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Contractive definitions and discontinuity at fixed point

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ABSTRACT

In this paper, we investigate some contractive definitions which are strong enough to generate a fixed point but do not force the mapping to be continuous at the fixed point. We also obtain a fixed point theorem for generalized nonexpansive mappings in metric spaces by employing Meir-Keeler type conditions.

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1. INTRODUCTION

The well-known Banach-Picard-Caccioppoli contraction principle states that:

Theorem 1.1. *If a self-mapping T of a complete metric space (X, d) satisfies the condition; $d(Tx, Ty) \leq ad(x, y), 0 \leq a < 1$, for each $x, y \in X$, then T has a unique fixed point. The Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n, (n = 0, 1, 2, ...)$ converges to x_* for any initial value $x_0 \in X$.*

It is known that the mapping T of Banach-Picard-Caccioppoli contraction is continuous in the entire domain of X.

In an interesting development, Kannan [9] proved the following theorem:

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Theorem 1.2 ([9]). *If a self-mapping T of a complete metric space (X, d) satisfies the condition:*

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], 0 \leq b < 1/2,$$

for each $x, y \in X$, then T has a unique fixed point.

The Kannan fixed point theorem gave rise to the famous question of continuity of contractive mappings at their fixed points. It may be observed that Kannan contractive condition does not require the continuity of the mapping T for the existence of the fixed point. However, a mapping T satisfying Kannan contractive condition turns out to be continuous at the fixed point. To see this, suppose that $z = Tz$ is a fixed point of T and $x_n \to z$. Then

$$d(Tx_n, z) = d(Tx_n, Tz) \leq b[d(x_n, Tx_n) + d(z, Tz)] \leq b[d(x_n, z) + d(z, Tx_n)],$$

that is, $(1 - b)d(Tx_n, z) \leq bd(x_n, z)$. This implies that $Tx_n \to z = Tz$ and T is continuous at the fixed point z.

Kannan's paper generated a far-flung interest in the study of fixed points of generalized contractive mappings and soon these were followed by a flood of papers involving contractive definitions, many of which did not require continuity of the mapping. Also, Kannan contractive condition contained the geometrically elegant idea of defining generalized contractions (generally referred to as contractive definitions in the literature) by replacing $d(x, y)$ in Theorem 1.1 above, by a convex combination of distances between the four points x, y, Tx and Ty. As a result of this, a large number of contractive definitions were soon introduced and studied by various researchers (for various contractive conditions see [3, 4, 15, 16, 17]).

One of the most interesting generalizations of the Banach-Picard-Caccioppoli contraction principle consists of replacing the Lipschitz constant k by some real valued function whose functional values are less than 1. In 1969, Boyd and Wang [2] initiated the work along these lines and proved the following theorem:

Theorem 1.3 ([2]). *Let T be a mapping of a complete metric space (X, d) into itself. Suppose there exists a function ϕ, upper semicontinuous from right from $\mathbb{R}_+$ into itself such that $d(Tx, Ty) \leq \phi(d(x, y))$, for all $x, y \in X$. If $\phi(t) < t$ for each $t > 0$, then T has a unique fixed point.*

Another noteworthy generalizations of both Banach-Picard-Caccioppoli contraction principle and Boyd and Wang fixed point theorem was obtained by Meir and Keeler [12] in 1969. They proved the following theorem:

Theorem 1.4 ([12]). *If a self-mapping T of a complete metric space (X, d) satisfies the condition:*

(i) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*
$\epsilon \leq d(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) < \epsilon$

then T has a unique fixed point.

A mapping satisfying Boyd and Wong or Meir-Keeler type condition is also continuous in the entire domain of X.

The following theorem was established by J. Matkowski [11] in 1975 as a generalization of Meir and Keeler fixed point theorem (see also [6]):

Theorem 1.5 ([11])**.** *If a self-mapping T of a complete metric space (X, d) satisfy the conditions:*

(i) $d(Tx, Ty) < d(x, y)$*, for all* $x, y \in X, x \neq y$*;*
(ii) *for a given* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that* $\epsilon < d(x, y) < \epsilon + \delta$ *implies* $d(Tx, Ty) \leq \epsilon$

then there exists exactly one fixed point of T; moreover, its domain of attraction coincides with the whole of X.

In [8] Jachymski listed some Meir-Keeler type conditions and established relations between them. Further he gave some new Meir-Keeler type conditions ensuring a convergence of the successive approximations (see also [5]).

In a survey paper of contractive definitions, Rhoades [17] compared 250 contractive definitions and showed that majority of the contractive definitions do not require the mapping to be continuous in the entire domain. However, in all the cases the mapping is continuous at the fixed point. He further demonstrated that the contractive definitions force the mapping to be continuous at the fixed point though continuity was neither assumed nor implied by the contractive definitions. The question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point was reiterated by Rhoades in [18] as an existing open problem.

The question of the existence of contractive mappings which are discontinuous at their fixed points was settled in the affirmative by Pant [13]. Recently, Bisht and Pant[1] also gave a contractive definition which does not force the map to be continuity at the fixed point.

In this note we provide more solutions to the open question of the existence of contractive definitions which are strong enough to generate a fixed point but which do not force the mapping to be continuous at the fixed point.

Recall that the set $O(x; T) = \{T^n x : n = 0, 1, 2, ...\}$ is called the orbit of the self-mapping T at the point $x \in X$.

Definition 1.6. A self-mapping T of a metric space (X, d) is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(x; T)$ (for some $x \in X$) $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

It is easy to check that every continuous self-mapping of a metric space is orbitally continuous, but converse need not be true.

2. Main results

In what follows we shall denote

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\};$$
$$N(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), a[d(x, Ty) + d(y, Tx)]/2\}, 0 \leq a < 1.$$

Theorem 2.1. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *$d(Tx, Ty) \leq \phi(N(x, y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*

(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) \leq \epsilon$.*

Suppose T is orbitally continuous. Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $lim_{x\to z} M(x, z) = 0$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = T x_n$ and $q_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \bigcup \{0\}$. Then by (i)

$$\begin{aligned} q_n &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \phi(N(x_{n-1}, x_n)) < N(x_{n-1}, x_n) \\ &= max\{q_n, q_{n-1}\} = q_{n-1}. \end{aligned}$$

Thus $\{q_n\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $q \geq 0$. If possible, suppose $q > 0$. Then there exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$q < q_n < q + \delta(q). \tag{2.1}$$

It follows from (ii) and $q_n < q_{n-1}$ that $q_n \leq q$, for $n \geq k$, which contradicts the above inequality. Thus we have $q = 0$.

We shall show that $\{x_n\}$ is a Cauchy sequence. Fix an $\epsilon > 0$. Without loss of generality, we may assume that $\delta(\epsilon) < \epsilon$. Since $q_n \to 0$, there exists $k \in \mathbb{N}$ such that $q_n < \frac{1}{2}\delta$, for $n \geq k$.

Following Jachymski [7, 8] we shall use induction to show that, for any $n \in \mathbb{N}$,

$$d(x_k, x_{k+n}) < \epsilon + \frac{1}{2}\delta. \tag{2.2}$$

Inequality (2.2) holds for $n = 1$. Assuming (2.2) is true for some n we shall prove it for $n + 1$. By the triangle inequlaity, we have

$$d(x_k, x_{k+n+1}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+n+1}). \tag{2.3}$$

Observe that it suffices to show that

$$d(x_{k+1}, x_{k+n+1}) \leq \epsilon. \tag{2.4}$$

To show it we shall prove that $M(x_k, x_{k+n}) \leq \epsilon + \delta$, where

$$\begin{aligned} M(x_k, x_{k+n}) =& max\{d(x_k, x_{k+n}), d(x_k, Tx_k), d(x_{k+n}, Tx_{k+n}), \\ & [d(x_k, Tx_{k+n}) + d(x_{k+n}, Tx_k)]/2\}. \end{aligned} \tag{2.5}$$

By the induction hypothesis, we get

$$d(x_k, x_{k+n}) < \epsilon + \frac{1}{2}\delta, d(x_k, x_{k+1}) < \frac{1}{2}\delta, d(x_{k+n}, x_{k+n+1}) < \frac{1}{2}\delta. \tag{2.6}$$

Also,

$$\frac{1}{2}[d(x_k, x_{k+n+1}) + d(x_{k+1}, x_{k+n})] \leq \frac{1}{2}[d(x_k, x_{k+n}) + d(x_{k+n+1}, x_{k+n}) + d(x_k, x_{k+1}) + d(x_k, x_{k+n})] < \epsilon + \delta.$$

Thus $M(x_k, x_{k+n}) < \epsilon + \delta$ so by (ii) $d(x_{k+1}, x_{k+n+1}) \leq \epsilon$, completing the induction. Hence (2.2) implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. Orbital continuity of T implies that $lim_{n\to\infty} Tx_n = Tz$. This yields $Tz = z$, that is, z is a fixed point of T. Uniqueness of the fixed point follows from (i).

Now, let T be continuous at the fixed point z and $x_n \to z$. Then $Tx_n \to Tz = z$. Hence

$$\lim_n M(x_n, z) = \lim_n max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz), [d(x_n, Tz) + d(z, Tx_n)]/2\} = 0.$$

On the other hand, if $\lim_{x_n \to z} M(x_n, z) = 0$, then $d(x_n, Tx_n) \to 0$ as $x_n \to z$. This implies that $Tx_n \to z = Tz$, i.e., T is continuous at z. This concludes the theorem. □

In the next theorem, we replace the orbital continuity of the mapping T by continuity condition on T^p, where p is a positive integer ≥ 2.

Theorem 2.2. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^p is continuous and for any $x, y \in X$;*

(i) *$d(Tx, Ty) \leq \phi(N(x, y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*
(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) \leq \epsilon$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $lim_{x\to z} M(x, z) = 0$.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Then following the proof of above theorem we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$ and $T^p x_n \to z$. By continuity of T^p, we have $T^p x_n \to T^p z$. This implies $T^p z = z$. We claim that $Tz = z$. For if $Tz \neq z$, we get

$d(Tz, z) = d(Tz, T^p z) \leq \phi(N(z, T^{p-1} z)) < N(z, T^{p-1} z) = d(T^p z, T^{p-1} z)$;
$d(T^p z, T^{p-1} z) \leq \phi(N(T^{p-1} z, T^{p-2} z)) < N(T^{p-1} z, T^{p-2} z) = d(T^{p-1} z, T^{p-2} z)$;
. . .
$d(T^2 z, Tz) \leq \phi(N(Tz, z)) < N(Tz, z) = d(Tz, z)$,

that is $z = Tz$ and z is a fixed point of T. Uniqueness of the fixed point follows from (i). □

Taking $M(x, y) = N(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), a[d(x, Ty) + d(y, Tx)]/2\}, 0 \leq a < 1$ we now state the following theorems:

Theorem 2.3. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *$d(Tx, Ty) \leq \phi(M(x, y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*
(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) \leq \epsilon$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $lim_{x\to z} M(x, z) = 0$.

Proof. It may be completed on the lines of the proof of Theorem 2.1 above. □

Theorem 2.4. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^p is continuous and for any $x, y \in X$;*

(i) *$d(Tx, Ty) \leq \phi(M(x, y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*
(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) \leq \epsilon$.*

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $lim_{x\to z} M(x, z) = 0$.

Proof. It may be completed on the lines of the proof of Theorem 2.2 above. □

Remark 2.5. The last part of Theorems 2.1 and 2.2 can alternatively be stated as: T is discontinuous at z iff $lim_{x\to z} M(x, z) \neq 0$.

The following example illustrates the above theorems:

Example 2.6. Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = 1 \text{ if } x \in [0, 1],\ T(x) = 0 \text{ if } x \in (1, 2].$$

Then T satisfies the conditions of Theorems 2.1 and 2.2 and has a unique fixed point $x = 1$ at which T is discontinuous. The mapping T satisfies the contractive condition (i) with $\phi(t) = 1$ for $t > 1$ and $\phi(t) = \frac{t}{2}$ for $t \leq 1$. Also, T satisfies condition (ii) with $\delta(\epsilon) = 1$ for $\epsilon \geq 1$ and $\delta(\epsilon) = 1 - \epsilon$ for $\epsilon < 1$. It can also be easily seen that $lim_{x\to 1} M(x, 1) \neq 0$ and T is discontinuous at the fixed point $x = 1$. However, T^p is continuous, since $T^p(x) = 1$ for all $x \in X (p \geq 2)$.

Theorem 2.7. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *$d(Tx, Ty) \leq \phi(N(x, y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*
(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) \leq \epsilon$.*

Then T has a unique fixed point. Moreover, T is continuous at z iff $lim_{x\to z} M(x, z)$ $= 0$.

Proof. Let x_0 be any point in X and let $x \neq Tx$. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Then following the proof of Theorem 2.1, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. We claim that $Tz = z$. For if $Tz \neq z$, we get

$$d(Tz, Tx_n) \leq \phi(max\{d(z, x_n), d(z, Tz), d(x_n, Tx_n), a[d(z, Tx_n)+d(x_n, Tz)]/2\}).$$

On letting $n \to \infty$ this yields, $d(Tz, z) \leq \phi(d(Tz, z)) < d(Tz, z)$, a contradiction. Thus z is a fixed point of T. Uniqueness of the fixed point follows from (i). □

The following theorem shows that power contraction allows the possibility of discontinuity at the fixed point. In the next theorem we denote:

$$M'(x,y) = max\{d(x,y), d(x,T^m x), d(y,T^m y), [d(x,T^m y) + d(y,T^m x)]/2\},$$
$$N'(x,y) = max\{d(x,y), d(x,T^m x), d(y,T^m y), a[d(x,T^m y) + d(y,T^m x)]/2\},$$
$$0 \leq a < 1$$

where $m \in \mathbb{N}$.

Theorem 2.8. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$:*

(i) *$d(T^m x, T^m y) \leq \phi(N'(x,y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*
(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M'(x,y) < \epsilon + \delta$ implies $d(T^m x, T^m y) \leq \epsilon$.*

Then T has a unique fixed point.

Proof. By Theorem 2.7, T^m has a unique fixed point $z \in X$; i.e., $T^m(z) = z$. Then $T(z) = T(T^m(z)) = T^m(T(z))$ and so $T(z)$ is a fixed point of T^m. Since the fixed point of T^m is unique, $Tz = z$. □

Taking $M'(x,y) = N'(x,y) = max\{d(x,y), d(x,T^m x), d(y,T^m y), a[d(x,T^m y)+ d(y,T^m x)]/2\}, 0 \leq a < 1$ we get the following result:

Theorem 2.9. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$:*

(i) *$d(T^m x, T^m y) \leq \phi(M'(x,y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*
(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M'(x,y) < \epsilon + \delta$ implies $d(T^m x, T^m y) \leq \epsilon$.*

Then T has a unique fixed point.

Proof. It may be completed following Theorem 2.7 above. □

Remark 2.10. Theorems 2.1, 2.2, and 2.3 unify and improve the results due to Bisht and Pant [1], Ćirić [5, 6], Jachymski [8], Kuczma et al. [10], Matkowski [11], and Pant [13].

Some consequences of the above proved theorems are the following corollaries which also generalize and extend the results of Jachymski [8], Kuczma et al. [10], Matkowski [11], and Pant [13].

Corollary 2.11. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that:*

(i) $d(Tx, Ty) < N(x, y)$, *for any* $x, y \in X$ *with* $M(x, y) > 0$;
(ii) *for a given* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that* $\epsilon < M(x, y) < \epsilon + \delta$ *implies* $d(Tx, Ty) \leq \epsilon$.

Suppose T is orbitally continuous. Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $lim_{x\to z} M(x, z) = 0$.

Corollary 2.12. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^p is continuous:*

(i) $d(Tx, Ty) < N(x, y)$, *for any* $x, y \in X$ *with* $M(x, y) > 0$;
(ii) *for a given* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that* $\epsilon < M(x, y) < \epsilon + \delta$ *implies* $d(Tx, Ty) \leq \epsilon$.

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $lim_{x\to z} M(x, z) = 0$.

3. Fixed points of nonexpansive mappings

In what follows we shall denote

$$P(x, y) = max\{d(x, y), b[d(x, Tx) + d(y, Ty)]/2, c[d(x, Ty) + d(y, Tx)]/2\}, 0 \leq b, c < 1.$$

Theorem 3.1. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *for a given* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that* $\epsilon < M(x, y) < \epsilon + \delta$ *implies* $d(Tx, Ty) \leq \epsilon$;
(ii) $d(Tx, Ty) \leq P(x, y)$.

Then T has a fixed point, say z, and $T^n x \to z$ for each $x \in X$.

Proof. Let x_0 be any point in X and let $x \neq Tx$. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$. Then following the proof of Theorem 2.1 we can easily prove that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$. Also $Tx_n \to z$. We claim that $Tz = z$. For if $Tz \neq z$, we get

$$d(Tz, Tx_n) \leq$$
$$max\{d(z, x_n), b[d(z, Tz) + d(x_n, Tx_n)]/2, c[d(z, Tx_n) + d(x_n, Tz)]/2\}.$$

On letting $n \to \infty$ this yields, $d(Tz, z) \leq max\{b[d(Tz, z)]/2, c[d(Tz, z)]/2\} < d(Tz, z)$, a contradiction since $0 \leq b, c < 1$. Thus z is a fixed point of T. □

Remark 3.2. Theorem 3.1 also remains true if we replace condition (ii) by the following condition:

(i). $d(Tx, Ty) \leq max\{d(x, y), d(x, Tx), d(y, Ty), b[d(x, Ty)+d(y, Tx)]/2\}, 0 \leq b < 1$.

The following example illustrates Theorem 3.1:

Example 3.3. Let $X = [-1, 1]$ and d be the usual metric on X. Define $T : X \to X$ by

$$T(x) = -|x|x \text{ for each } x.$$

Then T satisfies all the conditions of Theorem 3.1 and has a fixed point $x = 0$. The mapping T satisfies condition (i) with $\delta(\epsilon) = \frac{1}{4}(\sqrt{(2\epsilon)} - \epsilon))$ for $\epsilon < 2$ and $\delta(\epsilon) = \epsilon$ for $\epsilon \geq 2$. However, T does not satisfy the contractive condition $d(Tx, Ty) < max\{d(x, y), [d(x, Tx) + d(y, Ty)]/2, [d(x, Ty) + d(y, Tx)]/2\}$.

It may be observed that there exist a large number of Meir-Keeler type nonexpansive conditions which yield more than one fixed point. The following example illustrates this fact:

Example 3.4. Let $X = [0, 1]$ and d be the usual metric on X. Define $T : X \to X$ by

$$Tx = sgn(x) \text{ (the signum function), i.e., } T0 = 0, Tx = 1 \text{ if } x > 0.$$

Then T has two fixed points $x = 0$ and $x = 1$.

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References

[1] R. K. Bisht and R. P. Pant, A remark on discontinuity at fixed point, J. Math. Anal. Appl. 445 (2017), 1239–1242.

[2] D. W. Boyd and J. S. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (1969), 458–464.

[3] Lj. B. Ćirić, On contraction type mappings, Math. Balkanica 1 (1971), 52–57.

[4] Lj. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45, no. 2 (1974), 267–273.

[5] Lj. B. Ćirić, Fixed points of weakly contraction mappings, Publications de L'ÆInstitut Mathematique 20 (34) (1976), 79–84.

[6] Lj. B. Ćirić, A new fixed-point theorem for contractive mapping, Publications de l'Institut Mathematique 30 (44) (1981), 25–27.

[7] J. Jachymski, Common fixed point theorems for some families of maps, Indian J. Pure Appl. Math. 25 (1994), 925–937.

[8] J. Jachymski, Equivalent conditions and Meir-Keeler type theorems, J. Math. Anal. Appl. 194 (1995), 293–303.

[9] R. Kannan, Some results on fixed points-II, Amer. Math. Mon. 76 (1969) 405-408.

[10] M. Kuczma, B. Choczewski and R. Ger, Iterative Functional Equations, in: Encyclopedia of Mathematics and its Applications, Vol. 32, Cambridge Univ. Press, Cambridge, UK, 1990.

[11] J. Matkowski, Integrable solutions of functional equations, Diss. Math. 127 (1975) 1–68.
[12] A. Meir and E. Keeler, A theorem on contraction mappings, J. Math. Anal. Appl. 28 (1969), 326–329.
[13] R. P. Pant, Discontinuity and fixed points, J. Math. Anal. Appl. 240 (1999), 284–289.
[14] R. P. Pant, Non-expansive mappings and Meir-Keeler type conditions, J. Indian Math. Soc. 71 (2004), 239–244.
[15] M. Pacurar, Iterative Methods for Fixed Point Approximation. Risoprint, Cluj-Napoca, 2010.
[16] S. Reich, Some remarks concerning contraction mappings. Canad. Math. Bull. 14 (1971), 121–124.
[17] B. E. Rhoades, A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257–290.
[18] B. E. Rhoades, Contractive definitions and continuity, Contemporary Mathematics 72 (1988), 233–245.

