Proposition 2.15 ([23, 4]). *Digital homotopy is an equivalence relation among digitally continuous functions $f : X \to Y$.*

Definition 2.16 ([5]). Let $f : X \to Y$ be a (κ, κ')-continuous function and let $g : Y \to X$ be a (κ', κ)-continuous function such that

$$f \circ g \simeq_{\kappa',\kappa'} 1_X \text{ and } g \circ f \simeq_{\kappa,\kappa} 1_Y.$$

Then we say X and Y have the same (κ, κ')-homotopy type and that X and Y are (κ, κ')-homotopy equivalent, denoted $X \simeq_{\kappa,\kappa'} Y$ or as $X \simeq Y$ when κ and κ' are understood. If for some $x_0 \in X$ and $y_0 \in Y$ we have $f(x_0) = y_0$, $g(y_0) = x_0$, and there exists a homotopy between $f \circ g$ and 1_X that holds x_0 fixed, and a homotopy between $g \circ f$ and 1_Y that holds y_0 fixed, we say (X, x_0, κ) and (Y, y_0, κ') are pointed homotopy equivalent and that (X, x_0) and (Y, y_0) have the same pointed homotopy type, denoted $(X, x_0) \simeq_{\kappa,\kappa'} (Y, y_0)$ or as $(X, x_0) \simeq (Y, y_0)$ when κ and κ' are understood.

It is easily seen, from Proposition 2.15, that having the same homotopy type (respectively, the same pointed homotopy type) is an equivalence relation among digital images (respectively, among pointed digital images).

2.5. **Continuous and connectivity preserving multivalued functions.** Given sets X and Y, a *multivalued function* $f : X \to Y$ assigns a subset of Y to each point of x. We will write $f : X \multimap Y$. For $A \subset X$ and a multivalued function $f : X \multimap Y$, let $f(A) = \bigcup_{x \in A} f(x)$.

Definition 2.17 ([24])**.** A multivalued function $f : X \multimap Y$ is *connectivity preserving* if $f(A) \subset Y$ is connected whenever $A \subset X$ is connected.

As is the case with Definition 2.8, we can reformulate connectivity preservation in terms of adjacencies.

Theorem 2.18 ([12])**.** *A multivalued function $f : X \multimap Y$ is connectivity preserving if and only if the following are satisfied:*

- *For every $x \in X$, $f(x)$ is a connected subset of Y.*
- *For any adjacent points $x, x' \in X$, the sets $f(x)$ and $f(x')$ are adjacent.*

Definition 2.17 is related to a definition of multivalued continuity for subsets of $\mathbb{Z}^n$ given and explored by Escribano, Giraldo, and Sastre in [16, 17] based on subdivisions. (These papers make a small error with respect to compositions, that is corrected in [18].) Their definitions are as follows:

Definition 2.19. For any positive integer r, the *r-th subdivision* of $\mathbb{Z}^n$ is

$$\mathbb{Z}^n_r = \{(z_1/r, \ldots, z_n/r) \mid z_i \in \mathbb{Z}\}.$$

An adjacency relation κ on $\mathbb{Z}^n$ naturally induces an adjacency relation (which we also call κ) on $\mathbb{Z}^n_r$ as follows: $(z_1/r, \ldots, z_n/r), (z'_1/r, \ldots, z'_n/r)$ are adjacent in $\mathbb{Z}^n_r$ if and only if $(z_1, \ldots, z_n)$ and $(z'_1, \ldots, z'_n)$ are adjacent in $\mathbb{Z}^n$.

Given a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, the *$r$-th subdivision* of X is

$$S(X, r) = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n_r \mid (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor) \in X\}.$$

Let $E_r : S(X, r) \to X$ be the natural map sending $(x_1, \ldots, x_n) \in S(X, r)$ to $(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$.

Definition 2.20. For a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, a function $f : S(X, r) \to Y$ *induces a multivalued function* $F : X \multimap Y$ if $x \in X$ implies

$$F(x) = \bigcup_{x' \in E_r^{-1}(x)} \{f(x')\}.$$

Definition 2.21. A multivalued function $F : X \multimap Y$ is called *continuous* when there is some r such that F is induced by some single valued continuous function $f : S(X, r) \to Y$.

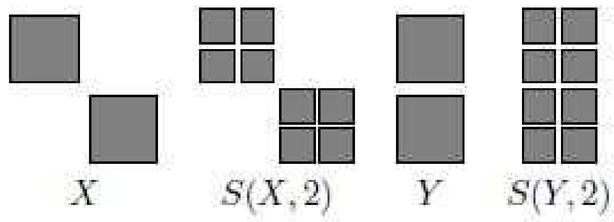

FIGURE 3. [12] Two images X and Y with their second subdivisions. (Subdivisions are drawn at half-scale.)

Note [12] that the subdivision construction (and thus the notion of continuity) depends on the particular embedding of X as a subset of $\mathbb{Z}^n$. In particular we may have $X, Y \subset \mathbb{Z}^n$ with X isomorphic to Y but $S(X, r)$ not isomorphic to $S(Y, r)$. E.g., in Figure 3, when we use 8-adjacency for all images, X and Y are isomorphic, each being a set of two adjacent points, but $S(X, 2)$ and $S(Y, 2)$ are not isomorphic since $S(X, 2)$ can be disconnected by removing a single point, while this is impossible in $S(Y, 2)$.

The definition of connectivity preservation makes no reference to X as being embedded inside of any particular integer lattice $\mathbb{Z}^n$.

Proposition 2.22 ([16, 17])**.** *Let $F : X \multimap Y$ be a continuous multivalued function between digital images. Then*

- *for all $x \in X$, $F(x)$ is connected; and*
- *for all connected subsets A of X, $F(A)$ is connected.*

Theorem 2.23 ([12])**.** *For $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, if $F : X \multimap Y$ is a continuous multivalued function, then F is connectivity preserving.*

The subdivision machinery often makes it difficult to prove that a given multivalued function is continuous. By contrast, many maps can easily be shown to be connectivity preserving.

2.6. **Other notions of multivalued continuity.** Other notions of continuity have been given for multivalued functions between graphs (equivalently, between digital images). We have the following.

Definition 2.24 ([27])**.** Let $F : X \multimap Y$ be a multivalued function between digital images.

- F has *weak continuity* if for each pair of adjacent $x, y \in X$, $f(x)$ and $f(y)$ are adjacent subsets of Y.

- F has *strong continuity* if for each pair of adjacent $x, y \in X$, every point of $f(x)$ is adjacent or equal to some point of $f(y)$ and every point of $f(y)$ is adjacent or equal to some point of $f(x)$.

Proposition 2.25 ([12]). *Let $F : X \multimap Y$ be a multivalued function between digital images. Then F is connectivity preserving if and only if F has weak continuity and for all $x \in X$, $F(x)$ is connected.*

Example 2.26 ([12]). If $F : [0,1]_{\mathbb{Z}} \multimap [0,2]_{\mathbb{Z}}$ is defined by $F(0) = \{0,2\}$, $F(1) = \{1\}$, then F has both weak and strong continuity. Thus a multivalued function between digital images that has weak or strong continuity need not have connected point-images. By Theorem 2.18 and Proposition 2.22 it follows that neither having weak continuity nor having strong continuity implies that a multivalued function is connectivity preserving or continuous.

Example 2.27 ([12]). Let $F : [0,1]_{\mathbb{Z}} \multimap [0,2]_{\mathbb{Z}}$ be defined by $F(0) = \{0,1\}$, $F(1) = \{2\}$. Then F is continuous and has weak continuity but does not have strong continuity.

Proposition 2.28 ([12]). *Let $F : X \multimap Y$ be a multivalued function between digital images. If F has strong continuity and for each $x \in X$, $F(x)$ is connected, then F is connectivity preserving.*

The following shows that not requiring the image of a point $F(p)$ to be connected can yield topologically unsatisfying consequences for weak and strong continuity.

Example 2.29 ([12]). Let X and Y be nonempty digital images. Let the multivalued function $f : X \multimap Y$ be defined by $f(x) = Y$ for all $x \in X$.

- f has both weak and strong continuity.
- f is connectivity preserving if and only if Y is connected.

As a specific example [12] consider $X = \{0\} \subset \mathbb{Z}$ and $Y = \{0,2\}$, all with c_1 adjacency. Then the function $F : X \multimap Y$ with $F(0) = Y$ has both weak and strong continuity, even though it maps a connected image surjectively onto a disconnected image.

2.7. **Shy maps and their inverses.**

Definition 2.30 ([5]). Let $f : X \to Y$ be a continuous surjection of digital images. We say f is *shy* if

- for each $y \in Y$, $f^{-1}(y)$ is connected, and
- for every $y_0, y_1 \in Y$ such that y_0 and y_1 are adjacent, $f^{-1}(\{y_0, y_1\})$ is connected.

Shy maps induce surjections on fundamental groups [5]. Some relationships between shy maps f and their inverses f^{-1} as multivalued functions were studied in [7, 12, 8]. Shyness as a factor or product property for the normal product adjacency was studied in [9]. We have the following.

Theorem 2.31 ([12, 8]). *Let $f : X \to Y$ be a continuous surjection between digital images. Then the following are equivalent.*

- *f is a shy map.*
- *For every connected $Y_0 \subset Y$, $f^{-1}(Y_0)$ is a connected subset of X.*
- *$f^{-1} : Y \multimap X$ is a connectivity preserving multi-valued function.*
- *$f^{-1} : Y \multimap X$ is a multi-valued function with weak continuity such that for all $y \in Y$, $f^{-1}(y)$ is a connected subset of X.*

2.8. **Other tools.** Other terminology we use includes the following. Given a digital image $(X, \kappa) \subset \mathbb{Z}^n$ and $x \in X$, the set of points adjacent to $x \in \mathbb{Z}^n$ and the neighborhood of x in $\mathbb{Z}^n$ are, respectively,

$$N_\kappa(x) = \{y \in \mathbb{Z}^n \mid y \text{ is } \kappa\text{-adjacent to } x\},$$
$$N^*_\kappa(x) = N_\kappa(x) \cup \{x\}.$$

3. Maps on products

In this section, we consider various product adjacencies with respect to continuity of functions.

3.1. **General properties.**

Definition 3.1. Let κ_1 and κ_2 be adjacency relations on a set X. We say κ_1 *dominates* κ_2, $\kappa_1 \geq_d \kappa_2$, or κ_2 *is dominated by* κ_1, $\kappa_2 \leq_d \kappa_1$, if for $x, x' \in X$, if x and x' are κ_1-adjacent then x and x' are κ_2-adjacent.

Example 3.2. We have the following comparisons of adjacencies.

- For $X \subset \mathbb{Z}^n$ and $1 \leq u \leq v \leq n$, $c_u \geq_d c_v$.
- For $\Pi_{i=1}^v (X_i, \kappa_i)$ and $1 \leq u \leq v \leq n$,
$$NP_u(\kappa_1, \ldots \kappa_v) \geq_d NP_v(\kappa_1, \ldots \kappa_v).$$
- For $\Pi_{i=1}^v (X_i, \kappa_i)$, $T(\kappa_1, \ldots \kappa_v) \geq_d NP_v(\kappa_1, \ldots \kappa_v)$.
- For $\Pi_{i=1}^v (X_i, \kappa_i)$, we have:
  - $NP_u(\kappa_1, \ldots, \kappa_v) \geq_d L(\kappa_1, \ldots, \kappa_v)$ for $1 \leq u \leq v$;
  - $T(\kappa_1, \ldots, \kappa_v) \geq_d L(\kappa_1, \ldots, \kappa_v)$;
  - $\times_{i=1}^v \kappa_i \geq_d L(\kappa_1, \ldots, \kappa_v)$.

Proof. These follow immediately from the definitions of these adjacencies. □

The next example shows that there are adjacencies that can be applied to the same set X such that neither dominates the other.

Example 3.3. In $X = \mathbb{Z}^6 = \mathbb{Z}^3 \times \mathbb{Z}^3$, neither of $T(c_2, c_2)$ nor $T(c_1, c_3)$ dominates the other.

Proof. Consider the points $p = (0,0,0,0,0,0)$ and $q = (1,1,0,1,1,0)$. We have $p \leftrightarrow_{T(c_2,c_2)} q$ but p and q are not $T(c_1, c_3)$-adjacent. Therefore $T(c_2, c_2)$ does not dominate $T(c_1, c_3)$.

Now consider $r = (1,0,0,1,1,1)$. We have $p \leftrightarrow_{T(c_1,c_3)} r$ but p and r are not $T(c_2, c_2)$-adjacent. Therefore $T(c_1, c_3)$ does not dominate $T(c_2, c_2)$. □

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Domination, and being dominated, are transitive relations among the adjacencies of a graph. I.e., we have the following.

Proposition 3.4. *Given adjacencies κ, λ, μ for a graph, if $\kappa \leq_d \lambda$ and $\lambda \leq_d \mu$, then $\kappa \leq_d \mu$.*

Proof. Elementary, and left to the reader. □

Proposition 3.5. *Let $f : X \to Y$ be a function.*

- *Let λ_1 and λ_2 be adjacency relations on Y. If f is (κ, λ_1) continuous and $\lambda_1 \geq_d \lambda_2$, then f is (κ, λ_2) continuous.*
- *Let κ_1 and κ_2 be adjacency relations on X. If f is (κ_1, λ)-continuous and $\kappa_1 \leq_d \kappa_2$, then f is (κ_2, λ)-continuous.*

Proof. The assertions follows from the definitions of continuity and the $\geq_d$ relation. □

Given functions $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$, $1 < i \leq v$, the function

$$\Pi_{i=1}^v f_i : \Pi_{i=1}^v X_i \to \Pi_{i=1}^v Y_i$$

is defined by

$$(\Pi_{i=1}^v f_i)(x_1, \ldots, x_v) = (f_1(x_1), \ldots, f_v(x_v)), \text{ where } x_i \in X_i.$$

3.2. **Normal product.** Here, we recall continuity properties of the normal product adjacency.

Theorem 3.6 ([9])**.** *Let $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$, $1 < i \leq v$. Then the product map*

$$f = \Pi_{i=1}^v f_i : (\Pi_{i=1}^v X_i, NP_v(\kappa_1, \ldots, \kappa_v)) \to (\Pi_{i=1}^v Y_i, NP_v(\lambda_1, \ldots, \lambda_v))$$

is continuous if and only if each f_i is continuous.

Theorem 3.7 ([9])**.** *Let $X = \Pi_{i=1}^v X_i$. Let $f_i : (X_i, \kappa_i) \to (Y_i \, \lambda_i)$, $1 \leq i \leq v$.*

- *For $1 \leq u \leq v$, if the product map $f = \Pi_{i=1}^v f_i : (X, NP_u(\kappa_1, \ldots, \kappa_v)) \to (\Pi_{i=1}^v Y_i, NP_u(\lambda_1, \ldots, \kappa_v))$ is an isomorphism, then for $1 \leq i \leq v$, f_i is an isomorphism.*
- *If f_i is an isomorphism for all i, then the product map $f = \Pi_{i=1}^v f_i : (X, NP_v(\kappa_1, \ldots, \kappa_v)) \to (\Pi_{i=1}^v Y_i, NP_v(\lambda_1, \ldots, \kappa_v))$ is an isomorphism.*

Theorem 3.8 ([22, 9])**.** *The projection maps $p_i : (\Pi_{j=1}^v X_j, NP_u(\kappa_1, \ldots, \kappa_v)) \to (X_i, \kappa_i)$ defined by $p_i(x_1, \ldots, x_v) = x_i$ for $x_i \in (X_i, \kappa_i)$, are all continuous, for $1 \leq u \leq v$.*

3.3. **Tensor product.** For the tensor product adjacency, we have the following.

Proposition 3.9. *Suppose $X = \Pi_{i=1}^v X_i$ has a pair of $T(\kappa_1, \ldots, \kappa_v)$-adjacent points. Then*

- *each X_i has 2 κ_i-adjacent points; and*

- *If $f : (X, T(\kappa_1, \ldots, \kappa_v)) \to (\Pi_{j=1}^{w} Y_j, T(\lambda_1, \ldots, \lambda_w))$ is continuous and not constant on some component of X, then for every j, Y_j has 2 λ_j-adjacent points.*

Proof. Let $p = (x_1, \ldots, x_v)$ and $p' = (x'_1, \ldots, x'_v)$ be $T(\kappa_1, \ldots, \kappa_v)$-adjacent in X. Then for each i, x_i and x'_i are κ_i-adjacent in X_i, which establishes the first assertion. Further, if f is as hypothesized, the continuity of f implies there are $T(\kappa_1, \ldots, \kappa_v)$-adjacent p, p' such that $f(p) = (y_1, \ldots, y_w)$ and $f(p') = (y'_1, \ldots, y'_w)$ are unequal, hence $T(\lambda_1, \ldots, \lambda_w)$-adjacent. Therefore, for all j, y_j and y'_j are λ_j-adjacent. □

It is easy to construct examples showing that the assertions obtained from Proposition 3.9 by substituting the normal product adjacency NP_v for T are false.

Theorem 3.10. *Let $X = \Pi_{i=1}^{v} X_i$, $Y = \Pi_{i=1}^{v} Y_i$. If the product map*

$$f = \Pi_{i=1}^{v} f_i : (X, T(\kappa_1, \ldots, \kappa_v)) \to (Y, T(\lambda_1, \ldots, \lambda_v))$$

is continuous, then for each i, $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ is continuous.

Proof. If x_i, x'_i are κ_i-adjacent in X_i, then $p = (x_1, \ldots, x_v)$ and $p' = (x'_1, \ldots, x'_v)$ are $T(\kappa_1, \ldots, \kappa_v)$-adjacent in X. Thus $f(p)$ and $f(p')$ are equal or $T(\lambda_1, \ldots, \lambda_v)$-adjacent in Y. This implies $f_i(x_i)$ and $f_i(x'_i)$ are equal or λ_i-adjacent in Y_i. Thus f_i is continuous. □

However, the converse to Theorem 3.10 is not generally true, as shown in the following.

Example 3.11. Let $f : [0, 1]_{\mathbb{Z}} \to [0, 1]_{\mathbb{Z}}$ be the identity function. Let $g : [0, 1]_{\mathbb{Z}} \to [0, 1]_{\mathbb{Z}}$ be the constant function $g(x) = 0$. Then, using Examples 2.12 and 2.11, f and g are each (c_1, c_1)-continuous, but $f \times g : [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}} \to [0, 1]_{\mathbb{Z}} \times [0, 1]_{\mathbb{Z}}$ is not $(T(c_1, c_1), T(c_1, c_1))$-continuous.

Proof. This follows from the observations that $(0, 0)$ and $(1, 1)$ are $T(c_1, c_1)$-adjacent, but $(f \times g)(0, 0) = (0, 0)$ and $(f \times g)(1, 1) = (1, 0)$ are neither equal nor $T(c_1, c_1)$-adjacent. □

A partial converse to Theorem 3.10 is obtained by using the following notion.

Definition 3.12. A continuous function $f : (X, \kappa) \to (Y, \lambda)$ is *locally one-to-one* if $f|_{N^*_\kappa(x,1)}$ is one-to-one for all $x \in X$.

Note any function between digital images that is one-to-one must be locally one-to-one.

Theorem 3.13. *Suppose $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ is continuous and locally one-to-one for $1 \leq i \leq v$. Then the product function $f = \Pi_{i=1}^{v} f_i : \Pi_{i=1}^{v} X_i \to \Pi_{i=1}^{v} Y_i$ is $(T(\kappa_1, \ldots, \kappa_v), T(\lambda_1, \ldots, \lambda_v))$-continuous and locally one-to-one.*

Proof. Suppose $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ is continuous and locally one-to-one for $1 \le i \le v$. Let $p = (x_1, \ldots, x_v)$ and $p' = (x'_1, \ldots, x'_v)$ be $T(\kappa_1, \ldots, \kappa_v)$-adjacent, where x_i and x'_i are κ_i-adjacent in X_i. Since f_i is continuous and locally one-to-one, we must have that $f_i(x_i)$ and $f_i(x'_i)$ are λ_i-adjacent in Y_i. Thus, $f(p)$ and $f(p')$ are $T(\lambda_1, \ldots, \lambda_v)$-adjacent, so f is continuous and locally one-to-one. □

Theorem 3.14. *Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. Then the product map*

$$f = \Pi_{i=1}^v f_i : (X, T(\kappa_1, \ldots, \kappa_v)) \to (Y, T(\lambda_1, \ldots, \lambda_v))$$

is an isomorphism if and only if each f_i is an isomorphism.

Proof. If f is an isomorphism, each f_i must be one-to-one and onto. Therefore, $f_i^{-1} : Y_i \to X_i$ is a single-valued function.

By Theorem 3.10, each f_i is continuous. Since $f^{-1} = \Pi_{i=1}^v f_i^{-1}$, it follows from Theorem 3.10 that each f_i^{-1} is continuous. Hence f_i is an isomorphism.

Conversely, if each f_i is an isomorphism, then f is one-to-one and onto, so $f^{-1} = \Pi_{i=1}^v f_i^{-1}$ is a single-valued function. By Theorem 3.13, f is continuous.

Similarly, f^{-1} is continuous. Therefore, f is an isomorphism. □

Theorem 3.15. *The projection maps $p_i : (\Pi_{i=1}^v X_i, T(\kappa_1, \ldots, \kappa_v)) \to (X_i, \kappa_i)$ defined by $p_i(x_1, \ldots, x_v) = x_i$ for $x_i \in X_i$ are all continuous.*

Proof. Let $p = (x_1, \ldots, x_v)$ and $p' = (x'_1, \ldots, x'_v)$ be $T(\kappa_1, \ldots, \kappa_v)$-adjacent in $\Pi_{i=1}^v X_i$, where $x_i, x'_i \in X_i$. Then for all indices i, $x_i = p_i(p)$ and $x'_i = p_i(p')$ are κ_i-adjacent. Thus, p_i is continuous. □

A seeming oddity is that a common method of injection that is often continuous, is not continuous when the tensor product adjacency is used, as shown in the following.

Proposition 3.16. *Let (X, κ) and (Y, λ) be digital images. Let $y \in Y$. If X has a pair of κ-adjacent points, then the function $f : X \to (X \times Y, T(\kappa, \lambda))$ defined by $f(x) = (x, y)$ is not continuous.*

Proof. This is because given κ-adjacent $x, x' \in X$, $f(x) = (x, y)$ and $f(x') = (x', y)$ are not $T(\kappa, \lambda)$-adjacent. □

3.4. **Cartesian product.**

Theorem 3.17. *Let $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ be functions between digital images, $1 \le i \le v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. Then the product function $f = \Pi_{i=1}^v f_i : X \to Y$ is $(\times_{i=1}^v \kappa_i, \times_{i=1}^v \lambda_i)$-continuous if and only if each f_i is continuous.*

Proof. Suppose f is continuous. Let $x_i \leftrightarrow_{\kappa_i} x'_i$ in X_i. Let $p = (x_1, \ldots, x_v)$, $p' = (x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_v)$. Then $p \leftrightarrow_{\times_{i=1}^v \kappa_i} p'$, so either $f(p) = f(p')$ or $f(p) \leftrightarrow_{\times_{i=1}^v \lambda_i} f(p')$. The former case implies $f_i(x_i) = f_i(x'_i)$ and the latter case implies $f_i(x_i) \leftrightarrow_{\lambda_i} f_i(x'_i)$. Hence f_i is continuous.

Suppose each f_i is continuous. Let p and p' be $\times_{i=1}^{v}\kappa_i$-adjacent points of X. Then there is only one index k in which p and p' differ, i.e., for some $x_i \in X_i$ and $x_k' \in X_k$, $p = (x_1, \ldots, x_v)$, $p' = (x_1, \ldots, x_{k-1}, x_k', x_{k+1}, \ldots, x_v)$, and $x_k \leftrightarrow_{\kappa_k} x_k'$. Then $f(p)$ and $f(p')$ have the same i^{th} coordinate for $i \neq k$, and have k^{th} coordinates of $f_k(x_k)$ and $f_k(x_k')$, respectively. Continuity of f_k implies either $f_k(x_k) = f_k(x_k')$ or $f_k(x_k) \leftrightarrow_{\kappa_k} f_k(x_k')$. Therefore, f is continuous. □

Theorem 3.18. *The projection maps* $p_i : (\Pi_{i=1}^{v} X_i, \times_{i=1}^{v}\kappa_i) \to (X_i, \kappa_i)$ *defined by* $p_i(x_1, \ldots, x_v) = x_i$ *for* $x_i \in X_i$ *are all continuous.*

Proof. This follows from Proposition 2.5 and Theorem 3.8. □

By contrast with Proposition 3.16, we have the following.

Proposition 3.19. *Let* (X_i, κ_i) *be digital images,* $1 \leq i \leq v$*. Let* $x_i \in X_i$*. The functions* $I_i : X_i \to (\Pi_{i=1}^{v} X_i, \times_{i=1}^{v}\kappa_i)$ *defined by*

$$I_i(x) = \begin{cases} (x, x_2, \ldots, x_v) & \text{for } i = 1; \\ (x_1, \ldots, x_{i-1}, x, x_{i+1} \ldots, x_v) & \text{for } 1 < i < v; \\ (x_1, \ldots, x_{v-1}, x) & \text{for } i = v, \end{cases}$$

are continuous.

Proof. This follows immediately from Definition 2.4. □

Theorem 3.20. *Let* $X = \Pi_{i=1}^{v} X_i$*,* $Y = \Pi_{i=1}^{v} Y_i$*. Then the product map*

$$f = \Pi_{i=1}^{v} f_i : (X, \times_{i=1}^{v}\kappa_i) \to (Y, \times_{i=1}^{v}\lambda_i)$$

is an isomorphism if and only if each f_i *is an isomorphism,.*

Proof. Suppose f is an isomorphism. Then it follows from Proposition 2.5 and Theorem 3.7 that f_i is an isomorphism.

Suppose each f_i is an isomorphism. Then f must be one-to-one and onto, and by Theorem 3.17, f is continuous. Similarly, $f^{-1} = \Pi_{i=1}^{v} f_i^{-1}$ is continuous. Therefore, f is an isomorphism. □

3.5. Lexicographic adjacency.

Theorem 3.21. *Suppose* $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ *is a function between digital images,* $1 \leq i \leq v$*. Let* $f = \Pi_{i=1}^{v} f_i : \Pi_{i=1}^{v} X_i \to \Pi_{i=1}^{v} Y_i$ *be the product function.*

- *If* f *is* $(L(\kappa_1, \ldots, \kappa_v), L(\lambda_1, \ldots, \lambda_v))$*-continuous, then each* f_i *is* (κ_i, λ_i)*-continuous. Further, if* f *is locally one-to-one, then each* f_i *is locally one-to-one.*
- *If each* f_i *is a continuous function that is locally one-to-one, then* f *is* $(L(\kappa_1, \ldots, \kappa_v), L(\lambda_1, \ldots, \lambda_v))$*-continuous.*

Proof. Suppose f is $(L(\kappa_1, \ldots, \kappa_v), L(\lambda_1, \ldots, \lambda_v))$-continuous. Let $x_i, x_i' \in X_i$ such that $x_i \leftrightarrow_{\kappa_i} x_i'$. Let $p_0 = (x_1, x_2, \ldots, x_v)$ and let

$$p_i = \begin{cases} (x_1', x_2, \ldots, x_v) & \text{for } i = 1; \\ (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_v) & \text{for } 1 < i < v; \\ (x_1, \ldots, x_{v-1}, x_v') & \text{for } i = v. \end{cases}$$

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Notice

(3.1) $\quad p_0$ and p_i differ only at index i and $p_0 \leftrightarrow_{L(\kappa_1,\ldots,\kappa_v)} p_i$ for $1 \le i \le v$.

Therefore, $f(p_0)$ and $f(p_i)$ are $L(\lambda_1,\ldots,\lambda_v)$-adjacent or equal. It follows from statement (3.1) that $f_i(x_i)$ and $f_i(x_i')$ are λ_i-adjacent or equal. Since $\{x_i, x_i'\}$ is an arbitrary set of κ_i-adjacent members of X_i, f_i is (κ_i,λ_i)-continuous. Further if f is locally one-to-one, then from statement (3.1), $f_i(x_i)$ and $f_i(x_i')$ are not equal, so f_i is locally one-to-one.

Suppose each f_i is continuous and locally one-to-one. Let $p, p' \in X = \Pi_{i=1}^v X_i$, where $p = (x_1,\ldots,x_v)$, $p' = (x_1',\ldots,x_v')$, for $x_i, x_i' \in X_i$. Assume $p \leftrightarrow_{L(\kappa_1,\ldots,\kappa_v)} p'$. Let k be the smallest index such that $x_k \leftrightarrow_{\kappa_k} x_k'$. Since f_k is locally one-to-one,

$$f_k(x_k) \leftrightarrow_{\lambda_k} f_k(x_k'). \tag{3.2}$$

- If $k = 1$, it follows from Definition 2.6 that $f(p) \leftrightarrow_{L(\lambda_1,\ldots,\lambda_v)} f(p')$.
- Otherwise, $i < k$ implies $x_i = x_i'$, hence $f_i(x_i) = f_i(x_i')$. Together with statement (3.2), this implies $f(p) \leftrightarrow_{L(\lambda_1,\ldots,\lambda_v)} f(p')$.

Then f is $(L(\kappa_1,\ldots,\kappa_v), L(\lambda_1,\ldots,\lambda_v))$-continuous, since p and p' were arbitrarily chosen. □

The following example illustrates the importance of the locally one-to-one hypothesis in Theorem 3.21.

Example 3.22. Let $X_i = [0,i]_{\mathbb{Z}}$ for $i \in \{1,2\}$. Let $f : X_1 \to X_2$ be the constant function with value 0. Then f and 1_{X_2} are (c_1,c_1) continuous. However, $f \times 1_{X_2} : X_1 \times X_2 \to X_2^2$ is not $(L(c_1,c_1), L(c_1,c_1))$-continuous.

Proof. Consider the points $p = (0,0)$ and $p' = (1,2)$. These points are $L(c_1,c_1)$-adjacent in $X_1 \times X_2$. However, $(f \times 1_{X_2})(p) = (0,0)$ and $(f \times 1_{X_2})(p') = (0,2)$ are neither equal nor $L(c_1,c_1)$-adjacent in X_2^2. □

Theorem 3.23. *Suppose $f_i : (X_i,\kappa_i) \to (Y_i,\lambda_i)$ is a function between digital images, $1 \le i \le v$. Let $f = \Pi_{i=1}^v f_i : \Pi_{i=1}^v X_i \to \Pi_{i=1}^v Y_i$ be the product function. Then f is an $(L(\kappa_1,\ldots,\kappa_v), L(\lambda_1,\ldots,\lambda_v))$-isomorphism if and only if each f_i is a (κ_i,λ_i)-isomorphism.*

Proof. This follows easily from Theorem 3.21. □

Proposition 3.24. *The projection map $p_1 : (\Pi_{i=1}^v X_i, L(\kappa_1,\ldots,\kappa_v)) \to (X_1,\kappa_1)$ is continuous.*

Proof. Let $p \leftrightarrow_{L(\kappa_1,\ldots,\kappa_v)} p'$ in $\Pi_{i=1}^v X_i$. Then $p = (x_1,\ldots,x_v)$, $p' = (x_1',\ldots,x_v')$ for some $x_i, x_i' \in X_i$, where either $x_1 = x_1'$ or $x_1 \leftrightarrow_{\kappa_1} x_1'$. Since $p_1(p) = x_1$ and $p_1(p') = x_1'$, it follows that p_1 is continuous. □

By contrast, we have the following.

Example 3.25. Let $v > 1$. The projection maps $p_i : ([0,2]_{\mathbb{Z}}^v, L(c_1,\ldots,c_1)) \to ([0,2]_{\mathbb{Z}}, c_1)$ are not continuous for $1 < i \le v$.

Proof. Let $x = (0, 0, \ldots, 0)$, $y = (1, 2, \ldots, 2)$. Then $x \leftrightarrow_{L(c_1,\ldots,c_1)} y$ in $[0, 2]_{\mathbb{Z}}^v$, but $i > 1$ implies $p_i(x) = 0$ and $p_i(y) = 2$, which are not c_1-adjacent in $[0, 2]_{\mathbb{Z}}$. The assertion follows. □

3.6. **More on isomorphisms.** We have the following.

Theorem 3.26. *Let $\sigma : \{i\}_{i=1}^v \to \{i\}_{i=1}^v$ be a permutation. Let $f_i : (X_i, \kappa_i) \to (Y_{\sigma(i)}, \lambda_{\sigma(i)})$ be an isomorphism of digital images, $1 \le i \le v$. Let $1 \le u \le v$. Let (κ, λ) be any of*

$$(NP_u(\kappa_1, \ldots, \kappa_v), NP_u(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(v)})),$$
$$(T(\kappa_1, \ldots, \kappa_v), T(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(v)})), \text{ or}$$
$$(\times_{i=1}^v \kappa_i, \times_{i=1}^v \lambda_{\sigma(i)}).$$

Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_{\sigma(i)}$. Then the function $f : X \to Y$ defined by

$$f(x_1, \ldots, x_v) = (f_1(x_1), \ldots, (f_v(x_v))$$

is an isomorphism.

Proof. It is easy to see that f is one-to-one and onto. Continuity of f and of f^{-1} follows easily from the definitions of the adjacencies under discussion. Thus, f is an isomorphism. □

The following example shows that the lexicographic adjacency does not yield a conclusion analogous to that of Theorem 3.26.

Example 3.27. Let $X_1 = \{0, 1\} \subset (\mathbb{Z}, c_1)$. Let $X_2 = \{0, 2\} \subset (\mathbb{Z}, c_1)$. Then $X = (X_1 \times X_2, L(c_1, c_1))$ and $Y = (X_2 \times X_1, L(c_1, c_1))$ are not isomorphic.

Proof. Observe that X is connected, since the 4 points of X form a path in the sequence

$$(0, 0),\ (1, 0),\ (0, 2),\ (1, 2)$$

(see Figure 2). However, Y is not connected, as there is no path in Y from $(0, 0)$ to $(2, 0)$. The assertion follows. □

4. Connectedness

In this section, we compare product adjacencies with respect to the property of connectedness.

Theorem 4.1 ([9]). *Let (X_i, κ_i) be digital images, $i \in \{1, 2, \ldots, v\}$. Then (X_i, κ_i) is connected for all i if and only $(\Pi_{i=1}^v X_i, NP_v(\kappa_1, \ldots, \kappa_v))$ is connected.*

Theorem 4.2. *Let (X_i, κ_i) be digital images, $i \in \{1, 2, \ldots, v\}$. If $\Pi_{i=1}^v X_i$ is $T(\kappa_1, \ldots, \kappa_v)$-connected, then X_i is κ_i-connected for all i.*

Proof. These assertions follow from Definition 2.8 and Theorem 3.15. □

However, the converse to Theorem 4.2 is not generally true, as shown by the following.

Example 4.3. Let $X = \{0\} \subset \mathbb{Z}$, $Y = [0,1]_{\mathbb{Z}} \subset \mathbb{Z}$. Then X and Y are each c_1-connected. However:

- $X \times Y = \{(0,0),(0,1)\}$ is not $T(c_1,c_1)$-connected.
- $Y \times Y$ has two $T(c_1,c_1)$-components, $\{(0,0),(1,1)\}$ and $\{(1,0),(0,1)\}$.

See also Figure 1(c), which illustrates that $MSC_8 \times [0,1]_{\mathbb{Z}}$ is not $T(c_2,c_1)$-connected, although MSC_8 is c_2-connected and $[0,1]_{\mathbb{Z}}$ is c_1-connected.

For the Cartesian product adjacency, we have the following.

Theorem 4.4. *Let (X_i,κ_i) be digital images, $i \in \{1,2,\ldots,v\}$. Then $\Pi_{i=1}^v X_i$ is $\times_{i=1}^v \kappa_i$-connected if and only if X_i is κ_i-connected for all i.*

Proof. Suppose $X = \Pi_{i=1}^v X_i$ is $\times_{i=1}^v \kappa_i$-connected. It follows from Proposition 3.18 that each X_i is κ_i-connected.

Suppose each X_i is κ_i-connected. Let $p = (x_1,\ldots,x_v)$ and $p' = (x'_1,\ldots,x'_v)$ be points of X such that $x_i, x'_i \in X_i$. There are κ_i-paths P_i in X_i from x_i to x'_i. If the functions I_i are as in Proposition 3.19, then it is easily seen that $\bigcup_{i=1}^v I_i(P_i)$ is a $\times_{i=1}^v \kappa_i$-path in X from p to p'. Since p and p' were arbitrarily chosen, it follows that X is $\times_{i=1}^v \kappa_i$-connected. □

Proposition 4.5. *Let (X,κ) and (Y,λ) be digital images, such that $|X| > 1$. Then $(X \times Y, L(\kappa,\lambda))$ is connected if and only if (X,κ) is connected.*

Proof. Suppose (X,κ) is connected. Let $p = (x,y)$, $p' = (x',y')$, with $x,x' \in X$, $y,y' \in Y$.

- If $x = x'$ then, since $|X| > 1$ and X is connected, there exists $x_0 \in X$ such that $x \leftrightarrow_\kappa x_0$. Therefore, $p \leftrightarrow_{L(\kappa,\lambda)} (x_0,y) \leftrightarrow_{L(\kappa,\lambda)} (x,y') = p'$.
- Suppose $x \neq x'$. Since X is connected, there is a path in X, $P = \{x_i\}_{i=0}^n$, such that
$$x = x_0 \leftrightarrow_\kappa x_1 \leftrightarrow_\kappa \ldots \leftrightarrow_\kappa x_{n-1} \leftrightarrow_\kappa x_n = x'.$$
Therefore,
$$p = (x_0,y) \leftrightarrow_{L(\kappa,\lambda)} (x_1,y') \leftrightarrow_{L(\kappa,\lambda)} (x_2,y') \leftrightarrow_{L(\kappa,\lambda)} \ldots$$
$$\leftrightarrow_{L(\kappa,\lambda)} (x_n,y') = p'.$$

Therefore, $(X \times Y, L(\kappa,\lambda))$ is connected.

Suppose (X,κ) is not connected. Then there exist $x,x' \in X$ such that x and x' are in distinct components of X. Let $y,y' \in Y$. By Definition 2.6, there is no path in $(X \times Y, L(\kappa,\lambda))$ from (x,y) to (x',y'). Therefore, $(X \times Y, L(\kappa,\lambda))$ is not connected. □

An argument similar to that used for the proof of Proposition 4.5 yields the following.

Theorem 4.6. *Let (X_i,κ_i) be digital images, $1 \le i \le v$. Suppose k is the smallest index for which $|X_k| > 1$. Then $(\Pi_{i=1}^v X_i, L(\kappa_1,\ldots,\kappa_v))$ is connected if and only if (X_k,κ_k) is connected.*

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5. Homotopy

5.1. **Tensor product.** In [9], it is shown that many homotopy properties are preserved by Cartesian products with the NP_v adjacency. We show that we cannot make analogous claims for the tensor product adjacency.

Example 5.1. There are digital images (X_i, κ_i) and (Y_i, λ_i) and continuous functions $f_i, g_i : X_i \to Y_i$, $i \in \{1, 2\}$, such that

$$f_i \simeq g_i \text{ but } f_1 \times f_2 \not\simeq_{T(\kappa_1,\kappa_2),T(\lambda_1,\lambda_2)} g_1 \times g_2.$$

Proof. We can use Example 4.3. E.g., if $X_1 = X_2 = Y_1 = Y_2 = [0, 1]_{\mathbb{Z}}$, $f_1 = f_2 : X_1 \to Y_1$ is the identity function, and $g_1 = g_2 : X_2 \to Y_2$ is the constant function taking the value 0, we have $f_1 \simeq_{c_1,c_1} g_1$ and $f_2 \simeq_{c_1,c_1} g_2$. As we saw in Example 4.3, $[0, 1]^2_{\mathbb{Z}}$ is not $T(c_1, c_1)$-connected, so its identity function $f_1 \times f_2$ is not homotopic to the constant function $g_1 \times g_2$. □

Example 5.2. There are digital images (X_i, κ_i) and (Y_i, λ_i) for $i \in \{1, 2\}$, such that X_i and Y_i have the same homotopy type, but $(X_1 \times X_2, T(\kappa_1, \lambda_1))$ and $(Y_1 \times Y_2, T(\kappa_2, \lambda_2))$ do not have the same homotopy type.

Proof. We saw in Example 4.3 that $[0, 1]^2_{\mathbb{Z}}$ is not $T(c_1, c_1)$-connected; however, it is trivial that $\{0\}^2 = \{(0, 0)\}$ is $T(c_1, c_1)$-connected. Therefore, we can take $X_1 = X_2 = [0, 1]_{\mathbb{Z}} \subset (\mathbb{Z}, c_1)$, $Y_1 = Y_2 = \{0\} \subset (\mathbb{Z}, c_1)$. □

5.2. **Cartesian product adjacency.**

Theorem 5.3. *Let $f_i, g_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ be continuous functions between digital images, $1 \le i \le v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$, $f = \Pi_{i=1}^v f_i : X \to Y$, $g = \Pi_{i=1}^v g_i : X \to Y$. Then $f \simeq_{\times_{i=1}^v \kappa_i, \times_{i=1}^v \lambda_i} g$ if and only if for all i, $f_i \simeq_{\kappa_i,\lambda_i} g_i$. Further, f and g are pointed homotopic if and only if for each i, f_i and g_i are pointed homotopic.*

Proof. Suppose $f \simeq_{\times_{i=1}^v \kappa_i, \times_{i=1}^v \lambda_i} g$. Then there is a homotopy

$$H : \Pi_{i=1}^v X_i \times [0, m]_{\mathbb{Z}} \to \Pi_{i=1}^v X_i$$

such that $H(p, 0) = f(p)$ and $H(p, m) = g(p)$ for all $p \in X$. Let $x_i \in X_i$ and let $H_i : X_i \times [0, m]_{\mathbb{Z}} \to Y_i$ be defined by

$$H_i(x, t) = p_i(H(I_i(x), t)),$$

where I_i is the continuous injection of Proposition 3.19 corresponding to the point $(x_1, \dots, x_v) \in X$ and p_i is the continuous projection map of Theorem 3.18. Then

$$H_i(x, 0) = p_i(f(I_i(x)) = f_i(x) \text{ and } H_i(x, m) = p_i(g(I_i(x)) = g_i(x).$$

Since the composition of continuous functions is continuous (Theorem 2.10), it follows that H_i is a homotopy from f_i to g_i. Further, if H holds some point p_0 of X fixed, then we can take $p_0 = (x_1, \dots, x_v)$ to be the point of X used in Proposition 3.19, and we can conclude that H_i holds $p_i(p) = x_i$ fixed.

Suppose for all i, $f_i \simeq_{\kappa_i,\lambda_i} g_i$. Let $H_i : X_i \times [0, m_i]_{\mathbb{Z}} \to Y_i$ be a (κ_i, λ_i)-homotopy from f_i to g_i. We execute these homotopies "one coordinate at a time," as follows. For $x = (x_1, \ldots, x_v) \in X$ such that $x_i \in X_i$, let $M_i = \sum_{k=1}^{i} m_i$ for all i and let $H : X \times [0, M_v]_{\mathbb{Z}} \to Y$ be defined by $H(x_1, \ldots, x_v, t) =$

- $(H_1(x_1, t), f_2(x_2), \ldots, f_v(x_v))$ if $0 \le t \le m_1$;
- $(g_1(x_1) \ldots, g_{j-1}(x_{j-1}), H_j(x_j, t - M_{j-1}), f_{j+1}(x_{j+1}), \ldots, f_v(x_v))$ if $M_{j-1} \le t \le M_j$;
- $(g_1(x_1) \ldots, g_{v-1}(x_{v-1}), H_v(x_v, t - M_{v-1}))$ if $M_{j-1} \le t \le M_j$.

It is easily seen that H is well defined and is a homotopy from f to g.

Further, if H_i holds x_i fixed, then H holds x fixed. □

Corollary 5.4. *Let (X_i, κ_i) and (Y_i, λ_i) be digital images, $1 \le i \le v$. Then $X = \Pi_{i=1}^{v} X_i$ and $Y = \Pi_{i=1}^{v} Y_i$ are $(\times_{i=1}^{v} \kappa_i, \times_{i=1}^{v} \lambda_i)$-(pointed) homotopy equivalent if and only if for each i, (X_i, κ_i) and (Y_i, λ_i) are (pointed) homotopy equivalent.*

Proof. This follows from Theorem 5.3. □

5.3. **Lexicographic adjacency.**

Theorem 5.5. *Let (X_i, κ_i) be digital images for $1 \le i \le v$. Let $X = \Pi_{i=1}^{v} X_i$. If there is a smallest index k such that $|X_k| > 1$, then $(X, L(\kappa_1, \ldots, \kappa_v))$ and (X_k, κ_k) have the same pointed homotopy type.*

Proof. For each $i \ne k$, let $x_i \in X_i$. Let $I_k : X_k \to X$ be the injection of Proposition 3.19. By choice of k, I_k is $(\kappa_k, L(\kappa_1, \ldots, \kappa_v))$-continuous. Also by choice of k, the projection map $p_k : (X, L(\kappa_1, \ldots, \kappa_v)) \to (X_k, \kappa_k)$ is continuous. Notice $p_k \circ I_k = 1_{X_k}$. Also, the function $H : X \times [0, 1]_{\mathbb{Z}} \to X$ defined for $p = (y_1, \ldots, y_v) \in X$ with $y_i \in X_i$ by

$$H(p, t) = \begin{cases} p & \text{if } t = 0; \\ (y_1, x_2, \ldots, x_v) & \text{if } t = 1 \text{ and } k = 1; \\ (x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_v) & \text{if } t = 1 \text{ and } 1 < k < v; \\ (x_1, \ldots, x_{v-1}, y_v) & \text{if } t = 1 \text{ and } k = v, \end{cases}$$

is easily seen from the choice of k to be a homotopy from 1_X to $I_k \circ p_k$ that holds fixed the point $(x_1, \ldots, x_v)$. The assertion follows. □

Corollary 5.6. *Let (X, κ) and (Y, λ) be digital images of different pointed homotopy types. If $|X| > 1$ and $|Y| > 1$, then $(X \times Y, L(\kappa, \lambda))$ and $(Y \times X, L(\lambda, \kappa))$ have different pointed homotopy types.*

Proof. This follows immediately from Theorem 5.5. □

Corollary 5.7. *Let (X_i, κ_i) and (Y_i, λ_i) be digital images, $1 \le i \le v$. Let $X = \Pi_{i=1}^{v} X_i$, $Y = \Pi_{i=1}^{v} Y_i$. Suppose there exist a smallest index j such that $|X_j| > 1$, and a smallest index k such that $|Y_k| > 1$. If (X_j, κ_j) and (Y_k, κ_k) have the same (pointed) homotopy type, then $(X, L(\kappa_1, \ldots, \kappa_v))$ and $(Y, L(\lambda_1, \ldots, \lambda_v))$ have the same (pointed) homotopy type.*

Proof. By Theorem 5.5, $(X, L(\kappa_1, \ldots, \kappa_v))$ and (X_j, κ_j) have the same pointed homotopy type, and (Y_k, λ_k) and $(Y, L(\lambda_1, \ldots, \lambda_v))$ have the same pointed homotopy type. Since we also have assumed (X_j, κ_j) and (Y_k, λ_k) have the same (pointed) homotopy type, the assertion follows from the transitivity of (pointed) homotopy type. □

6. Retractions

Definition 6.1 ([2, 3])**.** Let $Y \subset (X, \kappa)$. A (κ, κ)-continuous function $r : X \to Y$ is a *retraction*, and A is a *retract of* X, if $r(y) = y$ for all $y \in Y$.

Theorem 6.2 ([12])**.** *Let $A_i \subset (X_i, \kappa_i)$, $i \in \{1, \ldots, v\}$. Then A_i is a retract of X_i for all i if and only if $\Pi_{i=1}^v A_i$ is a retract of $(\Pi_{i=1}^v X_i, NP_v(\kappa_1, \ldots, \kappa_v))$.*

6.1. **Tensor product adjacency.** The following example shows that one of the assertions obtained by using the tensor product adjacency rather than NP_v in Theorem 6.2 is not generally valid.

Example 6.3. Let $X = \{(0,0),\ (1,0),\ (1,1)\} \subset (\mathbb{Z}^2, c_2)$. Observe that $X' = \{(0,0),\ (1,0)\}$ is a c_2-retract of X, and $\{0\}$ is a c_1-retract of $[0,1]_{\mathbb{Z}}$. However, $X' \times \{0\}$ is not a $T(c_2, c_1)$-retract of $X \times [0,1]_{\mathbb{Z}}$.

Proof. Note $X \times [0,1]_{\mathbb{Z}}$ is $T(c_2, c_1)$-connected, since

$$(0,0,0),\ (1,0,1),\ (1,1,0),\ (0,0,1),\ (1,0,0),\ (1,1,1)$$

is a listing of its points in a $T(c_2, c_1)$-path; but $X' \times \{0\} = \{(0,0,0), (1,0,0)\}$ is not $T(c_2, c_1)$-connected. The assertion follows. □

The question of whether $\Pi_{i=1}^v A_i$ being a retract of $(\Pi_{i=1}^v X_i, T(\kappa_1, \ldots, \kappa_v))$ implies A_i is a κ_i-retract of X_i, for all i, is unknown at the current writing.

6.2. **Cartesian product adjacency.** For the Cartesian product adjacency, we have the following analog of Theorem 6.2.

Theorem 6.4. *Suppose $A_i \subset (X_i, \kappa_i)$. Let $X = \Pi_{i=1}^v X_i$, $A = \Pi_{i=1}^v A_i$. Then there is a retraction $r_i : X_i \to A_i$, $1 \le i \le v$ if and only if there is a retraction $r : (X, \times_{i=1}^v \kappa_i) \to (A, \times_{i=1}^v \kappa_i)$.*

Proof. Suppose there is a retraction $r_i : X_i \to A_i$, $1 \le i \le v$. Let $r = \Pi_{i=1}^v r_i : X \to A$. Clearly $r(x) \in A$ for all $x \in X$, and $r(a) = a$ for all $a \in A$. By Theorem 3.17, r is continuous. Therefore, r is a retraction.

Conversely, suppose there exists a retraction $r : (X, \times_{i=1}^v \kappa_i) \to (A, \times_{i=1}^v \kappa_i)$. Let $r_i = p_i \circ r \circ I_i : (X_i, \kappa_i) \to (A_i, \kappa_i)$, where I_i is the injection of Proposition 3.19 and the x_i of Proposition 3.19 satisfies $x_i \in A_i$. Since composition preserves continuity, Theorem 3.18 and Proposition 3.19 imply r_i is continuous. Further, for $a_i \in A_i$ we clearly have $r_i(a_i) = a_i$. Thus, r_i is a retraction. □

6.3. **Lexicographic adjacency.** For the lexicographic adjacency, we do not have an analog of Theorem 6.2, as shown by the following example.

Example 6.5. $\{0\}$ is a c_1-retract of $[0,1]_{\mathbb{Z}}$ and $[1,4]_{\mathbb{Z}}$ is a c_1-retract of $[0,5]_{\mathbb{Z}}$. However, $A = \{0\} \times [1,4]_{\mathbb{Z}}$ is not an $L(c_1,c_1)$-retract of $X = [0,1]_{\mathbb{Z}} \times [0,5]_{\mathbb{Z}}$.

Proof. We give a proof by contradiction. Suppose there is an $L(c_1,c_1)$-retraction $r : [0,1]_{\mathbb{Z}} \times [0,5]_{\mathbb{Z}} \to \{0\} \times [1,4]_{\mathbb{Z}}$. Notice $p = (0,1) \leftrightarrow_{L(c_1,c_1)} (1,5) = p'$. Since $r(p) = p$, the continuity of r requires that $r(p') = p$ or $r(p') \leftrightarrow_{L(c_1,c_1)} p$, hence

$$r(p') \in \{p, (0,2)\}.$$

But also $p' \leftrightarrow_{L(c_1,c_1)} (0,4) = q$, and since $r(q) = q$, the continuity of r similarly requires that

$$r(p') \leftrightarrow_{L(c_1,c_1)} \{q, (0,3)\}.$$

Therefore,

$$r(p') \in \{p, (0,2)\} \cap \{q, (0,3)\} = \varnothing.$$

Since this is impossible, no such retraction r can exist. □

7. Approximate fixed point property

Some material in this section is quoted or paraphrased from [9, 10].

In both topology and digital topology,

- a *fixed point* of a continuous function $f : X \to X$ is a point $x \in X$ satisfying $f(x) = x$;
- if every continuous $f : X \to X$ has a fixed point, then X has the *fixed point property* (FPP).

However, a digital image X has the FPP if and only if X has a single point [10]. Therefore, it turns out that the *approximate fixed point property* is more interesting for digital images.

Definition 7.1 ([10]). A digital image (X,κ) has the *approximate fixed point property (AFPP)* if every continuous $f : X \to X$ has an *approximate fixed point*, i.e., a point $x \in X$ such that $f(x)$ is equal or κ-adjacent to x.

The following is a minor generalization of Theorem 5.10 of [10].

Theorem 7.2 ([9]). *Let (X_i,κ_i) be digital images, $1 \le i \le v$. Then for any $u \in \mathbb{Z}$ such that $1 \le u \le v$, if $(\Pi_{i=1}^{v} X_i, NP_u(\kappa_1,\ldots,\kappa_v))$ has the AFPP then (X_i,κ_i) has the AFPP for all i.*

Determining whether analogs of Theorem 7.2 for the tensor product adjacency, or for the Cartesian product adjacency, are generally true, appear to be difficult problems. The following examples show that the analogs of converses to Theorem 7.2 for the tensor product adjacency and for the Cartesian product adjacency are not generally true.

Example 7.3. Although $([0,1]_{\mathbb{Z}}, c_1)$ has the AFPP [25], $([0,1]_{\mathbb{Z}}^2, T(c_1,c_1))$ does not have the AFPP.

Proof. Consider the function $f : [0,1]_{\mathbb{Z}}^2 \to [0,1]_{\mathbb{Z}}^2$ defined by $f(a,b) = (1-a, b)$, i.e.,

$$f(0,0) = (1,0), \quad f(0,1) = (1,1), \quad f(1,0) = (0,0), \quad f(1,1) = (0,1).$$

One can easily check that f is continuous and has no approximate fixed point when the $T(c_1, c_1)$ adjacency is used. □

Example 7.4. Although $([0,1]_{\mathbb{Z}}, c_1)$ has the AFPP, $([0,1]_{\mathbb{Z}}^2, c_1 \times c_1)$ does not have the AFPP.

Proof. Consider the function $f : [0,1]_{\mathbb{Z}}^2 \to [0,1]_{\mathbb{Z}}^2$ defined by $f(a,b) = (1-a, 1-b)$, i.e.,

$$f(0,0) = (1,1), \quad f(0,1) = (1,0), \quad f(1,0) = (0,1), \quad f(1,1) = (0,0).$$

One can easily check that f is continuous and has no approximate fixed point when the $c_1 \times c_1$ adjacency is used. □

We have the following.

Theorem 7.5. *Let (X_i, κ_i) be digital images, $1 \le i \le v$. Suppose there is a smallest index k such that X_k is κ_k-connected and $|X_k| > 1$. If the product $(\Pi_{i=1}^v X_i, L(\kappa_1, \ldots, \kappa_v))$ has the AFPP property, then (X_k, κ_k) has the AFPP property.*

Proof. Let $X = \Pi_{i=1}^v X_i$.

Suppose the product $(X, L(\kappa_1, \ldots, \kappa_v))$ has the AFPP property. Let $g : X_k \to X_k$ be κ-continuous. Let $x_i \in X_i$. Notice this means $X_i = \{x_i\}$ for $i < k$. Let $X = \Pi_{i=1}^v X_i$. Let $G : X \to X$ be defined by

$$G(y_1, \ldots, y_v) = \begin{cases} (g(y_1), x_2, \ldots, x_v) & \text{if } k = 1; \\ (x_1, \ldots, x_{k-1}, g(y_k), x_{k+1}, \ldots, x_v) & \text{if } 1 < k < v; \\ (x_1, \ldots, x_{v-1}, g(y_v)) & \text{if } k = v. \end{cases}$$

Since g is κ_k-continuous, our choice of k implies G is $L(\kappa_1, \ldots, \kappa_v)$-continuous. By hypothesis, there is a $p = (y_1', \ldots, y_v') \in X$ with $y_i' \in X_i$ such that $G(p) = p$ or $G(p) \leftrightarrow p$. Therefore, either

$$g(y_k') = p_k(G(p)) = p_k(p) = y_k' \text{ or } g(y_k') \leftrightarrow_{\kappa_k} y_k'.$$

Thus, y_k' is an approximate fixed point for g. □

8. Multivalued functions

We study various product adjacencies with respect to properties of multivalued functions.

The following has an elementary proof.

Proposition 8.1. *Let $f : (X, \kappa) \to (Y, \lambda)$ be a single-valued function between digital images. Then the following are equivalent.*

- *f is continuous.*
- *As a multivalued function, f has weak continuity.*
- *As a multivalued function, f has strong continuity.*

For multivalued functions $F_i : X_i \multimap Y_i$, $1 \leq i \leq v$, define the product multivalued function

$$\Pi_{i=1}^v F_i : \Pi_{i=1}^v X_i \multimap \Pi_{i=1}^v Y_i$$

by

$$(\Pi_{i=1}^v F_i)(x_1, \ldots, x_v) = \Pi_{i=1}^v F_i(x_i).$$

8.1. **Weak continuity.** For NP_v, we have the following results.

Theorem 8.2 ([9]). *Let $F_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued functions for $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$, and $F = \Pi_{i=1}^v F_i : (X, NP_v(\kappa_1, \ldots, \kappa_v)) \multimap (Y, NP_v(\lambda_1, \ldots, \lambda_v))$. Then F has weak continuity if and only if each F_i has weak continuity.*

For the tensor product, we have the following.

Theorem 8.3. *For each index i such that $1 \leq i \leq v$, let $f_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be a multivalued map between digital images. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. If the product multivalued map*

$$f = \Pi_{i=1}^v f_i : (X, T(\kappa_1, \ldots, \kappa_v)) \multimap (Y, T(\lambda_1, \ldots, \lambda_v))$$

has weak continuity, then for each i, f_i has weak continuity.

Proof. For all indices i, let $x_i \leftrightarrow_{\kappa_i} x_i'$ in X_i. Then, in X, we have $p = (x_1, \ldots, x_v) \leftrightarrow_{T(\kappa_1, \ldots, \kappa_v)} p' = (x_1', \ldots, x_v')$. The weak continuity of f implies $f(p)$ and $f(p')$ are adjacent subsets of $(Y, T(\lambda_1, \ldots, \lambda_v))$. Therefore, there exist $y \in f(p)$ and $y' \in f(p')$ such that $y = y'$ or $y \leftrightarrow_{T(\lambda_1, \ldots, \lambda_v)} y'$.

Now, $y = (y_1, \ldots, y_v)$ where $y_i \in f_i(x_i)$, and $y' = (y_1', \ldots, y_v')$ where $y_i' \in f_i(x_i')$. If $y = y'$ then we have $y_i = y_i'$ for all indices i. If $y \leftrightarrow_{T(\lambda_1, \ldots, \lambda_v)} y'$ then we have $y_i \leftrightarrow_{\lambda_i} y_i'$ for all indices i. In either case, we have for all i that $f_i(x_i)$ and $f_i(x_i')$ are adjacent subsets of Y_i. It follows that each f_i has weak continuity. □

The converse of Theorem 8.3 is not generally true, as shown by the following.

Example 8.4. Let f and g be the single-valued functions of Example 3.11. By Proposition 8.1, f and g have weak continuity. However, Example 3.11 shows that $f \times g$ is not $(T(c_1, c_1), T(c_1, c_1))$-continuous, so by Proposition 8.1, $f \times g$ does not have $(T(c_1, c_1), T(c_1, c_1))$-weak continuity.

For the Cartesian product adjacency, we have the following.

Theorem 8.5. *Let $f_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued maps between digital images, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. Then the product multivalued map*

$$f = \Pi_{i=1}^v f_i : (X, \times_{i=1}^v \kappa_i) \multimap (Y, \times_{i=1}^v \lambda_i)$$

has weak continuity if and only if for each i, f_i has weak continuity.

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Proof. Suppose f has weak continuity. Let $x_i \leftrightarrow_{\kappa_i} x'_i$ in X_i. Let

$$x = (x_1, \ldots, x_v) \in X,$$

$$x' = (x_1, \ldots, x_{j-1}, x'_j, x_{j+1}, \ldots, x_v) \in X \text{ for some index } j.$$

We have $x \leftrightarrow_{\times_{i=1}^v \kappa_i} x'$. Therefore, there exist

$$y = (y_1, \ldots, y_v) \in f(x) = \Pi_{i=1}^v f_i(x_i),$$

$$y' = (y'_1, \ldots, y'_v) \in f(x') = \Pi_{i=1}^{j-1} f_i(x_i) \times f_j(x_j) \times \Pi_{i=j+1}^v f_i(x_i)$$

such that $y \leftrightarrow_{\times_{i=1}^v \lambda_i} y'$. Therefore, we have $y_j \in f_j(x_j)$, $y'_j \in f_j(x'_j)$, and $y_j = y'_j$ or $y_j \leftrightarrow_{\lambda_j} y'_j$. Thus, f_j has weak continuity.

Suppose each f_i has weak continuity. Let $p \leftrightarrow_{\times_{i=1}^v \kappa_i} p'$ in X, where $p = (x_1, \ldots, x_v)$, $p' = (x'_1, \ldots, x'_v)$, $x_i, x'_i \in X_i$, and, from the definition of the $\times_{i=1}^v \kappa_i$ adjacency, there is one index j such that $x_j \leftrightarrow_{\kappa_j} x'_j$ and for all indices $i \neq j$, $x_i = x'_i$ and therefore $f_i(x_i) = f_i(x'_i)$. Since f_j has weak continuity, there exist $y_j \in f_j(x_j)$ and $y'_j \in f_j(x'_j)$ such that $y_j = y'_j$ or $y_j \leftrightarrow_{\lambda_j} y'_j$. For $i \neq j$ we can take $y_i \in f_i(x_i)$. Then $y = (y_1, \ldots, y_v)$ and $y' = (y_1, \ldots, y_{j-1}, y'_j, y_{j+1}, \ldots, y_v)$ are equal or $\times_{i=1}^v \lambda_i$-adjacent, and we have $y \in f(p)$, $y' \in f(p')$. Therefore, f has weak continuity. □

For the lexicographic adjacency, Example 8.10 below shows there is no general product property for weak continuity, and Example 8.11 below shows there is not a general factor property for weak continuity.

8.2. Strong continuity.

Theorem 8.6 ([9]). *Let $F_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued functions for $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$, $F = \Pi_{i=1}^v F_i : (X, NP_v(\kappa_1, \ldots, \kappa_v)) \multimap (Y, NP_v(\lambda_1, \ldots, \lambda_v))$. Then F has strong continuity if and only if each F_i has strong continuity.*

For the tensor product adjacency, we have the following.

Theorem 8.7. *Let $f_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued maps between digital images, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. If the product multivalued map*

$$f = \Pi_{i=1}^v f_i : (X, T(\kappa_1, \ldots, \kappa_v)) \multimap (Y, T(\lambda_1, \ldots, \lambda_v))$$

has strong continuity, then for each i, f_i has strong continuity.

Proof. Let $x_i \leftrightarrow_{\kappa_i} x'_i$ in X_i. Let $p = (x_1, \ldots, x_v)$ and $p' = (x'_1, \ldots, x'_v)$. Note $p \leftrightarrow_{T(\kappa_1, \ldots, \kappa_v)} p'$ in X. Since f has strong continuity, for every $q = (y_1, \ldots, y_v) \in f(p) = \Pi_{i=1}^v f_i(x_i)$ where $y_i \in f_i(x_i)$, there exists $q' = (y'_1, \ldots, y'_v) \in f(p') = \Pi_{i=1}^v f_i(x'_i)$ where $y'_i \in f_i(x'_i)$ such that either $q = q'$ or $q \leftrightarrow_{T(\lambda_1, \ldots, \lambda_v)} q'$; and therefore $y_i = y'_i$ for all i or $y_i \leftrightarrow_{\lambda_i} y'_i$ for all i. Also, for every $r' = (r'_1, \ldots, r'_v) \in f(p')$ where $r'_i \in f_i(x'_i)$, there exists $r = (r_1, \ldots, r_v) \in f(p)$ where $r_i \in f_i(x_i)$ such that either $r = r'$ or $r \leftrightarrow_{T(\lambda_1, \ldots, \lambda_v)} r'$; and therefore $r_i = r'_i$ for all i or $r_i \leftrightarrow_{\lambda_i} r'_i$ for all i. Thus f_i has (κ_i, λ_i)-strong continuity. □

The converse of Theorem 8.7 is not generally true, as shown by the following.

Example 8.8. Let $f_1 : ([0,1]_{\mathbb{Z}}, c_1) \multimap ([0,1]_{\mathbb{Z}}, c_1)$ be defined by $f_1(x) = \{0\}$. Let $f_2 : ([0,1]_{\mathbb{Z}}, c_1) \multimap ([0,1]_{\mathbb{Z}}, c_1)$ be defined by $f_2(x) = \{x\}$. Then f_1 and f_2 both have strong continuity. However, $f_1 \times f_2$ does not have $(T(c_1,c_1), T(c_1,c_1))$-strong continuity.

Proof. It is easily seen that f_1 and f_2 both have strong continuity. However, in Example 8.4, we showed that $f_1 \times f_2$ does not have $(T(c_1,c_1), T(c_1,c_1))$-weak continuity. Therefore, $f_1 \times f_2$ does not have $(T(c_1,c_1), T(c_1,c_1))$-strong continuity. □

Theorem 8.9. *Let $f_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued maps between digital images, $1 \le i \le v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. Then the product multivalued map*

$$f = \Pi_{i=1}^v f_i : (X, \times_{i=1}^v \kappa_i) \multimap (Y, \times_{i=1}^v \lambda_i)$$

has strong continuity if and only if for each i, f_i has strong continuity.

Proof. Suppose f has strong continuity. Let $x_i \leftrightarrow_{\kappa_i} x_i'$ in X_i. Then

$$p = (x_1, \ldots, x_v) \leftrightarrow_{\times_{i=1}^v \kappa_i} (x_1, \ldots, x_{j-1}, x_j', x_{j+1}, \ldots, x_v) = p'$$

in X, for some index j. Since f has strong continuity, we must have that for every $q = (q_1, \ldots, q_v) \in f(p)$ there exists $q' = (q_1', \ldots, q_v') \in f(p')$ such that $q = q'$ or $q \leftrightarrow_{\times_{i=1}^v \lambda_i} q'$, so $q_i = q_i'$ or $q_i \leftrightarrow_{\lambda_i} q_i'$; and for every $r' = (r_1', \ldots, r_v') \in f(p')$ there exists $r = (r_1, \ldots, r_v) \in f(p)$ such that $r = r'$ or $r \leftrightarrow_{\times_{i=1}^v \lambda_i} r'$, so $r_i = r_i'$ or $r_i \leftrightarrow_{\lambda_i} r_i'$. Therefore, f_i has strong continuity.

Suppose for each i, f_i has strong continuity. Let $p = (x_1, \ldots, x_v)$ and $p' = (x_1', \ldots, x_v')$ with $x_i, x_i' \in X_i$ be such that $p \leftrightarrow_{\times_{i=1}^v \kappa_i} p'$. Then for some index j, $x_j \leftrightarrow_{\kappa_j} x_j'$ and for all indices $i \ne j$, $x_i = x_i'$. Therefore, $i \ne j$ implies there exists $q_i \in f_i(x_i) = f_i(x_i')$; and since f_j has strong continuity, for every $q_j \in f_j(x_j)$ there exists $q_j' \in f_j(x_j')$ such that $q_j = q_j'$ or $q_j \leftrightarrow_{\lambda_j} q_j'$. Let $q' = (q_1, \ldots, q_{j-1}, q_j', q_{j+1}, \ldots, q_v)$. Then $q = (q_1, \ldots, q_v) = q'$ or $q \leftrightarrow_{\times_{i=1}^v \lambda_i} q'$ with $q \in f(p)$, $q' \in f(p')$. Similarly, for every $r' \in f(p')$ there exists $r \in f(p)$ such that $r = r'$ or $r \leftrightarrow_{\times_{i=1}^v \lambda_i} r'$. Thus, f has strong continuity. □

For the lexicographic adjacency, the following shows there is not a general product property for weak or strong continuity.

Example 8.10. Let $f_1 : ([0,1]_{\mathbb{Z}}, c_1) \multimap ([0,1]_{\mathbb{Z}}, c_1)$ be the multivalued function $f_1(x) = \{0\}$. Let $f_2 : (\{0,2\}, c_1) \multimap (\{0,2\}, c_1)$ be the function $f_2(x) = \{x\}$. Then f_1 and f_2 have weak continuity and strong continuity, but $f_1 \times f_2$ lacks both $(L(c_1,c_1), L(c_1,c_1))$-weak continuity and $(L(c_1,c_1), L(c_1,c_1))$-strong continuity.

Proof. It is easy to see that f_1 and f_2 have weak continuity and strong continuity, and that $p = (0,0) \leftrightarrow_{L(c_1,c_1)} (1,2) = p'$. However

$$(f_1 \times f_2)(p) = \{(0,0)\} \text{ and } (f_1 \times f_2)(p') = \{(0,2)\},$$

are not $L(c_1,c_1)$-adjacent, so $f_1 \times f_2$ lacks $(L(c_1,c_1), L(c_1,c_1))$-weak continuity and therefore lacks $(L(c_1,c_1), L(c_1,c_1))$-strong continuity. □

For the lexicographic adjacency, the following shows there is not a general factor property for weak or strong continuity.

Example 8.11. Let $f_1 : ([0,1]_{\mathbb{Z}}, c_1) \multimap ([0,1]_{\mathbb{Z}}, c_1)$ be the multivalued function $f_1(x) = [0,1]_{\mathbb{Z}}$. Let $f_2 : ([0,1]_{\mathbb{Z}}, c_1) \multimap (\{0,2\}, c_1)$ be the multivalued function $f_2(x) = \{2x\}$. Then $f_1 \times f_2 : [0,1]_{\mathbb{Z}}^2 \multimap [0,1]_{\mathbb{Z}} \times \{0,2\}$ has $(L(c_1,c_1), L(c_1,c_1))$-weak and $(L(c_1,c_1), L(c_1,c_1))$-strong continuity, although f_2 lacks both weak and strong continuity.

Proof. It is easy to see that f_2 lacks weak and strong continuity. Since

$$(f_1 \times f_2)(0,0) = (f_1 \times f_2)(1,0) = \{(0,0),(1,0)\},$$

$$(f_1 \times f_2)(0,1) = (f_1 \times f_2)(1,1) = \{(0,2),(1,2)\},$$

it follows easily that $f_1 \times f_2$ has both $(L(c_1,c_1), L(c_1,c_1))$-weak continuity and $(L(c_1,c_1), L(c_1,c_1))$-strong continuity. □

8.3. **Continuous multifunctions.**

Lemma 8.12 ([9])**.** *Let $X \subset \mathbb{Z}^m$, $Y \subset \mathbb{Z}^n$. Let $F : (X, c_a) \multimap (Y, c_b)$ be a continuous multivalued function. Let $f : (S(X,r), c_a) \to (Y, c_b)$ be a continuous function that induces F. Let $s \in \mathbb{N}$. Then there is a continuous function $f_s : (S(X,rs), c_a) \to (Y, c_b)$ that induces F.*

For the NP_v adjacency, we have the following.

Theorem 8.13 ([9])**.** *Given multivalued functions $F_i : (X_i, c_{a_i}) \multimap (Y_i, c_{b_i})$, $1 \le i \le v$, each F_i is continuous if and only if the product multivalued function*

$$\Pi_{i=1}^v F_i : (\Pi_{i=1}^v X_i, NP_v(c_{a_1}, \ldots, c_{a_v})) \multimap (\Pi_{i=1}^v Y_i, NP_v(c_{b_1}, \ldots, c_{b_v}))$$

is continuous.

For the tensor product, since a single-valued function can be considered as multivalued, Example 3.11 shows there is no general product rule for the continuity of multivalued functions. However, we have the following.

Theorem 8.14. *Let $F_i : (X_i, c_{a_i}) \multimap (Y_i, c_{b_i})$ be a continuous multivalued function between digital images, $1 \le i \le v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$, $F = \Pi_{i=1}^v F_i : X \multimap Y$. If for some positive integer r and for all i there is a continuous locally one-to-one function $f_i : (S(X_i,r), c_{a_i}) \to (Y_i, c_{b_i})$ that generates F_i, then F is $(T(c_{a_1}, \ldots, c_{a_v}), T(c_{b_1}, \ldots, c_{b_v}))$-continuous and is generated by a function that is locally one-to-one.*

Proof. Let $f = \Pi_{i=1}^v f_i : \Pi_{i=1}^v S(X_i, r) \to Y$. It follows from Theorem 3.13 that f is $(T(c_{a_1}, \ldots, c_{a_v}), T(c_{b_1}, \ldots, c_{b_v}))$-continuous. Further, given $q \in F(p)$ where $p = (x_1, \ldots, x_v)$ for $x_i \in X_i$ and $q = (y_1, \ldots, y_v)$ where $y_i \in F_i(x_i)$, there exists $x_i' \in S(\{x_i\}, r) \subset S(X_i, r)$ such that $f_i(x_i') = y_i$. Therefore, $f(x_1', \ldots, x_v') = q$.

For $w \leftrightarrow_{T(c_{a_1}, \ldots, c_{a_v})} w'$ in $S(X,r) = \Pi_{i=1}^v S(X_i, r)$, we have $w = (w_1, \ldots, w_v)$ and $w' = (w_1', \ldots, w_v')$, where $w_i, w_i' \in S(X_i, r)$ and $w_i \leftrightarrow_{c_{a_i}} w_i'$. Since f_i is locally one-to-one and continuous, we have $f_i(w_i) \leftrightarrow_{c_{b_i}} f_i(w_i')$. It follows that

$f(w_1, \ldots, w_v) \leftrightarrow_{T(c_{b_1}, \ldots, c_{b_v})} f(w'_1, \ldots, w'_v)$. This allows us to conclude that f is $(T(c_{a_1}, \ldots, c_{a_v}), T(c_{b_1}, \ldots, c_{b_v}))$-continuous. Thus, f generates F.

Let $p' = (x'_1, \ldots, x'_v) \leftrightarrow_{T(\kappa_1, \ldots, \kappa_v)} p$ in X, where $x'_i \in X_i$. Since f_i is locally one-to-one, $f_i(x_i) \leftrightarrow_{\lambda_i} f_i(x'_i)$ for all i. Therefore, $f(p) \leftrightarrow_{T(\lambda_1, \ldots, \lambda_v)} f(p')$, so f is locally one-to-one. □

Deciding whether the converse of Theorem 8.14 is true appears to be a difficult problem.

For the Cartesian product adjacency, we have the following.

Theorem 8.15. *Let $F_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be a multivalued function between digital images, where $\kappa_i = c_{a_i}$, $\lambda_i = c_{b_i}$, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$, $F = \Pi_{i=1}^v F_i : X \multimap Y$. If each F_i is continuous, then F is $(\times_{i=1}^v \kappa_i, \times_{i=1}^v \lambda_i)$-continuous.*

Proof. Suppose each F_i is continuous. By Lemma 8.12, there exists $r \in \mathbb{N}$ and generating functions $f_i : S(X_i, r) \to Y_i$ of F_i.

We wish to show that $f = \Pi_{i=1}^v f_i$ generates F. Suppose $p \leftrightarrow_{\times_{i=1}^v \kappa_i} p'$ in $S(X, r)$. Then $p = (x_1, \ldots, x_v)$ and $p' = (x'_1, \ldots, x'_v)$ where $x_i, x'_i \in S(X_i, r)$ and $x_i = x'_i$ for all but one index j, with $x_j \leftrightarrow_{\kappa_j} x'_j$. Since each f_i is (κ_i, λ_i)-continuous, we have $f_j(x_j) = f_j(x'_j)$ or $f_j(x_j) \leftrightarrow_{\lambda_j} f_j(x'_j)$ and for all indices $i \neq j$ we have $f_i(x_i) = f_i(x'_i)$. Thus we have $f(p) = f(p')$ or $f(p) \leftrightarrow_{\times_{i=1}^v \lambda_i} f(p')$. Thus, f is $(\times_{i=1}^v \kappa_i, \times_{i=1}^v \lambda_i)$-continuous.

Let $y = (y_1, \ldots, y_v) \in F(X)$, where $y_i \in Y_i$. Then there exists $x_i \in S(X_i, r)$ such that $f_i(x_i) = y_i$. For $p = (x_1, \ldots, x_v)$, we have $f(p) = (y_1, \ldots, y_v)$. Thus, f generates F, so F is continuous. □

Deciding whether the converse of Theorem 8.15 is true appears to be a difficult problem.

For the lexicographic adjacency, there is no general product rule for the continuity of multivalued functions, as shown in Example 3.22 (since a single-valued function can be regarded as multivalued). However, we have the following.

Theorem 8.16. *Let $F_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be a continuous multivalued function between digital images, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$, $F = \Pi_{i=1}^v F_i : X \multimap Y$. If each F_i is generated by a function $f_i : (S(X_i, r), \kappa_i) \to Y_i$ that is locally one-to-one, then F is $(L(\kappa_1, \ldots, \kappa_v), L(\lambda_1, \ldots, \lambda_v))$-continuous.*

Proof. By Theorem 3.21, the single-valued function $f = \Pi_{i=1}^v f_i : \Pi_{i=1}^v S(X_i, r) \to Y$ is $(L(\kappa_1, \ldots, \kappa_v), L(\lambda_1, \ldots, \lambda_v))$-continuous. Further, given $y = (y_1, \ldots, y_v) \in F(X)$ with $y_i \in Y_i$, there exist $x'_i \in S(\{x_i\}, r) \subset S(X_i, r)$ such that $f_i(x'_i) = y_i$. Therefore, $y = f(x'_1, \ldots, x'_v) \in F(x_1, \ldots, x_v)$. Therefore, f generates F, and the assertion follows. □

The paper [16] has several results concerning the following notions.

Definition 8.17 ([16])**.** Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital image and $Y \subset X$. We say that Y is a κ*-retract of* X if there exists a κ-continuous multivalued function $F : X \multimap Y$ (a *multivalued κ-retraction*) such that $F(y) = \{y\}$ if $y \in Y$.

We generalize Theorem 6.2 as follows.

Theorem 8.18 ([9]). *For $1 \leq i \leq v$, let $A_i \subset (X_i, \kappa_i) \subset \mathbb{Z}^{n_i}$. Suppose $F_i : X_i \multimap A_i$ is a continuous multivalued function for all i. Then F_i is a multivalued κ_i-retraction for all i if and only if $F = \Pi_{i=1}^v F_i : \Pi_{i=1}^v X_i \multimap \Pi_{i=1}^v A_i$ is a multivalued $NP_v(\kappa_1, \ldots, \kappa_v)$-retraction.*

For the Cartesian product adjacency, we have the following.

Theorem 8.19. *Let $r_i : X_i \multimap A_i$ be multivalued κ_i-retractions, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $A = \Pi_{i=1}^v A_i$, $r = \Pi_{i=1}^v r_i : X \multimap A$. Then r is a $\times_{i=1}^v \kappa_i$-multivalued retraction.*

Proof. Since r_i is a multivalued retraction, we must have that $r_i(X_i) = A_i$ and $r_i(a_i) = \{a_i\}$ for all $a_i \in A_i$. Therefore, $r(X) = A$ and $r(a) = \{a\}$ for all $a \in A$. By Theorem 8.15, r is continuous, and therefore is a multivalued retraction. □

8.4. **Connectivity preserving multifunctions.**

Theorem 8.20 ([9]). *Let $f_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be a multivalued function between digital images, $1 \leq i \leq v$. Then the product map*

$$\Pi_{i=1}^v f_i : (\Pi_{i=1}^v X_i, NP_v(\kappa_1, \ldots, \kappa_v)) \multimap (\Pi_{i=1}^v Y_i, NP_v(\lambda_1, \ldots, \lambda_v))$$

is a connectivity preserving multifunction if and only if each f_i is a connectivity preserving multifunction.

The tensor product adjacency does not yield a similar result, as shown in the following.

Example 8.21. Consider $\{0\} \subset \mathbb{Z}$, $[0,1]_{\mathbb{Z}} \subset \mathbb{Z}$. The multivalued function $f : (\{0\}, c_1) \multimap ([0,1]_{\mathbb{Z}}, c_1)$ defined by $f(0) = [0,1]_{\mathbb{Z}}$ is connectivity preserving. However, $f \times f : \{0\}^2 = \{(0,0)\} \multimap [0,1]_{\mathbb{Z}}^2$ is not $(T(c_1, c_1), T(c_1, c_1))$-connectivity preserving.

Proof. This follows from the observations that $\{(0,0)\}$ has a single point, hence must be $T(c_1, c_1)$-connected; but, by Example 4.3, $(f \times f)(0,0) = [0,1]_{\mathbb{Z}}^2$ is not $T(c_1, c_1)$-connected. □

However, we have the following.

Theorem 8.22. *Let $f_i : (X_i, \kappa_i) \multimap (Y_i, \lambda_i)$ be multivalued functions, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. Suppose $f = \Pi_{i=1}^v f_i : X \multimap Y$ is $(T(\kappa_1, \ldots, \kappa_v), T(\lambda_1, \ldots, \lambda_v))$-connectivity preserving. Then each f_i is connectivity preserving.*

Proof. Let $p = (x_1, \ldots, x_v) \in X$, where $x_i \in X_i$. By assumption, $f(p) = \Pi_{i=1}^v f_i(x_i)$ is $T(\lambda_1, \ldots, \lambda_v)$-connected. From Theorem 4.2, it follows that $f_i(x_i)$ is λ_i-connected.

Suppose $x_i' \leftrightarrow_{\kappa_i} x_i$ in X_i. Then $p' = (x_1', \ldots, x_v') \leftrightarrow_{T(\kappa_1, \ldots, \kappa_v)} p$. Since f is connectivity preserving, $f(p')$ and $f(p)$ are $T(\lambda_1, \ldots, \lambda_v)$-adjacent subsets of

Y. This implies there exist $q' = (y'_1, \ldots, y'_v) \in f(p')$, $q = (y_1, \ldots, y_v) \in f(p)$ such that $q' \leftrightarrow_{T(\kappa_1,\ldots,\kappa_v)} q$ or $q' = q$. Therefore, for each index i, $y'_i \leftrightarrow_{\lambda_i} y_i$ or $y'_i = y_i$. Since $y'_i \in f_i(x'_i)$ and $y_i \in f_i(x_i)$, we have that $f_i(x'_i)$ and $f_i(x_i)$ are λ_i-adjacent subsets of Y_i.

From Theorem 2.18, f_i is connectivity preserving. □

For the Cartesian product adjacency, we have the following.

Theorem 8.23. *Let (X_i, κ_i) and (Y_i, λ_i) be digital images, for $1 \le i \le v$. Let $f_i : X_i \multimap Y_i$ be a multivalued function. Let $f = \Pi_{i=1}^v f_i : X = \Pi_{i=1}^v X_i \multimap Y = \Pi_{i=1}^v Y_i$ be the product function. Then f is $(\times_{i=1}^v \kappa_i, \times_{i=1}^v \lambda_i)$-connectivity preserving if and only if each f_i is connectivity preserving.*

Proof. Suppose f is connectivity preserving. Let $p = (x_1, \ldots, x_v) \in X$, where $x_i \in X_i$. Then $f(p) = \Pi_{i=1}^v f_i(x_i)$ is $\times_{i=1}^v \lambda_i$-connected. By Theorem 3.18, $f_i(x_i) = p_i(f(p))$ is λ_i-connected.

For any given index k, let $x_k \leftrightarrow_{\kappa_k} x'_k$ in X_k. For all indices $i \neq k$, let $x_i \in X_i$. Then $p = (x_1, \ldots, x_v)$ and $p' = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_v)$ are $\times_{i=1}^v \kappa_i$-adjacent. Since f is connectivity preserving, $f(p)$ and $f(p')$ are $\times_{i=1}^v \lambda_i$-adjacent subsets of Y. Therefore, Theorem 3.18 implies $f_k(x_k) = p_k(f(p))$ and $f_k(x'_k) = p_k(f(p'))$ are λ_k-adjacent subsets of Y_k. It follows from Theorem 2.18 that f_k is connectivity preserving. Since k was an arbitrarily selected index, f_i is connectivity preserving for all i.

Now suppose each f_i is connectivity preserving. Let $p = (x_1, \ldots, x_v) \in X$ where $x_i \in X_i$. Then $f(p) = \Pi_{i=1}^v f_i(x_i)$ is, by Theorem 4.4, $\times_{i=1}^v \lambda_i$-connected.

Suppose $p \leftrightarrow_{\times_{i=1}^v \lambda_i} p'$ in X. Then for some index k, $x_k \leftrightarrow_{\kappa_i} x'_k$ in X_k and for $i \neq k$ there exist $x_i \in X_i$ such that

$$p = (x_1, \ldots, x_v), \quad p' = (x_1, \ldots, x_{k-1}, x'_k, x_{k+1}, \ldots, x_v).$$

Since f_k is connectivity preserving, there exist $y_k \in f_k(x_k)$ and $y'_k \in f_k(x'_k)$ such that $y_k \leftrightarrow_{\lambda_k} y'_k$ or $y_k = y'_k$. For $i \neq k$, let $y_i \in f_i(x_i)$. Then $q = (y_1, \ldots, y_v) \in f(p)$ and $q' = (y_1, \ldots, y_{k-1}, y'_k, y_{k+1}, \ldots, y_v) \in f(p')$ are $\times_{i=1}^v \lambda_i$-adjacent or equal. Therefore, $f(p)$ and $f(q)$ are $\times_{i=1}^v \lambda_i$-adjacent subsets of Y. It follows from Theorem 2.18 that f is connectivity preserving. □

For lexicographic adjacency,

- Example 3.22 shows that there is no product property for connectivity preservation; and
- there is no factor property for connectivity preservation, as the following example shows.

Example 8.24. Let $f_1 : (\{0\}, c_1) \multimap ([0,1]_{\mathbb{Z}}, c_1)$ be the multivalued function $f_1(0) = [0,1]_{\mathbb{Z}}$. Let $f_2 : (\{0\}, c_1) \multimap (\{0,2\}, c_1)$ be the multivalued function $f_2(0) = \{0,2\}$. Then

$$f = f_1 \times f_2 : \{0\}^2 = \{(0,0)\} \multimap [0,1]_{\mathbb{Z}} \times \{0,2\}$$

is $(L(c_1,c_1), L(c_1,c_1))$-connectivity preserving, but f_2 is not (c_1,c_1)-connectivity preserving.

Proof. This follows from the observations that the single point $(0,0)$ is connected, and $f(0,0) = [0,1]_{\mathbb{Z}} \times \{0,2\}$ is $L(c_1,c_1)$-connected. □

9. Shy maps

We have the following.

Theorem 9.1. *Let $f:(X,\kappa) \to (Y,\lambda)$ be a shy map of digital images. Then f is an isomorphism if and only if f is locally one-to-one.*

Proof. It is obvious that if f is an isomorphism, then f is locally one-to-one.

To show the converse, we argue as follows. Since f is shy, we know f is a continuous surjection.

To show f is one-to-one, suppose there exist $x,x' \in X$ such that $y = f(x) = f(x') \in Y$. Since f is shy, $f^{-1}(y)$ is κ-connected. Therefore, if $x \neq x'$ then there is a path of distinct points $P = \{x_i\}_{i=1}^m \subset f^{-1}(y)$ such that $x = x_1$, $x_i \leftrightarrow x_{i+1}$ for $1 \leq i < m$, and $x_m = x'$. But since f is locally one-to-one, $f|_{N_\kappa^*(x)}$ is one-to-one, so $f(x_2) \neq f(x)$, contrary to the assumption $P \subset f^{-1}(y)$. Therefore, we must have $x = x'$, so f is one-to-one.

Since f is one-to-one, f^{-1} is one-to-one. Since f is shy, given $y \leftrightarrow y'$ in Y, $f^{-1}(\{y,y'\})$ is connected. Thus, f^{-1} is continuous. This completes the proof that f is an isomorphism. □

The following generalizes a result of [8].

Theorem 9.2 ([9]). *Let $f_i:(X_i,\kappa_i) \to (Y_i,\lambda_i)$ be a continuous surjection between digital images, $1 \leq i \leq v$. Then the product map*

$$\Pi_{i=1}^v f_i : (\Pi_{i=1}^v X_i, NP_v(\kappa_1,\ldots,\kappa_v)) \to (\Pi_{i=1}^v Y_i, NP_v(\lambda_1,\ldots,\lambda_v))$$

is shy if and only if each f_i is a shy map.

For the tensor product, we have the following.

Theorem 9.3. *Let $f_i:(X_i,\kappa_i) \to (Y_i,\lambda_i)$ be a surjection between digital images, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. If the product function*

$$f = \Pi_{i=1}^v f_i : (X, T(\kappa_1,\ldots,\kappa_v)) \to (Y, T(\lambda_1,\ldots,\lambda_v))$$

is shy, then f_i is shy for each i.

Proof. Since f is shy, it is continuous, so by Theorem 3.10, each f_i is continuous. Clearly, each f_i is a surjection.

Let $y_i \in Y_i$. Let $y = (y_1,\ldots,y_v) \in Y$. Since f is shy, $f^{-1}(y) = \Pi_{i=1}^v f_i^{-1}(y_i)$ is $T(\kappa_1,\ldots,\kappa_v)$-connected. By Theorem 4.2, $f_i(y_i)$ is κ_i-connected.

Let $y_i' \leftrightarrow_{\lambda_i} y_i$ in Y_i. Then $y' = (y_1',\ldots,y_v') \leftrightarrow_{T(\lambda_1,\ldots,\lambda_v)} y$. Since f is shy,

$$f^{-1}(\{y,y'\}) = f^{-1}(\{y\}) \cup f^{-1}(\{y'\}) = \Pi_{i=1}^v f_i^{-1}(y_i) \cup \Pi_{i=1}^v f_i^{-1}(y_i')$$

is $T(\kappa_1,\ldots,\kappa_v)$-connected. By Theorem 3.15,

$$p_i(f^{-1}(\{y,y'\})) = f_i^{-1}(y_i) \cup f_i^{-1}(y_i')$$

is κ_i-connected. From Definition 2.30, we conclude that f_i is a shy map. □

The converse to Theorem 9.3 is not generally true, as shown by the following.

Example 9.4. Let $f_1 : ([0,1]_{\mathbb{Z}}, c_1) \to (\{0\}, c_1)$ be the function $f_1(x) = 0$. Let $f_2 : ([0,1]_{\mathbb{Z}}, c_1) \to ([0,1]_{\mathbb{Z}}, c_1)$ be the function $f_2(x) = x$. Then f_1 and f_2 are shy, but $f_1 \times f_2 : ([0,1]_{\mathbb{Z}}^2, T(c_1,c_1)) \to (\{0\} \times [0,1]_{\mathbb{Z}}, T(c_1,c_1))$ is not shy.

Proof. That f_1 and f_2 are shy is easily seen. Further, $f_1 \times f_2$ is a surjection. Notice that $(0,0) \leftrightarrow_{T(c_1,c_1)} (1,1)$, but $(f_1 \times f_2)(0,0) = (0,0)$ and $(f_1 \times f_2)(1,1) = (0,1)$ are neither equal nor $T(c_1,c_1)$-adjacent. Therefore, $f_1 \times f_2$ is not $(T(c_1,c_1), T(c_1,c_1))$-continuous, hence is not $(T(c_1,c_1), T(c_1,c_1))$-shy. □

For the Cartesian product adjacency, we have the following.

Theorem 9.5. *Let $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ be a surjection between digital images, $1 \le i \le v$. Let $X = \Pi_{i=1}^v X_i$, $Y = \Pi_{i=1}^v Y_i$. Then the product function*

$$f = \Pi_{i=1}^v f_i : (X, \times_{i=1}^v \kappa_i) \to (Y, \times_{i=1}^v \lambda_i)$$

is shy if and only if f_i is shy for each i.

Proof. Suppose f is shy. Then clearly each f_i is a surjection, and by Theorem 3.17, f_i is continuous.

Let $y_i \in Y_i$. Let $y = (y_1, \ldots, y_v) \in Y$. Since f is shy, $f^{-1}(y) = \Pi_{i=1}^v f_i^{-1}(y_i)$ is $\times_{i=1}^v \kappa_i$-connected. By Theorem 3.18, the projection map p_i is continuous, so $p_i(f^{-1}(y)) = f_i^{-1}(y_i)$ is κ_i-connected.

Let $y' \in Y$ be such that $y' \leftrightarrow_{\times_{i=1}^v \lambda_i} y$. Then y' must be among the points $q_i = (y_1, \ldots, y_{i-1}, y_i', y_{i+1}, \ldots, y_v)$, where $y_i' \in Y_i$ satisfies $y_i' \leftrightarrow_{\lambda_i} y_i$. Since f is shy, $f^{-1}(\{y, q_i\}) = f^{-1}(y) \cup f^{-1}(q_i)$ is $\times_{i=1}^v \kappa_i$-connected. Since p_i is continuous,

$$p_i(f^{-1}(\{y, q_i\})) = p_i(f^{-1}(y) \cup f^{-1}(q_i)) = f_i^{-1}(y_i) \cup f_i^{-1}(y_i') = f_i^{-1}(\{y_i, y_i'\})$$

is κ_i-connected. This completes the proof that each f_i is shy.

Suppose each f_i is shy. Then clearly f is a surjection, and by Theorem 3.17, f is continuous.

Let $y_i \in Y_i$. Let $y = (y_1, \ldots, y_v) \in Y$. Since f_i is shy, $f_i^{-1}(y_i)$ is κ_i-connected. By Theorem 4.4,

$$f^{-1}(y) = \Pi_{i=1}^v f_i^{-1}(y_i) \text{ is } \times_{i=1}^v \kappa_i\text{-connected.} \tag{9.1}$$

Let $y' \in Y$ be such that $y' \leftrightarrow_{\times_{i=1}^v \lambda_i} y$. Then for some index i, $y' = (y_1, \ldots, y_{i-1}, y_i', y_{i+1}, \ldots, y_v)$, where $y_i' \in Y_i$ satisfies $y_i' \leftrightarrow_{\lambda_i} y_i$. Similarly,

$$f^{-1}(y') \text{ is } \times_{i=1}^v \kappa_i\text{-connected.} \tag{9.2}$$

Since f_i is shy, $f_i^{-1}(\{y_i, y_i'\})$ is connected, so there exist $x_i \in f_i^{-1}(y_i)$, $x_i' \in f_i^{-1}(y_i')$ such that $x_i \leftrightarrow_{\kappa_i} x_i'$ or $x_i = x_i'$. For indices $j \neq i$, let $x_j \in f_j^{-1}(y_j)$. Then $w = (x_1, \ldots, x_v)$ and $w' = (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_v)$ satisfy

$$w \in f^{-1}(y), \ w' \in f^{-1}(y'), \text{ and } w \leftrightarrow_{\times_{i=1}^v \kappa_i} w' \text{ or } w = w'. \tag{9.3}$$

From statements (9.1), (9.2), and (9.3), we conclude that $f^{-1}(\{y, y'\})$ is $\times_{i=1}^v \kappa_i$-connected. Therefore, f is shy. □

For the lexicographic adjacency, we have the following.

Theorem 9.6. *Let $f_i : (X_i, \kappa_i) \to (Y_i, \lambda_i)$ be functions between digital images, $1 \le i \le v$. Let $X = \Pi_{i=1}^{v} X_i$, $Y = \Pi_{i=1}^{v} Y_i$, $f = \Pi_{i=1}^{v} f_i : (X, L(\kappa_1, \ldots, \kappa_v)) \to (Y, L(\lambda_1, \ldots, \lambda_v))$. If each f_i is shy, then f is shy.*

Proof. Let $y = (y_1, \ldots, y_v) \in Y$, where $y_i \in Y_i$. Then $f^{-1}(y) = \Pi_{i=1}^{v} f_i^{-1}(y_i)$. Since each f_i is shy, $f_i^{-1}(y_i)$ is κ_i-connected. By Theorem 4.6, $f^{-1}(y)$ is $L(\kappa_1, \ldots, \kappa_v)$-connected.

Let $p = (y_1', \ldots, y_v') \leftrightarrow_{L(\lambda_1, \ldots, \lambda_v)} y$ in Y. Then for some smallest index k, $y_k' \leftrightarrow_{\lambda_k} y_k$ and if $k > 1$ then $y_i = y_i'$ for $i < k$. Since f_k is shy, $f_k^{-1}(\{y_k, y_k'\})$ is κ_k-connected. Further, if $k > 1$ then $f_i^{-1}(\{y_i, y_i'\}) = f_i^{-1}(y_i)$ is connected, since f_i is shy. Now,

$$f^{-1}(p) = \Pi_{i<k} f_i^{-1}(y_i) \times f_k^{-1}(y_k) \times \Pi_{i>k} f_i^{-1}(y_i), \tag{9.4}$$

$$f^{-1}(p') = \Pi_{i<k} f_i^{-1}(y_i') \times f_k^{-1}(y_k') \times \Pi_{i>k} f_i^{-1}(y_i') \tag{9.5}$$

By the shyness of the f_i and Theorem 4.6, each of $f^{-1}(p)$ and $f^{-1}(p')$ is $L(\kappa_1, \ldots, \kappa_v)$-connected. Further, since $y_i = y_i'$ for $i < k$ and, by shyness of f_k,

$$f_k^{-1}(\{y_k, y_k'\}) \text{ is } \kappa_k\text{-connected,} \tag{9.6}$$

from statements (9.4), (9.5), and (9.6) we can conclude that $f^{-1}(p)$ and $f^{-1}(p')$ are $L(\kappa_1, \ldots, \kappa_v)$-adjacent sets. Therefore, $f^{-1}(\{p, p'\}) = f^{-1}(p) \cup f^{-1}(p')$ is $L(\kappa_1, \ldots, \kappa_v)$-connected. Therefore, f is shy. □

The following shows that the converse of Theorem 9.6 is not generally true.

Example 9.7. Let $f_1 : ([0,1]_{\mathbb{Z}}, c_1) \to \{0\} \subset (\mathbb{Z}, c_1)$ be the function $f_1(x) = 0$. Let $f_2 : (\{0,2\}, c_1) \to \{0\} \subset (\mathbb{Z}, c_1)$ be the function $f_2(x) = 0$. Then

$$f_1 \times f_2 : ([0,1]_{\mathbb{Z}} \times \{0,2\}, L(c_1, c_1)) \to (\{(0,0)\}, L(c_1, c_1))$$

is shy, but f_2 is not shy.

Proof. Since $f_2^{-1}(0)$ is not connected, f_2 is not shy. However, $[0,1]_{\mathbb{Z}} \times \{0,2\}$ is $L(c_1, c_1)$-connected, as discussed in Example 3.27, so, from Definition 2.30, $f_1 \times f_2$ is shy. □

10. Further remarks

We have studied the tensor product, Cartesian product, and lexicographic adjacencies for finite Cartesian products of digital images. We have obtained many results for "product" and "factor" properties that parallel results obtained for extensions of the normal product adjacency in [9].

However, there are many properties known [9] for the normal product adjacency whose analogs for the adjacencies studied here are either false or we were not able to derive. By comparing the results of [9] with those of the current paper, it appears that the normal product adjacency is the adjacency that yields the most satisfying results for Cartesian products of digital images.

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