Applied General Topology

Appl. Gen. Topol. 18, no. 2 (2017), 301-316
doi:10.4995/agt.2017.7048

Quasi-uniform convergence topologies on function spaces - Revisited

Wafa Khalaf Alqurashi [a] and Liaqat Ali Khan [b,*]

[a] Department of Mathematics, Faculty of Science (Girls Section), King Abdulaziz University, P.O. Box 80203, Jeddah-21589, Saudi Arabia. (wafa-math@hotmail.com)
[b] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah-21589, Saudi Arabia. (lkhan@kau.edu.sa)

Communicated by H.-P. A. Künzi

Abstract

Let X and Y be topological spaces and $F(X,Y)$ the set of all functions from X into Y. We study various quasi-uniform convergence topologies $\mathcal{U}_{\mathcal{A}}$ ($\mathcal{A} \subseteq P(X)$) on $F(X,Y)$ and their comparison in the setting of Y a quasi-uniform space. Further, we study $\mathcal{U}_{\mathcal{A}}$-closedness and right K-completeness properties of certain subspaces of generalized continuous functions in $F(X,Y)$ in the case of Y a locally symmetric quasi-uniform space or a locally uniform space.

2010 MSC: *54C35; 54E15; 54C08.*

Keywords: *quasi-uniform space; topology of quasi-uniform convergence on a family of sets; locally uniform spaces, right K-completeness; quasi-continuous functions; somewhat continuous functions.*

1. Introduction

Let X and Y be two topological spaces, $F(X,Y)$ the set of all functions from X into Y and $C(X,Y)$ the set of all continuous functions in $F(X,Y)$. In the case of $Y = (Y,\mathcal{U})$, a uniform space, various uniform convergence topologies (such as $\mathcal{U}_X, \mathcal{U}_k, \mathcal{U}_p$) on $F(X,Y)$ and $C(X,Y)$ were systematically studied by Kelley ([17], Chapter 7). It is shown there that: (i) $\mathcal{U}_p \leq \mathcal{U}_k \leq \mathcal{U}_X$; (ii) $C(X,Y)$

*Corresponding author

Received 24 December 2016 – Accepted 18 February 2017

is $\mathcal{U}_X$-closed in $F(X,Y)$; (iii) if Y is complete, then $F(X,Y)$ is $\mathcal{U}_X$-complete, hence $C(X,Y)$ is also $\mathcal{U}_X$-complete.

Since every topological space is quasi-uniformizable ([8, 39]; [11], p. 27; [7], p. 34), we may assume that $Y=(Y,\mathcal{U})$ with $\mathcal{U}$ a quasi-uniformity. Main advantage of this assumption is that one can introduce various notions of Cauchy nets and completeness. In this setting, some quasi-uniform convergence topologies $\mathcal{U}_{\mathcal{A}}$ ($\mathcal{A}\subseteq P(X)$) on $F(X,Y)$ were first discussed by Naimpally [31]. In recent years, this topic has been further investigated by Papadopoulos [37, 38], Cao [6] and Kunzi and Romaguera [25, 26], among others. There is also a parallel notion of "set-open topologies" $S_{\mathcal{A}}$ ($\mathcal{A}\subseteq P(X)$) on $F(X,Y)$ which were introduced by Fox [12] and further developed by Arens [2], Arens-Dugundji [3], and more recently in the papers [4, 9, 23, 33, 34, 35, 36]. These $S_{\mathcal{A}}$ topologies are, in general, different from their corresponding uniform convergence topologies $\mathcal{U}_{\mathcal{A}}$ ($\mathcal{A}\subseteq P(X)$) even in the case of Y a metric space, but the two notions coincide in some other particular cases.

Regarding completeness in quasi-uniform spaces, the formulation of the notion of "Cauchy net" or "Cauchy filter" in such spaces has been fairly difficult, and has been approached by several authors (see, e.g., [1, 8, 10, 41, 42, 43, 44, 45]). We shall find it convenient to restrict ourselves to the notions of a "right K-Cauchy net" and "right K-complete space" on function spaces, as in [26].

In this paper, we consider various quasi-uniform convergence topologies on $F(X,Y)$ and study their comparison and equivalences. Further, we extend some results of above authors on closedness and completeness to more general classes of functions (not necessarily continuous). These include the subspaces of quasi-continuous, somewhat continuous and bounded functions [16, 22, 37, 40]. Here, we shall need to assume that Y is a locally symmetric quasi-uniform space or a locally uniform space (as appropriate), both notions being equivalent to Y a regular topological space [31, 46]. We have included multiple references for certain concepts for the convenience of readers to access the literature. Some open problems are also stated.

2. Preliminaries

Definition 2.1 ([17], p. 175-176)**.** Let Y be a non-empty set. For any $U,V\subseteq Y\times Y$, we define

$$\begin{aligned} U^{-1} &= \{(y,x):(x,y)\in U\} \\ U\circ V &= \{(x,y)\in Y\times Y:\exists\, z\in Y \text{ such that } (x,z)\in U \text{ and } (z,y)\in V\}. \end{aligned}$$

If $U=V$, we shall write $U\circ U=U^2$. If $U=U^{-1}$, then U is called *symmetric.* The subset $\Delta(Y)=\{(y,y):y\in Y\}$ of $Y\times Y$ is called the *diagonal* on Y. If $\triangle(Y)\subseteq U$, then clearly $U\subseteq U\circ U=U^2\subseteq U^3\subseteq$ ..... For any $x\in Y$, $A\subseteq Y$ and $U\subseteq Y\times Y$, let $U[x]=\{y\in Y:(x,y)\in U\}$ and $U[A]=\cup_{x\in A}U[x]$.

Definition 2.2. A family $\mathcal{U}$ of subsets of $Y\times Y$ is called a **quasi-uniformity** on Y [7, 11, 24, 29] if it satisfies the following conditions:

(QU_1) $\triangle(Y)\subseteq U$ for all $U\in\mathcal{U}$.

(QU_2) If $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.
(QU_3) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$.
(QU_4) If $U \in \mathcal{U}$, there is some $V \in \mathcal{U}$ such that $V^2 \subseteq U$.

In this case, the pair $(Y, \mathcal{U})$ is called a **quasi-uniform space**. If, in addition, $\mathcal{U}$ satisfies the symmetry condition:

(U_5) $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$,

then $\mathcal{U}$ is called a **uniformity** on Y and the pair $(Y, \mathcal{U})$ is called a **uniform space**.

The pair $(Y, \mathcal{U})$ is called a **semi-uniform space** [46] if $\mathcal{U}$ satisfies (QU_1)-(QU_3) and (U_5).

Definition 2.3. (i) A quasi-uniform space $(Y, \mathcal{U})$ is called **locally symmetric** if, for each $y \in Y$ and each $U \in \mathcal{U}$, there is a symmetric $V \in \mathcal{U}$ such that $V^2[y] \subseteq U[y]$ [11].

(ii) A semi-uniform space $(Y, \mathcal{U})$ is called **locally uniform** [46] if, for each $y \in Y$ and each $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^2[y] \subseteq U[y]$.

Definition 2.4 ([11], p. 2-3; [46], p. 436)**.** Let (Y, U) be a quasi-uniform space or a locally uniform space. Then the collection

$$T(\mathcal{U}) = \{G \subseteq Y : \text{for each } y \in G, \text{ there is } U \in \mathcal{U} \text{ such that } U[y] \subseteq G\}$$

is a topology, called the **topology induced by** $\mathcal{U}$ on Y. Equivalently, for each $y \in Y$, the collection $\mathcal{B}_y = \{U[y] : U \in \mathcal{U}\}$ forms a local base at y for the topology $T(\mathcal{U})$.

If (Y, τ) is a topological space, then a quasi-uniformity $\mathcal{U}$ on Y is said to be **compatible** with (Y, τ) provided $\tau = T(\mathcal{U})$. It is well-known that a topological space (Y, τ) is completely regular iff there exists a compatible uniformity $\mathcal{U}$ on Y. Csaszar [8] showed that every topological space has a compatible quasi-uniformity. In [39], Pervin greatly simplified Csaszar's proof by giving a direct method of constructing a compatible quasi-uniformity for an arbitrary topological space. For more information, see ([29], p. 14-16; [7], p. 34).

Definition 2.5. A net $\{y_\alpha : \alpha \in D\}$ in a topological space (Y, τ) is said to be **τ-convergent** to $y \in Y$ if, for each τ-open neighborhood G of y in Y, there exists an $\alpha_0 \in D$ such that $y_\alpha \in G$ for all $\alpha \geq \alpha_0$ ([17], p. 65-66). In particular, a net $\{y_\alpha : \alpha \in D\}$ in a quasi-uniform or locally uniform space $(Y, \mathcal{U})$ is said to be **$T(\mathcal{U})$-convergent** to $y \in Y$ if, for each $U \in \mathcal{U}$, there exists an $\alpha_0 \in D$ such that $y_\alpha \in U[y]$ for all $\alpha \geq \alpha_0$.

Definition 2.6 ([41, 26, 24, 7])**.** Let $(Y, \mathcal{U})$ be a quasi-uniform space. A net $\{y_\alpha : \alpha \in D\}$ in Y is called a **right K-Cauchy net** provided that, for each $U \in \mathcal{U}$, there exists some $\alpha_0 \in D$ such that

$$(y_\alpha, y_\beta) \in U \text{ for all } \alpha, \beta \in D \text{ with } \alpha \geq \beta \geq \alpha_0.$$

$(Y, \mathcal{U})$ is called **right K-complete** if each right K-Cauchy net is $T(\mathcal{U})$-convergent in Y (cf. [26], Lemma 1, p. 289).

Definition 2.7. Let $(Y,\mathcal{U})$ be a quasi-uniform space or a locally uniform space, and let $S \subseteq Y$. Then:

(i) S is called **precompact** [11, 29] if, given any $U \in \mathcal{U}$, there exists a finite set $F \subseteq Y$ such that $S \subseteq U[F]$.
(ii) S is called **totally bounded** [11, 29] if, given any $U \in \mathcal{U}$, there exists a finite cover $\{G_1, G_2, ..., G_n\}$ of S such that $\cup_{i=1}^{n}(G_i \times G_i) \subseteq U$.
(iii) S is **bounded** [30] if given any $U \in \mathcal{U}$, there exists an $m \in \mathbb{N}$ and a finite set $F \subseteq Y$, such that $S \subseteq U^m[F] = \cup_{y\in F} U^m[y]$.

Note. By ([29], p. 49; [30], p. 368), for any $S \subseteq (Y, U)$, a quasi-uniform space,

$$S \text{ is totally bounded } \Rightarrow S \text{ is precompact } \Rightarrow S \text{ is bounded},$$

but the converses need not be true ([11], p. 152; [29], p. 49). In fact, by [27], even a compact quasi-uniform space is not necessarily totally bounded. However, if (Y, U) is a uniform spaces, S is precompact iff S is totally bounded ([11], p. 52; [29], p. 49).

If (X, τ) is a topological space and $A \subseteq X$, the closure of A is denoted by $\overline{A}^{\tau}$ or τ-$cl(A)$ (or simply $\overline{A}$ or $cl(A)$); the interior of A is denoted by $\tau - int(A)$ (or simply $int(A)$). We shall denote the power set of X by $P(X)$.

3. Quasi-uniform convergence topologies on $F(X,Y)$

Let X be a topological space and $(Y,\mathcal{U})$ a quasi-uniform space, and let $\mathcal{A} = \mathcal{A}(X)$ be a certain collection of subsets of X which covers X. For any $A \in \mathcal{A}(X)$ and $U \in \mathcal{U}$, let

$$M_{A,U} = \{(f, g) \in F(X, Y) \times F(X, Y) : (f(x), g(x)) \in U \text{ for all } x \in A\}.$$

Then the collection $\{M_{A,U} : A \in \mathcal{A}(X) \text{ and } U \in \mathcal{U}\}$ forms a subbase for a quasi-uniformity, called the **quasi-uniformity of quasi-uniform convergence on the sets** in $\mathcal{A}(X)$ induced by $\mathcal{U}$. The resultant topology on $F(X,Y)$ is called the **topology of quasi-uniform convergence on the sets** in $\mathcal{A}(X)$ and is denoted by $\mathcal{U}_{\mathcal{A}}$ [25, 26].

(i) If $\mathcal{A} = \{X\}, \mathcal{U}_{\mathcal{A}}$ is called the **quasi-uniform convergence topology** on $F(X,Y)$ and is denoted by $\mathcal{U}_X$.
(ii) If $\mathcal{A} = K(X)$=$\{A \subseteq X : A \text{ is compact}\}$, $\mathcal{U}_{\mathcal{A}}$ is called the **quasi-uniform compact convergence topology** on $F(X,Y)$ and is denoted by $\mathcal{U}_k$
(iii) If $\mathcal{A} = \sigma K(X)$=$\{A \subseteq X : A \text{ is } \sigma\text{-compact}\}$, $\mathcal{U}_{\mathcal{A}}$ is called the **quasi-uniform σ-compact convergence topology** on $F(X,Y)$ and is denoted by $\mathcal{U}_\sigma$.
(iv) If $\mathcal{A} = \sigma_0(X)$=$\{A \subseteq X : A \text{ is countable}\}$, $\mathcal{U}_{\mathcal{A}}$ is called the **quasi-uniform countable convergence topology** on $F(X,Y)$ and is denoted by $\mathcal{U}_{\sigma_0}$.
(v) If $\mathcal{A} = K_0(X)$=$\{A \subseteq X : A \text{ is finite}\}$, $\mathcal{U}_{\mathcal{A}}$ is called the **quasi-uniform pointwise convergence topology** on $F(X,Y)$ and is denoted by $\mathcal{U}_p$.

Since each of the collection $\mathcal{A}$ in (i)-(v) is closed under finite unions, the collection $\{M_{A,U} : A \in \mathcal{A}(X) \text{ and } U \in \mathcal{U}\}$ actually forms a base for the topology $\mathcal{U}_{\mathcal{A}}$ (cf. [28], p. 7).

Lemma 3.1. *Let X be a topological space and $(Y,\mathcal{U})$ a quasi-uniform space, and let $A, B \subseteq X$ and $U, V \in \mathcal{U}$ be such that $M_{A,U} \subseteq M_{B,V}$.*

(i) If $B \neq \varnothing$, then $U \subseteq V$.

(ii) If $V \neq Y \times Y$, then $B \subseteq \overline{A}$.

Proof. (i) Suppose $B \neq \varnothing$, but $U \nsubseteq V$, and let $(a,b) \in U$ with $(a,b) \notin V$. Consider the constant functions $f, g : X \to Y$ defined by $f(x) = a$ $(x \in X)$, $g(x) = b$ $(x \in X)$. Then $(f,g) \in M_{A,U}$, but $(f,g) \notin M_{B,V}$. Indeed, if $A = \varnothing$, then $(f(A), g(A)) \in \varnothing \subseteq U$; if $A \neq \varnothing$ and $x \in A$, then $(f(x), g(x)) = (a,b) \in U$. Hence $(f,g) \in M_{A,U}$. On the other hand, since $B \neq \varnothing$, for any $x \in B$, $(f(x), g(x)) = (a,b) \notin V$, $(f,g) \notin M_{B,V}$. This contradicts $M_{A,U} \subseteq M_{B,V}$.

(ii) Suppose $B \nsubseteq \overline{A}$, and let $x_0 \in B \backslash \overline{A}$. Since $V \neq Y \times Y$, choose $c, d \in Y$ such that $(c,d) \notin V$. Fix $(p,q) \in U$. Define $f, g : X \to Y$ by

$$\begin{aligned} f(x) &= p \text{ if } x \in \overline{A},\ f(x) = c \text{ if } x \in X \backslash \overline{A}; \\ g(x) &= q \text{ if } x \in \overline{A},\ g(x) = d \text{ if } x \in X \backslash \overline{A}. \end{aligned}$$

Then $(f,g) \in M_{A,U}$, but $(f,g) \notin M_{B,V}$. Indeed, if $A = \varnothing$, then $(f(A), g(A)) \in \varnothing \subseteq U$; if $A \neq \varnothing$ and $x \in A$, then $(f(x), g(x)) = (p,q) \in U$. Hence $(f,g) \in M_{A,U}$. On the other hand, since $x_0 \in B$ and $(f(x_0), g(x_0)) = (c,d) \notin V$, $(f,g) \notin M_{B,V}$. This contradicts $M_{A,U} \subseteq M_{B,V}$. Therefore $B \subseteq \overline{A}$. $\square$

Theorem 3.2. *Let X be a Hausdorff topological space and $(Y,\mathcal{U})$ a quasi-uniform space. Let $\mathcal{U}_p, \mathcal{U}_{\sigma_0}, \mathcal{U}_\sigma, \mathcal{U}_k$ and $\mathcal{U}_X$ be the topologies on $F(X,Y)$ as defined above. Then*

(a) $\mathcal{U}_p \leq \mathcal{U}_k \leq \mathcal{U}_\sigma \leq \mathcal{U}_X$ and $\mathcal{U}_p \leq \mathcal{U}_{\sigma_0} \leq \mathcal{U}_\sigma$.

(b) $\mathcal{U}_k = \mathcal{U}_X$ iff X is compact.

(c) $\mathcal{U}_p = \mathcal{U}_k$ iff every compact subset of X is finite. In particular, if X is discrete, then $\mathcal{U}_p = \mathcal{U}_k$.

(d) $\mathcal{U}_\sigma = \mathcal{U}_X$ iff $X = \overline{A}$ for some σ-compact subset A of X.

(e) $\mathcal{U}_k = \mathcal{U}_\sigma$ iff every σ-compact subset of X is relatively compact.

(f) $\mathcal{U}_{\sigma_0} = \mathcal{U}_X$ iff X is separable.

(g) $\mathcal{U}_{\sigma_0} \leq \mathcal{U}_k$ iff every countable subset of X is relatively compact.

(h) $\mathcal{U}_{\sigma_0}$, $\mathcal{U}_\sigma$ and $\mathcal{U}_X$ have the same bounded sets in $F(X,Y)$.

Proof.

(a) Clearly, $K_0(X) \subseteq K(X) \subseteq \sigma K(X)$, and so $\mathcal{U}_p \leq \mathcal{U}_k \leq \mathcal{U}_\sigma \leq \mathcal{U}_X$ on $F(X,Y)$. Further, $K_0(X) \subseteq \sigma_0(X) \subseteq \sigma K(X)$, and so $\mathcal{U}_p \leq \mathcal{U}_{\sigma_0} \leq \mathcal{U}_\sigma \leq \mathcal{U}_X$ on $F(X,Y)$.

(b) Suppose $\mathcal{U}_X \leq \mathcal{U}_k$, and let $U \in \mathcal{U}$, with $U \neq Y \times Y$. Then there exist a compact subset K of X and a $V \in \mathcal{U}$ such that $M_{K,V} \subseteq M_{X,U}$. By Lemma 3.1(ii), $X \subseteq \overline{K} = K$ (since X is Hausdorff). Thus X is compact. Conversely, suppose X is compact. To show $\mathcal{U}_X \leq \mathcal{U}_k$, take arbitrary $M_{X,V} \in \mathcal{U}_X$. Taking $K = X$, which is compact, $M_{K,V} \in \mathcal{U}_k$ and $M_{X,V} \subseteq M_{K,V}$. Hence $M_{X,V} \in \mathcal{U}_k$, and so $\mathcal{U}_X \leq \mathcal{U}_k$.

(c) Suppose $\mathcal{U}_k \leq \mathcal{U}_p$, and let $K \subseteq X$ be a compact set and $U \in \mathcal{U}$, with $U \neq Y \times Y$. Then there exist a finite subset A of X and a $V \in \mathcal{U}$ such that $M_{A,V} \subseteq M_{K,U}$. By Lemma 3.1(ii), $K \subseteq \overline{A} = A$; hence K is finite. Conversely, suppose that every compact subset of X is finite. To show $\mathcal{U}_k \leq \mathcal{U}_p$, take arbitrary $M_{K,U} \in \mathcal{U}_k$ with $K \subseteq X$ a compact set. Then K is finite. Taking $A = K$, $M_{A,V} \in \mathcal{U}_p$ and $M_{A,U} \subseteq M_{K,U}$. Hence $M_{K,U} \in \mathcal{U}_p$, and so $\mathcal{U}_k \leq \mathcal{U}_p$.

In particular, if X is discrete, then every compact subset of X is finite and hence $\mathcal{U}_p = \mathcal{U}_k$.

(d) Suppose that $\mathcal{U}_X \leq \mathcal{U}_\sigma$, and let $U \in \mathcal{U}$, with $U \neq Y \times Y$. Then there exist a σ-compact set $A \subseteq X$ and a $V \in \mathcal{U}$ such that $M_{A,V} \subseteq M_{X,U}$. By Lemma 3.1(ii), $X = \overline{A}$, as required. Conversely, suppose $X = \overline{A}$ for some σ-compact subset A of X. To show $\mathcal{U}_X \leq \mathcal{U}_\sigma$, take arbitrary $M_{X,U} \in \mathcal{U}_X$. Clearly, $M_{A,U} \in \mathcal{U}_\sigma$ and $M_{X,U} \subseteq M_{\overline{A},U} \subseteq M_{A,U}$. Hence $M_{X,U} \in \mathcal{U}_\sigma$, and so $\mathcal{U}_X \leq \mathcal{U}_\sigma$.

(e) Suppose that $\mathcal{U}_\sigma \leq \mathcal{U}_k$ and let A be any σ-compact subset of X. If $U \in \mathcal{U}$, with $U \neq Y \times Y$, then there exist a compact set $B \subseteq X$ and a $V \in \mathcal{U}$ such that $M_{B,V} \subseteq M_{A,U}$. By Lemma 3.1(ii), $A \subseteq \overline{B}$, which implies that $\overline{A}$ is also compact. Conversely, suppose that every σ-compact subset of X is relatively compact. Take arbitrary $M_{A,U} \in \mathcal{U}_\sigma$ with A a σ-compact subset of X. Since $\overline{A}$ is compact, $M_{\overline{A},U} \in \mathcal{U}_k$ and clearly, $M_{\overline{A},U} \subseteq M_{A,U}$. Hence $M_{A,U} \in \mathcal{U}_k$, and so $\mathcal{U}_\sigma \leq \mathcal{U}_k$.

(f) Suppose $\mathcal{U}_X \leq \mathcal{U}_{\sigma_0}$. Then, for any $U \in \mathcal{U}$, with $U \neq Y \times Y$, $M_{X,U} \in \mathcal{U}_X$ and hence $M_{X,U} \in \mathcal{U}_{\sigma_0}$. So there exist a countable set $A \subseteq X$ and a $V \in \mathcal{U}$ such that $M_{A,V} \subseteq M_{X,U}$. By Lemma 3.1(ii), $X = \overline{A}$ and so X is separable. Conversely, suppose X is separable, and let $A \subseteq X$ be countable set such that $\overline{A} = X$. Take arbitrary $M_{X,U} \in \mathcal{U}_X$. Clearly, $M_{A,U} \in \mathcal{U}_{\sigma_0}$ and $M_{X,U} \subseteq M_{\overline{A},U} \subseteq M_{A,U}$. Hence $M_{X,U} \in \mathcal{U}_{\sigma_0}$, and so $\mathcal{U}_X \leq \mathcal{U}_{\sigma_0}$.

(g) Suppose that $\mathcal{U}_{\sigma_0} \leq \mathcal{U}_k$ and let A be any countable subset of X. If $U \in \mathcal{U}$, with $U \neq Y \times Y$, then there exist a compact set $B \subseteq X$ and a $V \in \mathcal{U}$ such that $M_{B,V} \subseteq M_{A,U}$. By Lemma 3.1(ii), $A \subseteq \overline{B}$, which implies that $\overline{A}$ is also compact. Conversely, suppose that every countable subset of X is relatively compact. To show $\mathcal{U}_{\sigma_0} \leq \mathcal{U}_k$, take arbitrary $M_{A,U} \in \mathcal{U}_{\sigma_0}$ with A a countable subset of X. Since $\overline{A}$ is compact, $M_{\overline{A},U} \in \mathcal{U}_k$ and clearly, $M_{\overline{A},U} \subseteq M_{A,U}$. Hence $M_{A,U} \in \mathcal{U}_k$, and so $\mathcal{U}_{\sigma_0} \leq \mathcal{U}_k$.

(h) Since $\mathcal{U}_{\sigma_0} \leq \mathcal{U}_X$, every $\mathcal{U}_X$-bounded subset of $F(X,Y)$ is easily seen to be $\mathcal{U}_{\sigma_0}$-bounded. In fact, let $S \subseteq F(X,Y)$ be $\mathcal{U}_X$-bounded set. Then for arbitrary $M_{A,U} \in \mathcal{U}_{\sigma_0}$ with A a countable subset of X and $U \in \mathcal{U}$, there exists an $m \in \mathbb{N}$ and a finite set $J \subseteq F(X,Y)$ such that $S \subseteq (M_{X,U})^m[J]$. Then $S \subseteq (M_{A,U})^m[J]$, showing that S is $\mathcal{U}_{\sigma_0}$-bounded. On the other hand, suppose that there exists a $\mathcal{U}_{\sigma_0}$-bounded set $T \subseteq F(X,Y)$ which is not $\mathcal{U}_X$-bounded. Then there exist a $V \in \mathcal{U}$ such that $T \nsubseteq (M_{X,V})^n[K] = M_{X,V^n}[K]$ for all $n \in \mathbb{N}$ and all finite sets $K \subseteq F(X,Y)$. Choose sequences $\{h_n\} \subseteq T, \{x_n\} \subseteq X$ such that $(f(x_n), h_n(x_n)) \notin V^n$ for all $n \in \mathbb{N}$ and all $f \in F(X,Y)$. Let $A = \{x_n\}$. Then $M_{A,V} \in \mathcal{U}_{\sigma_0}$, and $h_n \notin (M_{A,V})^n[f]$ for any $n \in \mathbb{N}$ and $f \in F(X,Y)$; hence $T \nsubseteq (M_{A,V})^n[K]$ for any $n \in \mathbb{N}$ and finite set $K \subseteq F(X,Y)$. Therefore T is not $\mathcal{U}_{\sigma_0}$-bounded, a contradiction.

□

Now, let X be a completely regular Hausdorff space and $Y = (E,\tau)$ a Hausdorff topological vector space (TVS, in short) over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$ with a base $\mathcal{W}_E(0)$ of balanced $\tau-$neighborhoods of 0 in E ([18], Theorem 5.1), and let $CB(X,E)$ denote the vector space of all continuous bounded functions from X into E. In this setting, the collection $\mathcal{V} = \{V_G : G \in \mathcal{W}_E(0)\}$ is a uniformity on E, where

$$V_H = \{(x,y) \in E \times E : x - y \in H\}.$$

For any $A \in \mathcal{A}(X)$ and $H \in \mathcal{W}_E(0)$, let

$$M^*_{A,V_H} = \{(f,g) \in CB(X,Y) \times CB(X,Y) : (f(x), g(x)) \in V_H \text{ for all } x \in A\}.$$

Then the collection $\{M^*_{A,V_H}(0) : A \in \mathcal{A}(X)$ and $H \in \mathcal{W}_E(0)\}$ forms a base of neighbourhood of 0 in $CB(X,E)$ for a linear topology, denoted by $t_{\mathcal{A}}$. Indeed, this follows from ([18], Corollary 8.2) and the fact that

$$\begin{aligned} M^*_{A,V_H}(0) &= \{g \in CB(X,Y) : (0,g) \in M^*_{A,V_H}\} \\ &= \{g \in CB(X,Y) : g(x) \in H \text{ for all } x \in A\} \\ &:= N_{cb}(A,H). \end{aligned}$$

The quasi-uniform topologies $\mathcal{U}_p, \mathcal{U}_{\sigma_0}, \mathcal{U}_\sigma, \mathcal{U}_k$ *and* $\mathcal{U}_X$ on $CB(X,Y)$ become the linear topologies, denoted by t_p, t_{σ_0}, t_σ, t_k and t_u in the terminology of [21]. Consequently, we can deduce the following from above two results:

Corollary 3.3 ([21], Lemma 3.2)**.** *Let X be a completely regular Hausdorff space and (E,τ) a Hausdorff TVS. Suppose that $A, B \subseteq X$ and that $G, H \in \mathcal{W}_E(0)$ are such that $N_{cb}(A,G) \subseteq N_{cb}(B,H)$. Then:*
(i) If $B \neq \varnothing$, then $G \subseteq H$.
(ii) If $W \neq E$, then $B \subseteq \overline{A}$. □

Corollary 3.4 ([14, 15]; [21], Theorem 3.3)**.** *Let X be a completely regular Hausdorff space and (E,τ) a Hausdorff TVS. Then:*
(i) $t_\sigma = t_u$ iff $X = \overline{A}$ for some σ-compact subset A of X.
(ii) $t_k = t_\sigma$ iff every σ-compact subset of X is relatively compact.
(iii) $t_{\sigma_0} = t_X$ iff X is separable.
(iv) $t_{\sigma_0} \leq t_k$ iff every countable subset of X is relatively compact.
(v) t_{σ_0}, t_σ and t_u have the same bounded sets in $CB(X,Y)$. □

Finally, in this section, we give a brief account of set-open topologies $S_{\mathcal{A}}$ ($\mathcal{A} \subseteq P(X)$) on $F(X,Y)$ and their comparison with the corresponding quasi-uniform convergence topologies $\mathcal{U}_{\mathcal{A}}$ ($\mathcal{A} \subseteq P(X)$) Let X and Y be topological spaces and let $\mathcal{A} \subseteq P(X)$. For any $A \in \mathcal{A}$ and any open set $H \subseteq Y$, let

$$N(A,H) = \{f \in F(X,Y) : f(A) \subseteq H\}.$$

Then the collection $\{N(A,H) : A \in \mathcal{A}, \text{ open sets } H \subseteq Y\}$ form a subbase for a topology on $F(X,Y)$, called the *set-open (or $\mathcal{A}$-open) topology* generated by $\mathcal{A}$ and denoted by $S_{\mathcal{A}}$. In particular, if $\mathcal{A} = \{X\}$ (resp. $K(X), \sigma K(X), \sigma_0(X), F(X)$), then $S_{\mathcal{A}}$ is called the **uniform** (resp. **compact-open**, **σ-compact-open**, **countable-open**, **point-open**) **topology** and denoted by S_u (resp. $S_k, S_\sigma, S_{\sigma_0}$, S_p). The relation between the set-open topology and the topology of uniform convergence on a family $\mathcal{A} \subseteq K(X)$ was investigated by Kelley ([17], p. 230) and McCoy and Ntantu ([28], p. 9) in the case of Y a uniform space (see also [23]). These $S_{\mathcal{A}}$ topologies are, in general, different from their corresponding uniform convergence topologies $\mathcal{U}_{\mathcal{A}}$ even in the case of Y a metric space. More recently, there has been a renewed interest on the problem for coincidence of these two notions and some interesting partial answers have been obtained in [33, 34, 4, 35, 36].

4. Closedness and completeness in function spaces

The results of this section are motivated by those given in [17, 31, 26] regarding the closedness and completeness of $C(X,Y)$ and $C_{\mathcal{A}}(X,Y)$ in $(F(X,Y), \mathcal{U}_X)$. It is well-known (e.g., [17, 32]) that $C(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$ but not necessarily $\mathcal{U}_p$-closed. Later, some authors obtained variants of these results for spaces of quasi-continuous, somewhat continuous and bounded functions in the case of Y a uniform space [16, 22, 37, 40]. In this section, we examine their $\mathcal{U}_{\mathcal{A}}$-closedness and right K-completeness in the setting of Y a locally symmetric quasi-uniform or locally uniform spaces.

Let X be a topological space and $(Y, \mathcal{U})$ a quasi-uniform space. Let $\{f_\alpha : \alpha \in D\}$ be a net in $F(X,Y)$ and $A \subseteq X$. We recall from [25, 26] that:

(i) $\{f_\alpha\}$ is said to be **right K-Cauchy** in $(F(X,Y), \mathcal{U}_A)$ if, for any $U \in \mathcal{U}$, there exists an index $\alpha_0 \in D$ such that $(f_\alpha, f_\beta) \in M_{A,U}$ for all $\alpha \geq \beta \geq \alpha_0$.

(ii) $\{f_\alpha\}$ is said to be **$\mathcal{U}_A$-convergent** to $f \in F(X,Y)$ if, for any $U \in \mathcal{U}$, there exists an index $\alpha_0 \in D$ such that $(f, f_\alpha) \in M_{A,U}$ for all $\alpha \geq \alpha_0$. In this case, we shall write $f_\alpha \xrightarrow{\mathcal{U}_A} f$.

(iii) $(F(X,Y), \mathcal{U}_A)$ is called **right K-complete** if each right K-Cauchy net in $(F(X,Y), \mathcal{U}_A)$ is *$\mathcal{U}_A$-convergent* to some function in $F(X,Y)$.

Lemma 4.1. *Let X be a topological space and $(Y, \mathcal{U})$ a quasi-uniform space, and let $A \subseteq X$. Let $\{f_\alpha : \alpha \in D\}$ be a net in $F(X,Y)$ such that*

(a) $\{f_\alpha : \alpha \in D\}$ is a right K-Cauchy net in $(F(X,Y), \mathcal{U}_A)$

(b) $f_\alpha(x) \to f(x)$ for each $x \in A$ (i.e. $f_\alpha \xrightarrow{\mathcal{U}_P} f$ on A).

Then $f_\alpha \xrightarrow{\mathcal{U}_A} f$.

Proof. Let $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Since $\{f_\alpha\}$ is right-K-Cauchy, there exists $\alpha_0 \in D$ such that $(f_\alpha, f_\beta) \in M_{A,U}$ for all $\alpha \geq \beta \geq \alpha_0$. We claim that $(f, f_\gamma) \in M_{A,U}$ for all $\gamma \geq \alpha_0$. Fix any $\gamma \geq \alpha_0$ and $x_0 \in A$. Since $\{f_\alpha(x_0) : \alpha \in D\} \longrightarrow f(x_0)$, also its subnet $\{f_\alpha(x_0) : \alpha \geq \gamma\} \longrightarrow f(x_0)$. So there is $\alpha(x_0) \in D$ with $\alpha(x_0) \geq \gamma$ such that $(f(x_0), f_{\alpha(x_0)}(x_0)) \in V$. Since $\alpha(x_0) \geq \gamma \geq \alpha_0$,

$$(f(x_0), f_{\alpha(x_0)}(x_0)) \in V,\ (f_{\alpha(x_0)}(x_0), f_\gamma(x_0)) \in V;$$

hence $(f(x_0), f_\gamma(x_0)) \in V \circ V \subseteq U$. □

The following result is due to Kunzi and Romaguera ([26], Proposition 1). Using Lemma 4.1, we can present a somewhat shorter proof of this result for reader's benefit.

Theorem 4.2. *Let X be a topological space and $Y = (Y,\mathcal{U})$ a right K-complete quasi-uniform space, and let $\mathcal{A} \subseteq P(X)$ which covers X. Then $(F(X,Y),\mathcal{U}_\mathcal{A}\ )$ is right K-complete.*

Proof. Let $\{f_\alpha : \alpha \in D\}$ be a right K-Cauchy net in $(F(X,E),\mathcal{U}_\mathcal{A})$, and let $U \in \mathcal{U}$ and $x \in X$ be fixed. Since $\mathcal{A}$ covers X, $x \in A_x$ for $A_x \in \mathcal{A}$. There exists $\alpha_0 \in D$ such that for each $\alpha \geq \beta \geq \alpha_0$,

$$(f_\alpha, f_\beta) \in M_{A_x,U} \text{ for all } \alpha \geq \beta \geq \alpha_0.$$

In particular, $(f_\alpha(x), f_\beta(x)) \in U$ for all $\alpha \geq \beta \geq \alpha_0$, and so $\{f_\alpha(x) : \alpha \in D\}$ is a right K-Cauchy net in Y. Since $(Y,\mathcal{U})$ a right K-complete, $\{f_\alpha(x) : \alpha \in D\}$ is $T(\mathcal{U})$-convergent to a point $f(x) \in Y$. Hence we have an $f \in F(X,Y)$ such that $f_\alpha \xrightarrow{\mathcal{U}_p} f$. Consequently, by Lemma 4.1, $f_\alpha \xrightarrow{\mathcal{U}_\mathcal{A}} f$. Thus $(F(X,Y),\mathcal{U}_\mathcal{A}\ )$ is complete. □

We next obtain variants of some results in [26] for spaces of quasi-continuous, somewhat continuous and bounded functions in the setting of locally symmetric quasi-uniform and locally uniform spaces.

A subset A of topological space X is called **semi-open** (or **quasi-open)** if there exists an open set G such that $G \subseteq A \subseteq cl(G)$; equivalently, A is **semi-open** iff $A \subset cl(int(A))$. A function $f : X \to Y$ is said to be **quasi-continuous** [22, 40] if $f^{-1}(H)$ is semi-open in X for each open set H in Y, or equivalently, if, for each point $x \in X$ and for each open set $H \subseteq Y$ containing $f(x)$, there exists a semi-open set $A \subseteq X$ such that $x \in A$ and $f(A) \subseteq H$. Let $Q(X,Y)$ denote the set of all quasi-continuous functions from X into Y.

Theorem 4.3. *Let X be a topological space and $(Y,\mathcal{U})$ a locally symmetric quasi-uniform space, and let $A \subseteq X$. Let $\{f_\alpha : \alpha \in D\}$ be a net in $Q(X,Y)$ which is $\mathcal{U}_A$-convergent to f. Then $f \in Q(X,Y)$.*

Proof. Let $x_0 \in X$ and suppose H is any $T(\mathcal{U})$-open set containing $f(x_0)$ in Y. We need to show that there exists a semi-open set G containing x_0 in X such that $f(G \cap A) \subseteq H$. By definition of $T(\mathcal{U})$, there exists a $U \in \mathcal{U}$ such that $U[f(x_0)] \subseteq H$. By local symmetry, choose a symmetric $V \in \mathcal{U}$ such that

$V^2[f(x_0)] \subseteq U[f(x_0)]$. Next choose a closed $W \in \mathcal{U}$ such that $W^2 \subseteq V$ Since $f_\alpha \xrightarrow{\mathcal{U}_A} f$, there exists $\alpha_0 \in D$ such that $(f, f_\alpha) \in M_{A,W}$ for all $\alpha \geq \alpha_0$. In particular, we can write

$$(f(z), f_{\alpha_0}(z)) \in W \subseteq W^2 \subseteq V \text{ for all } z \in X.$$

Since f_{α_0} is quasi-continuous at x_0, there exists a semi-open set G containing x_0 in X such that

$$f_{\alpha_0}(z) \subseteq V[f(x_0)] \text{ for all } z \in G \cap A.$$

Finally, let $z \in G \cap A$. Then, using symmetry of V, we obtain

$$f(z) \in V^{-1}[f_{\alpha_0}(z)] = V[f_{\alpha_0}(z)] \subseteq V[V[f(x_0)]] \subseteq U[f(x_0)] \subseteq H.$$

Therefore, $f(G \cap A) \subseteq H$; hence $f \in Q(X,Y)$. $\square$

Theorem 4.4. *Let X be a topological space and $(Y,\mathcal{U})$ a locally symmetric quasi-uniform space. Then:*

(a) $Q(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$.
(b) If Y is right K-complete, then $(Q(X,Y),\mathcal{U}_X)$ is right K-complete.

Proof. (a) This follows from Theorem 4.3.

(b) Suppose Y is right K-complete. Let $\{f_\alpha : \alpha \in D\}$ be a right K-Cauchy net in $(Q(X,Y),\mathcal{U}_X)$. Let $U \in \mathcal{U}$ and let $x_0 \in X$ be fixed. Since $\{f_\alpha : \alpha \in D\}$ be a right K-Cauchy net, there exists $\alpha_0 \in D$ such that $(f_\alpha, f_\beta) \in M_{X,U}$ for all $\alpha \geq \beta \geq \alpha_0$. In particular

$$(f_\alpha(x_0), f_\beta(x_0)) \in U \text{ for all } \alpha \geq \beta \geq \alpha_0.$$

and so $\{f_\alpha(x_0) : \alpha \in D\}$ is a right K-Cauchy net in Y. Since Y is right K-complete, $\{f_\alpha(x_0)\}$ is $T(\mathcal{U})$-convergent to a point $f(x_0) \in Y$. Hence we have a function $f \in F(X,Y)$ such that $f_\alpha \xrightarrow{\mathcal{U}_p} f$. Consequently, by Lemma 4.1, $f_\alpha \xrightarrow{\mathcal{U}_X} f$, and, by part (a), $f \in Q(X,Y)$. Thus $(Q(X,Y),\mathcal{U}_X)$ is K-complete. $\square$

Corollary 4.5 ([22], Theorem 3.1; [40], Theorem 2.2)**.** *Let X be a topological space and $(Y,\mathcal{U})$ a uniform space. Then:*

(a) $Q(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$.
(b) If Y is complete, then $(Q(X,Y),\mathcal{U}_X)$ is complete.

A function $f : X \to Y$ is said to be **somewhat continuous** [13] if for each open set V in Y such that $f^{-1}(V) \neq \varnothing$, there exists a nonempty open set U in X such that $U \subset f^{-1}(V)$; or equivalently, if, for any $M \subseteq X$, M is dense in X implies $f(M)$ is dense in $f(X)$ ([13], p. 6) . Let $SW(X,Y)$ denote the set of all somewhat continuous functions from X into Y.

Theorem 4.6. *Let X be a topological space and $(Y,\mathcal{U})$ a locally symmetric quasi-uniform space. Let $\{f_\alpha : \alpha \in D\}$ be a net in $SW(X,Y)$ which is $\mathcal{U}_X$-convergent to f. Then $f \in SW(X,Y)$.*

Proof. Let M be a dense subset of X. We need to show that $f(M)$ is dense in $f(X)$. Let $y_0 \in f(X)$, and let H be an open neighborhood of y_0 in $f(X)$. (We need to show that $H \cap f(M) \neq \varnothing$.) There exists a $G \in T(\mathcal{U})$ such that $H = G \cap f(X)$. Choose $x_0 \in X$ such that $f(x_0) = y_0$. Since $G \in T(\mathcal{U})$, there exists $U \in \mathcal{U}$ such that $U[y_0] \subseteq G$. By local symmetry, choose a symmetric $V \in \mathcal{U}$ such that $V^2[y_0] \subseteq U[y_0]$. Since $(Y,\mathcal{U})$ is a quasi-uniform space, choose a closed $W \in \mathcal{U}$ such that $W^2 \subseteq V$. Since $f_\alpha \xrightarrow{\mathcal{U}_X} f$, there exists $\alpha_0 \in D$ such that $f_\alpha \in M_{X,W}[f]$ for all $\gamma \geq \alpha_0$. In particular,

$$(f(z), f_{\alpha_0}(z)) \in W \subseteq W^2 \subseteq V \text{ for all } z \in X.$$

Since f_{α_0} is somewhat continuous, $f_{\alpha_0}(M)$ is dense in $f_{\alpha_0}(X)$. Since $W[f_{\alpha_0}(x_0)]$ is a neighbohood of $f_{\alpha_0}(x_0)$ in the $T(\mathcal{U})$-topology, there exists some $m \in M$ such that $f_{\alpha_0}(m) \in W[f_{\alpha_0}(x_0)]$. Then

$$(f(x_0), f_{\alpha_0}(x_0)) \in W \text{ and } (f_{\alpha_0}(x_0), f_{\alpha_0}(m)) \in W;$$

hence $(f(x_0), f_{\alpha_0}(m)) \in W \circ W \subseteq V$. Since $m \in M \subseteq X$, $(f(m), f_{\alpha_0}(m)) \in V$. Hence $(f(x_0), f(m)) \in V \circ V^{-1} = V^2$. Consequently,

$$f(m) \in V^2[f(x_0)] \subseteq U[f(x_0)] = U[y_0] \subseteq G \subseteq H.$$

Thus $f(m) \in H$, and so $H \cap f(M) \neq \varnothing$. Hence $f(M)$ is dense in $f(X)$, showing that $f \in SW(X,Y)$. $\square$

Theorem 4.7. *Let X be a topological space and $(Y,\mathcal{U})$ a locally symmetric quasi-uniform space. Then:*

(a) $SW(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$.
(b) If Y is right K-complete, then $(SW(X,Y),\mathcal{U}_X)$ is right K-complete.

Proof. (a) This follows from Theorem 4.6.

(b) Suppose Y is right K-complete. Let $\{f_\alpha : \alpha \in D\}$ be a right K-Cauchy net in $(SW(X,Y),\mathcal{U}_X)$. Let $U \in \mathcal{U}$ and let $x_0 \in X$ be fixed. There exists $\alpha_0 \in D$ such that $(f_\alpha, f_\beta) \in M_{X,U}$ for all $\alpha \geq \beta \geq \alpha_0$. In particular

$$(f_\alpha(x_0), f_\beta(x_0)) \in U \text{ for all } \alpha \geq \beta \geq \alpha_0,$$

and so $\{f_\alpha(x_0) : \alpha \in D\}$ is a right K-Cauchy net in Y. Since Y is right K-complete, $\{f_\alpha(x_0)\}$ is $T(\mathcal{U})$-convergent to a point $f(x_0) \in Y$. Hence we have a function $f \in F(X,Y)$ such that $f_\alpha \xrightarrow{\mathcal{U}_p} f$. Consequently, by Lemma 4.1, $f_\alpha \xrightarrow{\mathcal{U}_X} f$, and, by part (a), $f \in SW(X,Y)$. Thus $(SW(X,Y),\mathcal{U}_X)$ is right K-complete. $\square$

Corollary 4.8 ([16], Theorem 1, p. 32)**.** *Let X be a topological space and $(Y,\mathcal{U})$ a uniform space. Then:*

(a) $SW(X,Y)$ is $\mathcal{U}_X$-closed in $F(X,Y)$.
(b) If Y is complete, then $(SW(X,Y),\mathcal{U}_X)$ is complete.

We next present analogues of some results from [37] for functions having range as precompact or bounded sets.

Theorem 4.9. *Let X be a non-empty set and $(Y,\mathcal{U})$ a locally uniform space. Let $\mathcal{A} \subseteq P(X)$ and $B_{\mathcal{A}}(X,Y) \subseteq F(X,Y)$ the set of all functions which are bounded on each member of $\mathcal{A}$. Let $\{f_\alpha : \alpha \in D\}$ be a net in $B_{\mathcal{A}}(X,Y)$ which is $U_{\mathcal{A}}$-convergent to f. Then $f \in B_{\mathcal{A}}(X,Y)$. (Hence $B_{\mathcal{A}}(X,Y)$ is $\mathcal{U}_{\mathcal{A}}$-closed in $F(X,Y)$.)*

Proof. Let $U \in \mathcal{U}$ be symmetric and $A \in \mathcal{A}(X)$. Choose an $\alpha_0 \in D$ such that $(f, f_{\alpha_0}) \in M_{A,U}$ for all $\alpha \geq \alpha_0$. In particular, this implies that for each $x \in A$,

$$f(x) \in U^{-1}[f_{\alpha_0}(x))] = U[f_{\alpha_0}(x))] \subseteq U[f_{\alpha_0}(A))];$$

hence $f(A) \subseteq U[f_{\alpha_0}(A)]$. But $f_{\alpha_0} \in B_{\mathcal{A}}(X,Y)$, so there exists an integer $m \geq 1$ and a finite set $F \subseteq Y$, such that $f_{\alpha_0}(A) \subseteq U^m[F]$. Thus

$$f(A) \subseteq U[f_{\alpha_0}(A)] \subseteq U[U^m[F] = U^{m+1}[F],$$

which means that $f \in B(X,Y)$. □

Problem 4.10. *The authors do not know whether or not the above result can be established for $(Y,\mathcal{U})$ a locally symmetric quasi-uniform space.*

Theorem 4.11. *Let X be a non-empty set and $(Y,\mathcal{U})$ a uniform space, and let $\mathcal{A} \subseteq P(X)$ and $PC_{\mathcal{A}}(X,Y) \subseteq F(X,Y)$ be the set of all functions which have precompact range on each member of $\mathcal{A}$. Let $\{f_\alpha : \alpha \in D\}$ be a net in $PC_{\mathcal{A}}(X,Y)$ which is $\mathcal{U}_{\mathcal{A}}$-convergent to f. Then $f \in PC_{\mathcal{A}}(X,Y)$. (Hence $PC_{\mathcal{A}}(X,Y)$ is $\mathcal{U}_{\mathcal{A}}$-closed in $F(X,Y)$ and is $\mathcal{U}_{\mathcal{A}}$-complete if $(Y,\mathcal{U})$ is complete.)*

Proof. Let $U \in \mathcal{U}$ and $A \in \mathcal{A}(X)$. Choose symmetric $W \in \mathcal{U}$ with $W \circ W \subseteq U$. There exists a $\alpha_0 \in D$ such that for each $\alpha \geq \alpha_0$, $f_\alpha \in M_{A,W}[f]$. In particular, $(f, f_{\alpha_0}) \in M_{A,W}$, which implies that

$$f(A) \subseteq W^{-1}[f_{\alpha_0}(A)] = W[f_{\alpha_0}(A)].$$

But $f_{\alpha_0} \in PC_{\mathcal{A}}(X,Y)$, so there exists a finite set $F \subseteq Y$, such that $f_{\alpha_0}(A) \subseteq W[F]$. Thus

$$f(A) \subseteq (W \circ W)[F] \subseteq U[F],$$

which means that $f \in PC_{\mathcal{A}}(X,Y)$. □

Problem 4.12. *The authors do not know whether or not the above result can be established for $(Y,\mathcal{U})$ a locally symmetric quasi-uniform space or a locally uniform space.*

Next we consider the notion of functions $f \in F(X,Y)$ which are "small off compact set". First, let $Y = E$, a TVS over $\mathbb{K}$ (=$\mathbb{R}$ or $\mathbb{C}$) with $\mathcal{W}_E(0)$ a base of balanced neighborhoods of 0 in E. Recall that: a function $f : X \to E$ is called **small off compact set** (or **vanish at infinity**) [5, 18, 20] if, for each $G \in \mathcal{W}_E(0)$, there exists a compact set $K \subseteq X$ such that $f(x) \in G$ for all $x \in X \backslash K$.

Note that if $f \in F(X,E)$ is small off compact set, then given any $G \in \mathcal{W}_E(0)$, there exists a compact set $K \subseteq X$ such that

$$f(x) - f(y) \in G \text{ for all } x, y \in X \backslash K.$$

In fact, choose a balanced (or symmetric) $H \in \mathcal{W}_E(0)$ such that $H + H \subseteq G$. Since $f \in F_0(X, E)$, there exists a compact set $K \subseteq X$ such that $f(x) \in H$ for all $x \in X \backslash K$. Then, for any $x, y \in X \backslash K$, $f(x) - f(y) \in H - H = H + H \subseteq G$.

Motivated by this observation, we can introduce the notion of "small off compact set" in the setting of quasi-uniform spaces, as follows. Let X be a topological space and $(Y, \mathcal{U})$ a quasi-uniform space.

(i) A function $f : X \to Y$ is said to be **small off compact set** if, for each $U \in \mathcal{U}$, there exists a compact set $K \subseteq X$ such that $(f(x), f(y)) \in U$ for all $x, y \in X \backslash K$.

(ii) $f : X \to Y$ is said to have **compact support** if there exists a compact set $K \subseteq X$ such that $(f(x), f(y)) \in U$ for all $x, y \in X \backslash K$ and all $U \in \mathcal{U}$.

Note that a quasi-uniform space $(Y, \mathcal{U})$ is T_1 iff $\cap\{U : U \in \mathcal{U}\} = \Delta(Y)$ holds ([11], p. 6; [29], p. 36). Hence, in this case, the condition in (ii) is equivalent to: $f(x) = f(y)$ for all $x, y \in X \backslash K$.

Let $F_0(X, Y)$ (resp. $F_{00}(X, Y)$) denote the subset of $F(X, Y)$ consisting of those functions which are small of compact set (resp. have compact support), and let $C_0(X, Y) = F_0(X, Y) \cap C(X, Y)$ and $C_{00}(X, Y) = F_{00}(X, Y) \cap C(X, Y)$. Clearly, $F_{00}(X, Y) \subseteq F_0(X, Y)$ and $C_{00}(X, Y) \subseteq C_0(X, Y)$.

Lemma 4.13. $C_0(X, Y) \subseteq B(X, Y)$.

Proof. Let $f \in C_0(X, Y)$, and let $U \in \mathcal{U}$. There exists a compact set $K \subseteq X$ such that $(f(x), f(y)) \in U$ for all $x, y \in X \backslash K$. In particular, for a fixed $x_0 \in X \backslash K$,

$$f(y) \in U[f(x_0)] \text{ for all } y \in X \backslash K.$$

Since $f(K)$ is compact and hence bounded in Y ([30], p. 368), there exists an integer m and a finite set $F \subseteq Y$, such that $f(K) \subseteq U^m[F]$. Taking $F_1 = F \cup \{f(x_0)\}$, we have $f(X) \subseteq U^m[F_1]$, which means that $f \in B(X, Y)$. □

Theorem 4.14. *Let X be a topological space and $(Y, \mathcal{U})$ a uniform space. Then both $F_0(X, Y)$ and $C_0(X, Y)$ are $\mathcal{U}_X$-closed in $F(X, Y)$.*

Proof. Let $f \in F(X, Y)$ with $f \in \mathcal{U}_X - cl(F_0(X, Y))$. Let $U \in \mathcal{U}$. Choose a symmetric $V \in \mathcal{U}$ such that $V^3 \subseteq U$. There exists $g \in F_0(X, Y)$ such that $(f(x), g(x)) \in V$ for all $x \in X$. There exists a compact set $K \subseteq X$ such that $(g(x), g(y)) \in V$ for all $x, y \in X \backslash K$. Then, for any $x, y \in X \backslash K$,

$$(f(x), g(x)) \in V, \ (g(x), g(y)) \in V, \ (g(y), f(y)) \in V^{-1}.$$

Hence

$$(f(x), f(y) \in V \circ V \circ V^{-1} = V \circ V \circ V \subseteq U \text{ for all } x, y \in X \backslash K.$$

Therefore $f \in F_0(X, Y)$, and so $F_0(X, Y)$ is $\mathcal{U}_X$-closed in $F(X, Y)$. By ([17], Theorem 7.9), $C(X, Y)$ is $\mathcal{U}_X$-closed in $F(X, Y)$. Thus $C_0(X, Y)$ is also $\mathcal{U}_X$-closed in $F(X, Y)$. □

Remark 4.15. If X is not locally compact, then $C_0(X, Y)$ may consist of only constant functions. For example, if $X = \mathcal{Q}$ (rationals), then $C_0(X, \mathbb{R}) = \{0\}$ ([20], p. 12).

Problem 4.16. *If X is locally compact and $Y = E$, a topological vector space, then it is well-known that $C_{00}(X, E)$ is $\mathcal{U}_k$-dense in $C(X, E)$ and also $\mathcal{U}_X$-dense in $C_0(X, E)$ ([5], p. 96-98; [18], p. 81; [19], Theorem 3.2; [20], Theorem 1.1.10). We leave an open problem that whether or not these denseness results hold for Y a uniform space.*

Acknowledgements. *The authors wish to thank Professors H. P. A. Künzi, R. A. McCoy and A. V. Osipov for communicating to us useful information of some concepts used in this paper, and the referee for suggesting a number of improvements to the original version of the paper.*

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