APPLIED GENERAL TOPOLOGY

Existence of common fixed points of improved F-contraction on partial metric spaces

MUHAMMAD NAZAM ^{a}, MUHAMMAD ARSHAD ^{a} AND MUJAHID ABBAS ^{b}

^{a} Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan (nazim.phdma47@iiu.edu.pk, marshadzia@iiu.edu.pk)
^{b} Department of Mathematics and Applied Mathematics, University of Pretoria, Lynnwood road, Pretoria 0002, South Africa. (abbas.mujahid@gmail.com)

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ABSTRACT

Following the approach of F-contraction introduced by Wardowski [13], in this paper, we introduce improved F-contraction of rational type in the framework of partial metric spaces and used it to obtain a common fixed point theorem for a pair of self mappings. We show, through example, that improved F-contraction is more general than F- contraction and guarantees fixed points in those cases where F-contraction fails to provide. Moreover, we apply this fixed point result to show the existence of common solution of the system of integral equations.

1. INTRODUCTION

Matthews [11] introduced the concept of partial metric spaces and proved an analogue of Banach's fixed point theorem in partial metric spaces. In fact, a partial metric space is a generalization of metric space in which the self distances $p(r_1, r_1)$ of elements of a space may not be zero and follows the inequality $p(r_1, r_1) \leq p(r_1, r_2)$. After this remarkable contribution, many authors took interest in partial metric spaces and its topological properties and presented

several well known fixed point results in the framework of partial metric spaces (see [1, 2, 3, 4, 12] and references therein).

In 1922, Banach presented a landmark fixed point result (Banach Contraction Principle). This result proved a gateway for the fixed point researchers and opened a new door in metric fixed point theory. A number of efforts have been made to enrich and generalize Banach Contraction Principle (see [6, 7] and references therein). Following Banach, in 2012, Wardowski [13] presented a new contraction (known as F - contraction). Since 2012, a number of fixed point results have been established by using F-contraction (see [5, 9, 10] and references therein). Recently, Vetro et al.[8] proved some fixed point theorems for Hardy-Rogers-type self-mappings in complete metric spaces and complete ordered metric spaces for F- contractions.

In this article, following Wardowski [13] and Vetro et al.[8], we prove a common fixed point theorem for a pair of self mappings satisfying improved F-contraction of rational type in complete partial metric spaces. An example is constructed to illustrate this result. We apply the mentioned theorem to show the existence of solution of system of Volterra type integral equations.

2. PRELIMINARIES

Throughout this paper, we denote $(0, \infty)$ by $\mathbb{R}^+$, $[0, \infty)$ by $\mathbb{R}_0^+$, $(-\infty, +\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$. Following concepts and results will be required for the proofs of main results.

Definition 2.1 ([13]). A mapping $T : M \to M$, is said to be F-contraction if it satisfies following condition

$$(2.1) \qquad (d(T(r_1), T(r_2)) > 0 \Rightarrow \tau + F(d(T(r_1), T(r_2)) \leq F(d(r_1, r_2))),$$

for all $r_1, r_2 \in M$ and some $\tau > 0$. Where $F : \mathbb{R}^+ \to \mathbb{R}$ is a function satisfying following properties.

- (F_1) : F is strictly increasing.
- (F_2) : For each sequence $\{r_n\}$ of positive numbers $\lim_{n \to \infty} r_n = 0$ if and only if $\lim_{n \to \infty} F(r_n) = -\infty$.
- (F_3) : There exists $\theta \in (0, 1)$ such that $\lim_{\alpha \to 0^+} (\alpha)^\theta F(\alpha) = 0$.

Wardowski [13] established the following result using F-contraction.

Theorem 2.2 ([13]). *Let (M, d) be a complete metric space and $T : M \to M$ be a F-contraction. Then T has a unique fixed point $\upsilon \in M$ and for every $r_0 \in M$ the sequence $\{T^n(r_0)\}$ for all $n \in \mathbb{N}$ is convergent to υ.*

We denote by Δ_F, the set of all functions satisfying the conditions $(F_1) - (F_3)$.

Example 2.3 ([13]). Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by the formula $F(\alpha) = \ln(\alpha)$. It is clear that F satisfies $(F_1) - (F_3)$ for any $\kappa \in (0, 1)$. Each mapping $T : M \to M$ satisfying (2.1) is a F-contraction such that

$$d(T(r_1), T(r_2)) \leq e^{-\tau} d(r_1, r_2), \text{ for all } r_1, r_2 \in M, T(r_1) \neq T(r_2).$$

Obviously, for all $r_1, r_2 \in M$ such that $T(r_1) = T(r_2)$, the inequality

$$d(T(r_1), T(r_2)) \leq e^{-\tau} d(r_1, r_2)$$

holds, that is T is a Banach contraction.

Remark 2.4. From (F_1) and (2.1) it is easy to conclude that every F-contraction is necessarily continuous.

Definition 2.5 ([11])**.** Let M be a nonempty set and if the function $p : M \times M \to \mathbb{R}_0^+$ satisfies following properties,

(p_1) $r_1 = r_2 \Leftrightarrow p(r_1, r_1) = p(r_1, r_2) = p(r_2, r_2)$,
(p_2) $p(r_1, r_1) \leq p(r_1, r_2)$,
(p_3) $p(r_1, r_2) = p(r_2, r_1)$,
(p_4) $p(r_1, r_3) \leq p(r_1, r_2) + p(r_2, r_3) - p(r_2, r_2)$.

for all $r_1, r_2, r_3 \in M$. Then p is called a partial metric on M and the pair (M, p) is known as partial metric space.

In [11], Matthews proved that every partial metric p on M induces a metric $d_p : M \times M \to \mathbb{R}_0^+$ defined by

(2.2) $\qquad d_p(r_1, r_2) = 2p(r_1, r_2) - p(r_1, r_1) - p(r_2, r_2)$;

for all $r_1, r_2 \in M$.

Notice that a metric on a set M is a partial metric p such that $p(r, r) = 0$ for all $r \in M$ and $p(r_1, r_2) = 0$ implies $r_1 = r_2$ (using (p_1) and (p_2)).

Matthews [11] established that each partial metric p on M generates a T_0 topology $\tau(p)$ on M. The base of topology $\tau(p)$ is the family of open p-balls $\{B_p(r, \epsilon) : r \in M, \ \epsilon > 0\}$, where $B_p(r, \epsilon) = \{r_1 \in M : p(r, r_1) < p(r, r) + \epsilon\}$ for all $r \in M$ and $\epsilon > 0$. A sequence $\{r_n\}_{n \in \mathbb{N}}$ in (M, p) converges to a point $r \in M$ if and only if $p(r, r) = \lim_{n \to \infty} p(r, r_n)$.

Definition 2.6 ([11])**.** Let (M, p) be a partial metric space.

(1) A sequence $\{r_n\}_{n \in \mathbb{N}}$ in (M, p) is called a Cauchy sequence if $\lim_{n, m \to \infty} p(r_n, r_m)$ exists and is finite.
(2) A partial metric space (M, p) is said to be complete if every Cauchy sequence $\{r_n\}_{n \in \mathbb{N}}$ in M converges, with respect to $\tau(p)$, to a point $r \in X$ such that $p(r, r) = \lim_{n, m \to \infty} p(r_n, r_m)$.

The following lemma will be helpful in the sequel.

Lemma 2.7 ([11])**.**

(1) *A sequence r_n is a Cauchy sequence in a partial metric space (M, p) if and only if it is a Cauchy sequence in metric space (M, d_p)*
(2) *A partial metric space (M, p) is complete if and only if the metric space (M, d_p) is complete.*
(3) *A sequence $\{r_n\}_{n \in \mathbb{N}}$ in M converges to a point $r \in M$, with respect to $\tau(d_p)$ if and only if $\lim_{n \to \infty} p(r, r_n) = p(r, r) = \lim_{n, m \to \infty} p(r_n, r_m)$.*
(4) *If $\lim_{n \to \infty} r_n = \upsilon$ such that $p(\upsilon, \upsilon) = 0$ then $\lim_{n \to \infty} p(r_n, r) = p(\upsilon, r)$ for every $r \in M$.*

In the following example, we shall show that there are mappings which are not F-contractions in metric spaces, nevertheless, such mappings follow the conditions of F-contraction in partial metric spaces.

Example 2.8. Let $M = [0, 1]$ and define partial metric by $p(r_1, r_2) = \max\{r_1, r_2\}$ for all $r_1, r_2 \in M$. The metric d induced by partial metric p is given by $d(r_1, r_2) = |r_1 - r_2|$ for all $r_1, r_2 \in M$. Define the mappings $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ and T by

$$
T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0, 1); \\[2mm] 0 & \text{if } r = 1 \end{cases}
$$

Then T is not a F- contraction in a metric space (M, d). Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, $d(T(r_1), T(r_2)) > 0$ and we have

$$
\begin{aligned}
\tau + F\left(d(T(r_1), T(r_2))\right) &\leq F\left(d(r_1, r_2)\right), \\
\tau + F\left(d(T(1), T(\tfrac{5}{6}))\right) &\leq F\left(d(1, \tfrac{5}{6})\right), \\
\tau + F\left(d(0, \tfrac{1}{6})\right) &\leq F\left(\tfrac{1}{6}\right), \\
\frac{1}{6} &< \frac{1}{6},
\end{aligned}
$$

which is a contradiction for all possible values of τ. Now if we work in partial metric space (M, p), we get a positive answer that is

$$
\begin{aligned}
\tau + F\left(p(T(r_1), T(r_2))\right) &\leq F\left(p(r_1, r_2)\right) \text{ implies} \\
\tau + F\left(\frac{1}{6}\right) &\leq F(1),
\end{aligned}
$$

which is true. Similarly, for all other points in M our claim proves true.

3. MAIN RESULT

We begin with following definitions.

Definition 3.1. Let (M, p) be a partial metric space. The mapping $T : M \to M$ is called a improved F-contraction of rational type, if for all $m_1, m_2 \in M$, we have

$$
(3.1) \qquad \tau + F(p(T(m_1), T(m_2))) \leq F\left(N(m_1, m_2)\right),
$$

for some $F \in \Delta_F$, $\tau > 0$ and

$$
N(m_1, m_2) = \max \left\{ \begin{array}{l} p(m_1, m_2), \dfrac{p(m_1, T(m_1))p(m_2, T(m_2))}{1 + p(m_1, m_2)}, \\[3mm] \dfrac{p(m_1, T(m_1))p(m_2, T(m_2))}{1 + p(T(m_1), T(m_2))} \end{array} \right\}.
$$

Definition 3.2. Let (M, p) be a partial metric space. The mappings $S, T :$ $M \to M$ are called a pair of improved F-contraction of rational type, if for all $m_1, m_2 \in M$, we have
(3.2)
$$(p(S(m_1), T(m_2)) > 0 \text{ implies } \tau + F(p(S(m_1), T(m_2))) \leq F\left(\mathbb{M}(m_1, m_2)\right),$$

for some $F \in \Delta_F$, $\tau > 0$ and

$$\mathbb{M}(m_1, m_2) = \max \left\{ \begin{array}{c} p(m_1, m_2), \dfrac{p(m_1, S(m_1))p(m_2, T(m_2))}{1 + p(m_1, m_2)}, \\ \dfrac{p(m_1, S(m_1))p(m_2, T(m_2))}{1 + p(S(m_1), T(m_2))} \end{array} \right\}.$$

The following theorem is one of our main results.

Theorem 3.3. *Let (M, p) be a complete partial metric space and $S, T : M \to M$ be a pair of mappings such that*

(1) *S or T is a continuous mapping,*
(2) *(S, T) is a pair of improved F-contraction of rational type.*

Then there exists a common fixed point υ of the pair (S, T) in M such that $p(\upsilon, \upsilon) = 0$.

Proof. We begin with the following observation:

$\mathbb{M}(m_1, m_2) = 0$ if and only if $m_1 = m_2$ is a common fixed point of (S, T).

Indeed, if $m_1 = m_2$ is a common fixed point of (S, T), then $T(m_2) = T(m_1) = m_1 = m_2 = S(m_2) = S(m_1)$ and

$$\begin{aligned} \mathbb{M}(m_1, m_2) &= \max \left\{ \begin{array}{c} p(m_1, m_1), \dfrac{p(m_1, S(m_1))p(m_2, T(m_2))}{1 + p(m_1, m_2)}, \\ \dfrac{p(m_1, S(m_1))p(m_2, T(m_2))}{1 + p(S(m_1), T(m_2))} \end{array} \right\}, \\ &= p(m_1, m_1). \end{aligned}$$

From contractive condition (3.2), we get

$$\tau + F(p(m_1, m_1)) = \tau + F(p(S(m_1), T(m_2))) \leq F\left(p(m_1, m_1)\right).$$

This is only possible if $p(m_1, m_1) = 0$, which entails $\mathbb{M}(m_1, m_1) = 0$. Conversely, if $\mathbb{M}(m_1, m_2) = 0$, it is easy to check that $m_1 = m_2$ is a fixed point of S and T.
In order to find common fixed points of S and T for the situation when $\mathbb{M}(r_1, r_2) > 0$ for all $r_1, r_2 \in M$ with $r_1 \neq r_2$, we construct an iterative sequence $\{r_n\}$ of points in M such a way that, $r_{2i+1} = S(r_{2i})$ and $r_{2i+2} = T(r_{2i+1})$ where $i = 0, 1, 2, \ldots$. Assume that $p(S(r_{2i}), T(r_{2i+1})) > 0$, then from contractive condition (3.2), we get

$$F\left(p(r_{2i+1}, r_{2i+2})\right) = F\left(p(S(r_{2i}), T(r_{2i+1}))\right) \leq F\left(\mathbb{M}(r_{2i}, r_{2i+1})\right) - \tau,$$

for all $i \in \mathbb{N} \cup \{0\}$, where

$$\mathbb{M}(r_{2i}, r_{2i+1}) = \max \left\{ \begin{array}{c} p(r_{2i}, r_{2i+1}), \dfrac{p(r_{2i}, S(r_{2i}))p(r_{2i+1}, T(r_{2i+1}))}{1 + p(r_{2i}, r_{2i+1})}, \\ \dfrac{p(r_{2i}, S(r_{2i}))p(r_{2i+1}, T(r_{2i+1}))}{1 + p(S(r_{2i}), T(r_{2i+1}))} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} p(r_{2i}, r_{2i+1}), \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1 + p(r_{2i}, r_{2i+1})}, \\ \dfrac{p(r_{2i}, r_{2i+1})p(r_{2i+1}, r_{2i+2})}{1 + p(r_{2i+1}, r_{2i+2})} \end{array} \right\}$$

$$\leq \max \{p(r_{2i}, r_{2i+1}), p(r_{2i+1}, r_{2i+2})\}.$$

For if $\mathbb{M}(r_{2i}, r_{2i+1}) \leq p(r_{2i+1}, r_{2i+2})$, then

$$F\left(p(r_{2i+1}, r_{2i+2})\right) \leq F\left(p(r_{2i+1}, r_{2i+2})\right) - \tau,$$

which is a contradiction due to F_1. Therefore,

$$F\left(p(r_{2i+1}, r_{2i+2})\right) \leq F\left(p(r_{2i}, r_{2i+1})\right) - \tau,$$

for all $i \in \mathbb{N} \cup \{0\}$. Hence,

$$(3.3) \qquad F\left(p(r_{n+1}, r_{n+2})\right) \leq F\left(p(r_n, r_{n+1})\right) - \tau,$$

for all $n \in \mathbb{N} \cup \{0\}$. Following (3.3), we obtain

$$F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_{n-2}, r_{n-1})\right) - 2\tau.$$

Repeating these steps we get,

$$(3.4) \qquad F\left(p(r_n, r_{n+1})\right) \leq F\left(p(r_0, r_1)\right) - n\tau.$$

From (3.4), we obtain $\lim_{n \to \infty} F\left(p(r_n, r_{n+1})\right) = -\infty$. Since $F \in \Delta_F$,

$$(3.5) \qquad \lim_{n \to \infty} p(r_n, r_{n+1}) = 0.$$

From the property (F_3) of F-contraction, there exists $\kappa \in (0, 1)$ such that

$$(3.6) \qquad \lim_{n \to \infty} \left((p(r_n, r_{n+1}))^\kappa F\left(p(r_n, r_{n+1})\right)\right) = 0.$$

Following (3.4), for all $n \in \mathbb{N}$, we obtain
$$(3.7)$$
$$\left(p(r_n, r_{n+1})\right)^\kappa \left(F\left(p(r_n, r_{n+1})\right) - F\left(p(r_0, x_1)\right)\right) \leq -\left(p(r_n, r_{n+1})\right)^\kappa n\tau \leq 0.$$

Considering (3.5), (3.6) and letting $n \to \infty$ in (3.7), we have

$$(3.8) \qquad \lim_{n \to \infty} \left(n \left(p(r_n, r_{n+1})\right)^\kappa\right) = 0.$$

Since (3.8) holds, there exists $n_1 \in \mathbb{N}$, such that $n\left(p(r_n, r_{n+1})\right)^\kappa \leq 1$ for all $n \geq n_1$ or,

$$(3.9) \qquad p(r_n, r_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1.$$

Using (3.9), we get for $m > n \geq n_1$,

$$
\begin{aligned}
p(r_n, r_m) &\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + ... + p(r_{m-1}, r_m) \\
&\quad - \sum_{j=n+1}^{m-1} p(r_j, r_j) \\
&\leq p(r_n, r_{n+1}) + p(r_{n+1}, r_{n+2}) + p(r_{n+2}, r_{n+3}) + ... + p(r_{m-1}, r_m) \\
&= \sum_{i=n}^{m-1} p(r_i, r_{i+1}) \\
&\leq \sum_{i=n}^{\infty} p(r_i, r_{i+1}) \\
&\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}
$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ entails $\lim_{n,m \to \infty} p(r_n, r_m) = 0$. Hence $\{r_n\}$ is a Cauchy sequence in (M, p). Due to Lemma 2.7, $\{r_n\}$ is a Cauchy sequence in (M, d_p). Since (M, p) is a complete partial metric space, so (M, d_p) is a complete metric space and as a result there exists $v \in M$ such that $\lim_{n \to \infty} d_p(r_n, v) = 0$, moreover, by Lemma 2.7

$$(3.10) \qquad \lim_{n \to \infty} p(v, r_n) = p(v, v) = \lim_{n,m \to \infty} p(r_n, r_m).$$

Since $\lim_{n,m \to \infty} p(r_n, r_m) = 0$, from (3.10) we deduce that

$$(3.11) \qquad p(v, v) = 0 = \lim_{n \to \infty} p(v, r_n).$$

Now from (3.11) it follows that $r_{2n+1} \to v$ and $r_{2n+2} \to v$ as $n \to \infty$ with respect to $\tau(p)$. The continuity of T implies

$$v = \lim_{n \to \infty} r_n = \lim_{n \to \infty} r_{2n+1} = \lim_{n \to \infty} r_{2n+2} = \lim_{n \to \infty} T(r_{2n+1}) = T(\lim_{n \to \infty} r_{2n+1}) = T(v),$$

and from contractive (3.2), we have

$$
\begin{aligned}
\tau + F(p(v, S(v))) &= \tau + F(p(S(v), T(v))) \\
&\leq F(\mathbb{M}(v, v)) = F(p(v, v)).
\end{aligned}
$$

This implies that $p(v, S(v)) = 0$ and due to (p_1), (p_2) we conclude that $v = S(v)$. Thus we have $S(v) = T(v) = v$. Hence (S, T) has a common fixed point v. Now we show that v is the unique common fixed point of S and T. Assume the contrary, that is, there exists $\omega \in M$ such that $v \neq \omega$ and $\omega = T(\omega)$. From the contractive condition (3.2), we have

$$(3.12) \qquad \tau + F(p(S(v), T(\omega))) \leq F(\mathbb{M}(v, \omega)),$$

where

$$\mathbb{M}(\upsilon,\omega) = \max \left\{ \begin{array}{c} p(\upsilon,\omega), \dfrac{p(\upsilon,S(\upsilon))p(\omega,T(\omega))}{1+p(\upsilon,y)}, \\ \dfrac{p(\upsilon,S(\upsilon))p(\omega,T(\omega))}{1+p(S(\upsilon),T(\omega))} \end{array} \right\}.$$

From (3.12), we have

(3.13) $$\tau + F(p(\upsilon,\omega)) \leq F\left(p(\upsilon,\omega)\right),$$

The inequality (3.13), leads to

$$p(\upsilon,\omega) < p(\upsilon,\omega),$$

which is a contradiction. Hence, $\upsilon = \omega$ and υ is a unique common fixed point of a pair (S,T). □

The following example illustrates Theorem 3.3 and shows that condition (3.2) is more general than contractivity condition given by Wardowski ([13]).

Example 3.4. Let $M = [0,1]$ and define $p(r_1, r_2) = \max\{r_1, r_2\}$, then (M,p) is a complete partial metric space. Moreover, define $d(r_1, r_2) = |r_1 - r_2|$, so, (M,d) is a complete metric space. Define the mappings $S, T : M \to M$ as follows:

$$T(r) = \left\{ \begin{array}{ll} \dfrac{r}{5} & \text{if } r \in [0,1); \\ \\ 0 & \text{if } r = 1 \end{array} \right. \quad \text{and } S(r) = \dfrac{3r}{7} \text{ for all } r \in M$$

Clearly, S, T are self mappings. Define the function $F : R^+ \to R$ by $F(r) = \ln(r)$, for all $r \in R^+ > 0$. Let $r_1, r_2 \in M$ such that $p(S(r_1), T(r_2)) > 0$ and suppose that $r_1 \leq r_2$. Then

$$\mathbb{M}(r_1, r_2) = \max \left\{ r_2, \frac{r_1 r_2}{1+r_1}, \frac{r_1 r_2}{1+\max\left\{\frac{3r_1}{7}, \frac{r_2}{5}\right\}} \right\}.$$

Since $\frac{r_1}{1+r_1} < 1$ and $\frac{r_1}{1+\max\left\{\frac{3r_1}{7}, \frac{r_2}{5}\right\}} < 1$, we have that $\mathbb{M}(r_1, r_2) = r_2$. In a similar way, if $r_1 \geq r_2$, we obtain that $\mathbb{M}(r_1, r_2) = r_1$, i.e., $\mathbb{M}(r_1, r_2) = p(r_1, r_2)$. Let $\tau \leq \ln(\frac{7}{3})$. Then

$$\begin{aligned} \tau + (p(S(r_1), T(r_2))) &= \tau + \ln\left(\max\left\{\frac{3r_1}{7}, \frac{r_2}{5}\right\}\right) \\ &\leq \ln(\frac{7}{3}) + \ln\left(\max\left\{\frac{3p(r_1, r_2)}{7}, \frac{p(r_1, r_2)}{5}\right\}\right) \\ &= \ln(\frac{7}{3}) + \ln\left(\frac{3p(r_1, r_2)}{7}\right) = \ln\left(p(r_1, r_2)\right) \\ &= F\left(\mathbb{M}(r_1, r_2)\right). \end{aligned}$$

Thus, the contractive condition (3.2) is satisfied for all $r_1, r_2 \in M$. Hence, all the hypotheses of the Theorem 3.3 are satisfied, note that (S,T) have a unique

common fixed point $r = 0$. As we have seen in Example 2.8, T is not a F-contraction in (M, d) and consequently we can not apply Theorem 2.2.

Corollary 3.5. *Let (M, p) be a complete partial metric space and $T : M \to M$ be a mapping such that*

    (1) *T is a continuous mapping,*
    (2) *T is an improved F-contraction of rational type.*

    Then T has a unique fixed point v in M such that $p(v, v) = 0$.

Proof. Setting $S = T$ in Theorem 3.3, we obtain required result. $\qquad\square$

Remark 3.6. If we set $N(r_1, r_2) = \max\{p(r_1, r_2), p(r_1, T(r_1)), p(r_2, T(r_2))\}$ in inequality (3.1), Corollary 3.5 remains true. Similarly, by setting

$$\mathbb{M}(r_1, r_2) = \max\{p(r_1, r_2), p(r_1, T(r_1)), p(r_2, S(r_2))\}$$

in inequality (3.2), Theorem 3.3 remains true.

4. APPLICATION OF THEOREM 3.3

In this part of paper, we shall apply Theorem 3.3 to show the existence of common solution of the system of Volterra type integral equations. Such system is given by the following equations.

$$(4.1) \qquad u(t) \quad = \quad f(t) + \int_0^t K_n(t, s, u(s))ds,$$

$$(4.2) \qquad w(t) \quad = \quad f(t) + \int_0^t J_n(t, s, w(s))ds,$$

for all $t \in [0, a]$, and $a > 0$. We shall show, by using Theorem 3.3, that the solution of integral equations (4.1) and (4.2) exists. Let $C([0, a], \mathbb{R})$ be the space of all continuous functions defined on $[0, a]$. For $u \in C([0, a], \mathbb{R})$, define sup norm as: $\|u\|_\tau = \sup_{t \in [0, a]} \{u(t)e^{-\tau t}\}$, where $\tau > 0$. Let $C([0, a], \mathbb{R})$ be endowed with the partial metric

$$(4.3) \qquad p_\tau(u, v) = d_\tau(u, v) + c_n = \sup_{t \in [0, a]} \| \, |u(t) - v(t)| \, e^{-\tau t}\|_\tau + c_n$$

for all $u, v \in C([0, a], \mathbb{R})$ and $\{c_n\}$ is a sequence of positive real numbers such that $\lim_{n \to \infty} c_n = 0$. Obviously, $C([0, a], \mathbb{R}, \| \cdot \|_\tau)$ is a Banach space.

Now we prove the following theorem to ensure the existence of solution of system of integral equations.

Theorem 4.1. Assume the following conditions are satisfied.

(i) $K_n, J_n : [0, a] \times [0, a] \times \mathbb{R} \to \mathbb{R}$ and $f, g : [0, a] \to \mathbb{R}$ are continuous.

(ii) Define

$$Su(t) = f(t) + \int_0^t K_n(t, s, u(s))ds,$$

$$Tu(t) = f(t) + \int_0^t J_n(t, s, u(s))ds,$$

and when $n \to \infty$ there exists $\tau \geq 1$ such that

$$|K_n(t, s, u) - J_n(t, s, v)| \leq \tau e^{-\tau}[\mathbb{M}(u, v)]$$

for all $t, s \in [0, a]$ and $u, v \in C([0, a], \mathbb{R})$, where

$$\mathbb{M}(u, v) = \max \left\{ \begin{array}{c} p(u(t), v(t)), \dfrac{p(u(t), Su(t))p(v(t), Tv(t))}{1 + p(u(t), v(t))}, \\ \dfrac{p(u(t), Su(t))p(v(t), Tv(t))}{1 + p(Su(t), Tv(t))} \end{array} \right\}.$$

Then the system of integral equations given in (4.1) and (4.2) has a solution.

Proof. Following assumption (ii), we have

$$
\begin{aligned}
p(Su(t), Tv(t)) &= d(Su(t), Tv(t)) + c_n \\
&= \int_0^t |K_n(t, s, u(s) - J_n(t, s, v(s)))|\, ds + c_n \\
&\leq \int_0^t \tau e^{-\tau}([\mathbb{M}(u, v)]e^{-\tau s})e^{\tau s}ds \ \text{(by taking limit } n \to \infty) \\
&\leq \int_0^t \tau e^{-\tau}\|\mathbb{M}(u, v)\|_\tau e^{\tau s}ds \\
&\leq \tau e^{-\tau}\|\mathbb{M}(u, v)\|_\tau \int_0^t e^{\tau s}ds \\
&\leq \tau e^{-2\tau}\|\mathbb{M}(u, v)\|_\tau \frac{1}{\tau}e^{\tau t} \\
&\leq e^{-\tau}\|\mathbb{M}(u, v)\|_\tau e^{\tau t}.
\end{aligned}
$$

This implies

$$p(Su(t), Tv(t))e^{-\tau t} \leq e^{-\tau}\|\mathbb{M}(u, v)\|_\tau,$$

That is

$$\|p(Su(t), Tv(t))\|_\tau \leq e^{-\tau}\|\mathbb{M}(u, v)\|_\tau,$$

and consequently,

$$\tau + \ln \|p(Su(t), Tv(t))\|_\tau \le \ln \|\mathbb{M}(u,v)\|_\tau.$$

So conditions of Theorem 3.3 are satisfied. Hence the system of integral equations given in (4.1) and (4.2) have a unique common solution. $\quad\square$

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