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Fixed points of α-Θ-Geraghty type and Θ-Geraghty graphic type contractions

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Abstract

In this paper, by using the concept of the α-Garaghty contraction, we introduce the new notion of the α-Θ-Garaghty type contraction and prove some fixed point results for this contraction in partial metric spaces. Also, we give some examples and applications to illustrate the main results.

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1. Introduction

In 1922, Banach [4] proved a theorem, which is called *Banach's fixed point theorem*, to show the existence of a solution for an integral equation. In fact,

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Banach's fixed point theorem plays an important role in several branches of mathematics and applied sciences because of its importance and usefulness to show the existence and uniqueness of solutions of many kinds of nonlinear problems.

Especially, in 1973, Geraghty [9] generalized Banach's fixed point theorem as follows:

Theorem G. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Suppose that there exists $\beta \in \mathfrak{F}$ such that, for all $x, y \in X$,*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

where $\mathfrak{F}$ denotes the family of all functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the following condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \implies \lim_{n\to\infty} t_n = 0.$$

Then T has a unique fixed point $z \in X$ and $\{T^n x\}$ converges to the point z for each $x \in X$.

Since Geraghty's fixed point theorem, some authors have studied this theorem in several ways (see [11, 23, 21, 8, 25, 7]). On the other hand, in 2012 and 2013, Samet et al. [27] and Hussain et al. [13] introduced the concept of α-admissible mappings in metric spaces and proved some fixed point theorems for these mappings. Subsequently, in 2013, Abdeljawad [1] introduced a pair of α-admissible mappings satisfying new sufficient contractive conditions, which are different from those in [27, 13], and obtained fixed point and common fixed point theorems. Afterward, some authors have obtained fixed point theorems for some kinds of α-admissible mappings (see [27, 8, 13, 24, 2, 3, 10]).

On the other hand, in 2014, Jleli et al. [17] introduced a class Θ of all the functions satisfying the following conditions:

- (Θ1) θ is nondecreasing;
- (Θ2) for any sequence $\{t_n\}$ in $(0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
- (Θ3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$;
- (Θ4) θ is continuous.

Also, they generalized Banach's fixed point theorem in generalized metric spaces (see Branciari [6], sometime, a generalized metric space is called a *Branciari metric space*) as follows:

Theorem JS. *Let (X, d) be a complete generalized metric space and $T : X \to X$ be a mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$d(Tx, Ty) \neq 0 \implies \theta(d(Tx, Ty)) \leq [\theta(d(x, y))]^k$$

for all $x, y \in X$. Then T has a unique fixed point in X.

Also, in 2014, Jleli et al. [16] established a new fixed point theorem, which is an extension of their recent result, Theorem JS. Recently, in 2016, Liu et al. [20] introduced the notion of a Θ-type contraction and a Θ-type Suzuki contraction and established some new fixed point theorems for such kinds of contractions in complete metric spaces.

Motivated by the above results, in this paper, we introduce the notion of an α-Θ-Geraghty type contraction and prove some common fixed point theorems for this contraction in complete partial metric spaces. Moreover, we give some examples and applications to illustrate our main results.

2. Preliminaries

In this section, we give some definitions, examples and fundamental results.

Definition 2.1 ([22]). Let X be a nonempty set and $p : X \times X \to \mathbb{R}^+$ be a mapping satisfying following conditions: for all $x, y, z \in X$,

(PM1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$;
(PM2) $p(x,x) \leq p(x,y)$;
(PM3) $p(x,y) = p(y,x)$;
(PM4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

Then p is called a *partial metric* on X and the pair (X,p) is called a *partial metric space*.

In 1995, Matthews [22] proved that every partial metric p on X induces a metric $d_p : X \times X \to \mathbb{R}^+$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

for all $x, y \in X$. Notice that a metric on a set X is a partial metric d such that $d(x,x) = 0$ for all $x \in X$.

Definition 2.2 ([22]). Let (X,p) be a partial metric space.

1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) is said to be *convergent* to a point $x \in X$ if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.
2. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) is called a *Cauchy sequence* in X if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
3. A partial metric space (X,p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Definition 2.3 ([27]). Let $S : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be two mappings. S is said to be *α-admissible* if

$$\alpha(x,y) \geq 1 \implies \alpha(Sx,Sy) \geq 1$$

for all $x, y \in X$.

Example 2.4 ([19]). Consider $X = [0,\infty)$ and define two mappings $S : X \to X$, $\alpha : X \times X \to [0,\infty)$ by $Sx = 2x$ for all $x, y \in X$ and

$$\alpha(x,y) = \begin{cases} e^{y/x}, & \text{if } x \geq y,\ x \neq 0, \\ 0, & \text{if } x < y. \end{cases}$$

Then S is α-admissible.

Definition 2.5 ([1]). Let $S, T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. The pair (S, T) is said to be *α-admissible* if

$$\alpha(x, y) \geq 1 \implies \alpha(Sx, Ty) \geq 1, \ \ \alpha(Tx, Sy) \geq 1$$

for all $x, y \in X$.

Example 2.6. Let $X = [0, \infty)$ and define the mappings $S, T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = 2x$, $Tx = x^2$ for all $x, y \in X$ and

$$\alpha(x, y) = \begin{cases} e^{xy}, & \text{if } x, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then the pair (S, T) is α-admissible.

Definition 2.7 ([12]). Let $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be two mappings. S is called a *triangular α-admissible mapping* if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Sy) \geq 1$ for all $x, y \in X$;
(T2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$ for all $x, y, z \in X$.

Example 2.8 ([12]). Let $X = \mathbb{R}$ and define the mappings $S : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = \sqrt[3]{x}$ and $\alpha(x, y) = e^{x-y}$ for all $x, y \in X$. Then S is a triangular α-admissible mapping. Indeed, if $\alpha(x, y) = e^{x-y} \geq 1$, then $x \geq y$, which implies $Sx \geq Sy$, that is, $\alpha(Sx, Sy) = e^{Sx-Sy} \geq 1$. Also, if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $x - z \geq 0$ and $z - y \geq 0$, that is, $x - y \geq 0$ and so $\alpha(x, y) = e^{x-y} \geq 1$.

Definition 2.9. [1] Let $S, T : X \times X$ and $\alpha : X \times X \to [0, \infty)$ be three mappings. The pair (S, T) is said to be *triangular α-admissible* if

(T1) $\alpha(x, y) \geq 1$ implies $\alpha(Sx, Ty) \geq 1$ and $\alpha(Tx, Sy) \geq 1$ for all $x, y \in X$;
(T2) $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$ for all $x, y, z \in X$.

Example 2.10. Let $X = \mathbb{R}$ and define the mappings $S, T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ by $Sx = \sqrt{x}$, $Tx = x^2$ and $\alpha(x, y) = e^{xy}$ for all $x, y \in X$. Then the pair (S, T) is triangular α-admissible.

Definition 2.11 ([26]). Let $S : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$ be three mappings. S is called an *α-admissible mapping* with respect to η if

$$\alpha(x, y) \geq \eta(x, y) \implies \alpha(Sx, Sy) \geq \eta(Sx, Xy)$$

for all $x, y \in X$.

Note that, if we take $\eta(x, y) = 1$, then Definition 2.11 reduces to Definition 2.7 (see [27]). Also, if we take $\alpha(x, y) = 1$, then we say that S is an *η-subadmissible mapping*.

Example 2.12. Let $X = [0, \infty)$ and $S : X \to X$ be a mapping defined by $Sx = \frac{x}{2}$ for all $x \in X$. Also, define the mappings $\alpha, \eta : X \times X \to [0, \infty)$ by $\alpha(x, y) = 3$ and $\eta(x, y) = 1$ for all $x, y \in X$. Then S is α-admissible mapping with respect to η.

Lemma 2.13 ([22]).

(1) *A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.*

(2) *A sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to $\tau(d_p)$ if and only if*

$$\lim_{n\to\infty} p(x,x_n) = p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

(3) *If $\lim_{n\to\infty} x_n = v$ such that $p(v,v)=0$, then $\lim_{n\to\infty} p(x_n,y) = p(v,y)$ for all $y \in X$.*

Lemma 2.14 ([18, 7]). *Let (X,d) be a metric space and $S : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ for each $n \geq 0$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \geq 0$ with $n < m$.*

Lemma 2.15 ([1]). *Let (X,d) be a metric space and $S,T : X \to X$ be triangular α-admissible mappings. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,Sx_0) \geq 1$. Define a sequence $\{x_n\}$ in X by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for each $n \geq 0$. Then we have $\alpha(x_n,x_m) \geq 1$ for all $m,n \geq 0$ with $n < m$.*

In the sequel, we denote by $\tilde{\Theta}$ the set of all the functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

$(\Theta1)'$ θ is non-decreasing and continuous;
$(\Theta2)'$ $\inf_{t\in(0,\infty)} \theta(t) = 1$.

Example 2.16. It is obvious that the following functions belong to $\tilde{\Theta}$:

(1) $\theta_1(t) := e^{e^{-\frac{1}{t^p}}}$ for all $p > 0$;
(2) $\theta_2(t) := 1 + t$ for all $t > 0$;
(3) $\theta_3(t) := e^{\sqrt{t}}$ for all $t > 0$;
(4) $\theta_4(t) := 2 - \frac{2}{\pi}\arctan(\frac{1}{t^\alpha})$ for all $0 < \alpha < 1$ and $t > 0$.

3. Main results

In this section, we prove some fixed point theorems for α-Θ-Geraghty type contractions in complete partial metric spaces.

First, we begin with the following definition:

Definition 3.1. Let (X,p) be a partial metric space and $S,T : X \to X$, $\alpha : X \times X \to [0,\infty)$ be three mappings.

(1) The pair (S,T) is called the *modified α-Θ-Geraghty type contraction* if there exist $\theta \in \tilde{\Theta}$, $k \in (0,1)$ and $\beta \in \mathfrak{F}$ such that

$$\alpha(x,y)\theta(p(Sx,Ty)) \leq [\theta(\beta(M(x,y))M(x,y))]^k \tag{3.1}$$

for all $x, y \in X$, where

$$M(x,y) = \max\{p(x,y), p(x,Sx), p(y,Ty)\}.$$

(2) If $S = T$ in (1), then T is called a *generalized* α-Θ-*Geraghty type contraction* if there exist $\theta \in \tilde{\Theta}$, $k \in (0,1)$ and $\beta \in \mathfrak{F}$ such that

$$\alpha(x,y)\theta(p(Tx,Ty)) \leq [\theta(\beta(N(x,y))N(x,y))]^k, \tag{3.2}$$

for all $x, y \in X$, where

$$N(x,y) = \max\{p(x,y), p(x,Tx), p(y,Ty)\}.$$

The following theorem is our main result in this paper:

Theorem 3.2. *Let (X,p) be a complete partial metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping. Suppose that $S,T : X \times X$ are two continuous mappings satisfying the following conditions:*

(i) *the pair (S,T) is the modified α-Θ-Geraghty type contraction;*
(ii) *the pair (S,T) is triangular α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*

Then S and T have a unique common fixed point $z \in X$.

Proof. First, we prove that $M(x,y) = 0$ if and only if $x = y$ is a common fixed point of the mappings S and T. In fact, if $x = y$ is a common fixed point of (S,T), then $Ty = Tx = x = y = Sy = Sx$ and

$$M(x,y) = \max\{p(x,x), p(x,x), p(x,x)\} = p(x,x).$$

From the condition (3.1), it follows that

$$\theta(p(x,x)) = \theta(p(Sx,Ty)) \leq \alpha(x,y)\theta(p(Sx,Ty)) \leq [\theta(\beta(M(x,y))M(x,y))]^k.$$

It is only possible if $p(x,x) = 0$, which implies that $M(x,y) = 0$. Conversely, if $M(x,y) = 0$, then, using (PM1) and (PM2), it is easy to prove that $x = y$ is a fixed point of S and T.

On the other hand, if $M(x,y) > 0$, we construct an iterative sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}$$

for each $n \geq 0$. We observe that, if $x_n = x_{n+1}$, then x_n is a common fixed point of the mappings S and T. So, assume that $x_n \neq x_{n+1}$ for each $n \geq 0$. Since $\alpha(x_0, x_1) \geq 1$ and (S,T) is triangular α-admissible, using Lemma 2.15, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{3.3}$$

for each $n \geq 0$. Thus we have

$$\begin{aligned}\theta(p(x_{2n+1}, x_{2n+2})) &= \theta(p(Sx_{2n}, Tx_{2n+1})) \leq \alpha(x_{2n}, x_{2n+1})\theta(p(Sx_{2n}, Tx_{2n+1})) \\ &\leq [\theta(\beta(M(x_{2n}, x_{2n+1}))M(x_{2n}, x_{2n+1}))]^k\end{aligned} \tag{3.4}$$

for each $n \geq 0$. Now, also, we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, Sx_{2n}), p(x_{2n+1}, Tx_{2n+1})\} \\ &= \max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} \\ &= \max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} \end{aligned}$$

for each $n \geq 0$.

If $M(x_{2n}, x_{2n+1}) = p(x_{2n+1}, x_{2n+2})$ for each $n \geq 0$, then it follows from (3.4) that

$$\theta(p(x_{2n+1}, x_{2n+2})) \leq [\theta(\beta(p(x_{2n+1}, x_{2n+2}))p(x_{2n+1}, x_{2n+2}))]^k,$$

which implies that

$$\ln[\theta(p(x_{2n+1}, x_{2n+2}))] \leq k \ln[\theta(\beta(p(x_{2n+1}, x_{2n+2}))p(x_{2n+1}, x_{2n+2}))].$$

This is a contradiction to $k \in (0, 1)$. Thus we have $M(x_{2n}, x_{2n+1}) = p(x_{2n}, x_{2n+1})$ for each $n \geq 0$ and so it follows from (3.4) that

$$\begin{aligned} \theta(p(x_{2n+1}, x_{2n+2})) &\leq [\theta(\beta(p(x_{2n}, x_{2n+1}))p(x_{2n}, x_{2n+1}))]^k \\ &< [\theta(p(x_{2n}, x_{2n+1}))]^k \\ &< \theta(p(x_{2n}, x_{2n+1})) \end{aligned} \tag{3.5}$$

and so

$$\theta(p(x_{2n+1}, x_{2n+2})) < \theta(p(x_{2n}, x_{2n+1})). \tag{3.6}$$

This implies that

$$\theta(p(x_{n+1}, x_{n+2})) < \theta(p(x_n, x_{n+1})) \tag{3.7}$$

for each $n \geq 0$. Taking $n \to \infty$ in (3.7), we have

$$\theta(p(x_n, x_{n+1})) \to 1. \tag{3.8}$$

Thus, from (Θ2), it follows that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.9}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in X. Suppose that $\{x_n\}$ is not a Cauchy sequence in X, that is, there exists $\varepsilon > 0$, we can find the sequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ such that, for all $k \geq 1$, if $m_k > n_k > k$, then

$$p(x_{m_k}, x_{n_k}) \geq \varepsilon, \quad p(x_{m_k}, x_{n_{k-1}}) < \varepsilon.$$

So, we have

$$\begin{aligned} \varepsilon &\leq p(x_{m_k}, x_{n_k}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) - p(x_{n_{k-1}}, x_{n_{k-1}}) \\ &\leq p(x_{m_k}, x_{n_{k-1}}) + p(x_{n_{k-1}}, x_{n_k}) \\ &< \varepsilon + p(x_{n_{k-1}}, x_{n_k}), \end{aligned}$$

that is,

$$\varepsilon < \varepsilon + p(x_{n_{k-1}}, x_{n_k}).$$

Thus, from (3.9) and the above inequality, it follows that

$$\lim_{k\to\infty} p(x_{m_k}, x_{n_k}) = \varepsilon. \tag{3.10}$$

By the triangle inequality, we have

$$\begin{aligned} p(x_{m_k}, x_{n_k}) &\le p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) - p(x_{m_{k+1}}, x_{m_{k+1}}) \\ &\le p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_k}) \\ &\le p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}) - p(x_{n_{k+1}}, x_{n_{k+1}}) \\ &\le p(x_{m_k}, x_{m_{k+1}}) + p(x_{m_{k+1}}, x_{n_{k+1}}) + p(x_{n_{k+1}}, x_{n_k}) \end{aligned}$$

and

$$\begin{aligned} p(x_{m_{k+1}}, x_{n_{k+1}}) &\le p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) - p(x_{m_k}, x_{m_k}) \\ &\le p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_{k+1}}) \\ &\le p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}) - p(x_{n_k}, x_{n_k}) \\ &\le p(x_{m_{k+1}}, x_{m_k}) + p(x_{m_k}, x_{n_k}) + p(x_{n_k}, x_{n_{k+1}}). \end{aligned}$$

Taking $k \to \infty$, it follows from (3.9) and (3.10) that

$$\lim_{k\to\infty} p(x_{m_{k+1}}, x_{n_{k+1}}) = \varepsilon.$$

By Lemma 2.15, since $\alpha(x_{n_k}, x_{m_{k+1}}) \ge 1$, we obtain

$$\begin{aligned} \theta(p(x_{n_{k+1}}, x_{m_{k+2}})) &= \theta(p(Sx_{n_k}, Tx_{m_{k+1}})) \\ &\le \alpha(x_{n_k}, x_{m_{k+1}})\theta(p(Sx_{n_k}, Tx_{m_{k+1}})) \\ &\le [\theta(\beta(M(x_{n_k}, x_{m_{k+1}}))M(x_{n_k}, x_{m_{k+1}}))]^k \\ &< [\theta(M(x_{n_k}, x_{m_{k+1}}))]^k \\ &< \theta(M(x_{n_k}, x_{m_{k+1}})). \end{aligned}$$

By using (3.8) and taking $k \to \infty$, we conclude that

$$\lim_{k\to\infty} \theta(p(x_{n_k}, x_{m_{k+1}})) = 1$$

and so $\lim_{k\to\infty} p(x_{n_k}, x_{m_{k+1}}) = 0 < \varepsilon$, which is a contradiction. Therefore, we have

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence in (X, p). Since X is complete, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$ and so $x_{2n+1} \to z$ and $x_{2n+2} \to z$. Since S and T are continuous, we have $Tx_{2n+1} \to Tz$ and $Sx_{2n+2} \to Sz$. Hence, from the definition of the sequence $\{x_n\}$, we have $z = Sz$. Similarly, we have $z = Tz$, that is, $Sz = Tz = z$. Therefore, z is a common fixed point of S and T.

Now, we show that z is the unique common fixed point of the mappings S and T. Assume the contrary, that is, there exists $w \in X$ such that $z \ne w$ and $w = Tw$. From (3.1), we have

$$\theta(p(z, w)) \le [\theta(\beta(M(z, w))M(z, w))]^k < [\theta(M(z, w))]^k < \theta(M(z, w)),$$

that is,

$$p(z,w) < M(z,w).$$

But, we have

$$\begin{aligned} M(z,w) &= \max\{p(z,w), p(z,Sz), p(w,Tw)\} \\ &= p(z,w). \end{aligned}$$

This means that $p(z,w) < p(z,w)$, which is a contradiction and so $p(z,w) = 0$. Therefore, z is a unique common fixed point of S and T. This completes the proof. □

In Theorem 3.2, it is possible to remove the continuity of the mappings S and T by replacing the following condition:

(A) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to z \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, z) \geq 1$ for all $k \geq 0$.

Theorem 3.3. *Let (X,p) be a complete partial metric space, $\alpha : X \times X \to [0,\infty)$ be a function. Suppose that $S,T : X \times X$ are two mappings satisfying the following conditions:*

(i) *the pair (S,T) is the modified α-Θ-Geraghty type contraction;*
(ii) *the pair (S,T) is triangular α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*
(iv) (A) *holds.*

Then S and T have a unique common fixed point $z \in X$.

Proof. Following the proof lines of Theorem 3.2, we know that $x_{2n+1} \to z$ and $x_{2n+2} \to z$ as $n \to \infty$.

Now, we show that z is a common fixed point of S and T. Due to the condition (iv), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{2n_k}, z) \geq 1$ for all $k \geq 1$. Using (3.1), we have

$$\begin{aligned} \theta(p(x_{2n_k+1}, Tz)) &= \theta(p(Sx_{2n_k}, Tz)) \\ &\leq \alpha(x_{2n_k}, z)\theta(p(Sx_{2n_k}, Tz)) \\ &\leq [\theta(\beta(M(x_{2n_k}, z))M(x_{2n_k}, z))]^k \end{aligned}$$

and so

$$\theta(p(x_{2n_k+1}, Tz)) \leq [\theta(\beta(M(x_{2n_k}, z))M(x_{2n_k}, z))]^k, \tag{3.11}$$

where

$$M(x_{2n_k}, z) = \max\{p(x_{2n_k}, z), p(x_{2n_k}, Sx_{2n_k}), p(z, Tz)\}.$$

Taking $k \to \infty$, we have

$$\lim_{k\to\infty} M(x_{2n_k}, z) = \max\{p(z,Sz), p(z,Tz)\}. \tag{3.12}$$

Case I. Suppose that $\lim_{k\to\infty} M(x_{2n_k}, z) = p(z, Tz)$ and $p(z, Tz) > 0$. From (3.12), for sufficiently large k, we have $M(x_{2n_k}, z) > 0$, which implies that

$$\beta(M(x_{2n_k}, z)) < M(x_{2n_k}, z)$$

and so

$$[\theta(\beta(M(x_{2n_k}, z))M(x_{2n_k}, z))]^k < [\theta(M(x_{2n_k}, z))]^k < \theta(M(x_{2n_k}, z)).$$

Then we have

$$\theta(p(x_{2n_k+1}, Tz)) < \theta(M(x_{2n_k}, z)),$$

which implies that

$$p(x_{2n_k+1}, Tz) < M(x_{2n_k}, z).$$

Taking $k \to \infty$ in the above inequality, we obtain

$$p(z, Tz) < p(z, Tz),$$

which is a contradiction. So we obtain that $p(z, Tz) = 0$. By (PM1) and (PM2), we have $z = Tz$.

Case II. Suppose that $\lim_{k\to\infty} M(x_{2n_k}, z) = p(z, Sz)$. Similarly, from Case I, we obtain $z = Sz$. Thus, from two cases, we have $z = Tz = Sz$. Therefore, z is a common fixed point of S and T.

□

If $S = T$ and $M(x, y) = \max\{p(x, y), p(x, Sx), p(y, Sy)\}$ in Theorem 3.2 and 3.3, then we have the following corollaries:

Corollary 3.4. *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that $S : X \times X$ is a continuous mapping satisfying the following conditions:*

(i) *S is a generalized α-Θ-Geraghty type contraction;*
(ii) *S is triangular α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*

Then S has a unique fixed point $z \in X$.

Corollary 3.5. *Let (X, p) be a complete partial metric space and $\alpha : X \times X \to [0, \infty)$ be a function. Suppose that $S : X \times X$ is a mapping satisfying the following conditions:*

(i) *S is a generalized α-Θ-Geraghty type contraction;*
(ii) *S is triangular α-admissible;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*
(iv) (A) *holds.*

Then S has a unique fixed point $z \in X$.

If $M(x, y) = \max\{p(x, y), p(x, Sx), p(y, Sy)\}$ and $p(x, x) = 0$ for all $x \in X$ in Theorem 3.2 and 3.3, then we have the following corollaries:

Corollary 3.6. *Let (X,p) be a complete metric space and $\alpha : X \times X \to [0,\infty)$ be a function. Suppose that $S : X \times X$ is a continuous mapping satisfying the following conditions:*

(i) *S is a generalized α-Θ-Geraghty type contraction;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*

Then S has a unique fixed point $z \in X$.

Corollary 3.7. *Let (X,p) be a complete metric space and $\alpha : X \times X \to [0,\infty)$ be a function. Suppose that $S : X \times X$ is a mapping satisfying the following conditions:*

(i) *S is a generalized α-Θ-Geraghty type contraction;*

(ii) *S is triangular α-admissible;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$;*

(iv) (A) *holds.*

Then S has a unique fixed point $z \in X$.

Definition 3.8. Let (X,p) be a partial metric space, $S,T : X \to X$ be two mappings and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions.

(1) The pair (S,T) is called the *modified (α,η)-Θ-Geraghty type contraction* if there exist $\theta \in \tilde{\Theta}$, $k \in (0,1)$ and $\beta \in \mathfrak{F}$ such that

$$\alpha(x,y) \geq \eta(x,y) \implies \theta(p(Sx,Ty)) \leq [\theta(\beta(M(x,y))M(x,y))]^k \tag{3.13}$$

for all $x,y \in X$, where

$$M(x,y) = \max\{p(x,y), p(x,Sx), p(y,Ty)\}.$$

(2) If $S = T$ in (1), then S is called a *generalized (α,η)-Θ-Geraghty type contraction* if there exist $\theta \in \tilde{\Theta}$, $k \in (0,1)$ and $\beta \in \mathfrak{F}$ such that

$$\alpha(x,y) \geq \eta(x,y) \Longrightarrow \theta(p(Sx,Sy)) \leq [\theta(\beta(N(x,y))N(x,y))]^k \tag{3.14}$$

for all $x,y \in X$, where

$$N(x,y) = \max\{p(x,y), p(x,Sx), p(y,Sy)\}.$$

Theorem 3.9. *Let (X,p) be a complete partial metric space and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. Suppose that $S,T : X \times X$ are two continuous mappings satisfying the following conditions:*

(i) *the pair (S,T) is the improved (α,η)-Θ-Geraghty type contraction;*

(ii) *the pair (S,T) is triangular α-admissible with respect to η;*

(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$.*

Then S and T have a unique common fixed point $z \in X$.

Proof. Let $x_1 \in X$ be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Then, iteratively, we can construct a sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1} \tag{3.15}$$

for each $n \geq 0$. By the conditions (ii) and (iii), we have $\alpha(Sx_0, Tx_1) \geq \eta(Sx_0, Tx_1)$ and so $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$, which implies that $\alpha(Sx_1, Tx_2) \geq \eta(Sx_1, Tx_2)$. By induction, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \geq 0$ and so, by (i), we have

$$\begin{aligned} \theta(p(x_{2n+1}, x_{2n+2})) &= \theta(p(Sx_{2n}, Tx_{2n+1})) \\ &\leq \alpha(x_{2n}, x_{2n+1})\theta(p(Sx_{2n}, Tx_{2n+1})) \\ &\leq [\theta(\beta(M(x_{2n}, x_{2n+1}))M(x_{2n}, x_{2n+1}))]^k \end{aligned} \tag{3.16}$$

for all $n \geq 0$. Now, we have

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, Sx_{2n}), p(x_{2n+1}, Tx_{2n+1})\} \\ &= \max\{p(x_{2n}, x_{2n+1}), p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} \\ &= \max\{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

If $M(x_{2n}, x_{2n+1}) = p(x_{2n+1}, x_{2n+2})$ for all $n \geq 0$, then, from (3.16), we have

$$\theta(p(x_{2n+1}, x_{2n+2})) \leq [\theta(\beta(p(x_{2n+1}, x_{2n+2}))p(x_{2n+1}, x_{2n+2}))]^k,$$

which implies that

$$\ln[\theta(p(x_{2n+1}, x_{2n+2}))] \leq k \ln[\theta(\beta(p(x_{2n+1}, x_{2n+2}))p(x_{2n+1}, x_{2n+2}))].$$

This is a contradiction to $k \in (0, 1)$. So, we have $M(x_{2n}, x_{2n+1}) = p(x_{2n}, x_{2n+1})$ for all $n \geq 0$. Thus it follows from (3.4) that

$$\begin{aligned} \theta(p(x_{2n+1}, x_{2n+2})) &\leq [\theta(\beta(p(x_{2n}, x_{2n+1}))p(x_{2n}, x_{2n+1}))]^k \\ &< [\theta(p(x_{2n}, x_{2n+1}))]^k \\ &< \theta(p(x_{2n}, x_{2n+1})) \end{aligned} \tag{3.17}$$

and so

$$\theta(p(x_{2n+1}, x_{2n+2})) < \theta(p(x_{2n}, x_{2n+1})). \tag{3.18}$$

This implies that

$$\theta(p(x_{n+1}, x_{n+2})) < \theta(p(x_n, x_{n+1})) \tag{3.19}$$

for all $n \geq 0$. Taking $n \to \infty$ in (3.19), we have

$$\theta(p(x_n, x_{n+1})) \to 1 \tag{3.20}$$

and so, from (Θ2),

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.21}$$

Therefore, as in the proof lines of Theorem 3.2, we can get the conclusion. This completes the proof. □

It is possible to remove the continuity of the mappings S and T in Theorem 3.9 by replacing the following condition:

(B) If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \geq 0$ and $x_n \to z \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, z) \geq \eta(x_{n_k}, z)$ for all $k \geq 0$.

Theorem 3.10. *Let (X,p) be a complete partial metric space and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. Suppose that $S,T : X \times X$ are two mappings satisfying the following conditions:*

(i) *the pair (S,T) is the modified (α,η)-Θ-Geraghty type contraction;*
(ii) *the pair (S,T) is triangular α-admissible with respect to η;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*
(iv) (B) *holds.*

Then (S,T) has a unique common fixed point $z \in X$.

Proof. Following the proof lines of Theorem 3.3 and 3.9, we can get the conclusion. □

If $S = T$ and $M(x,y) = \max\{p(x,y), p(x,Sx), p(y,Sy)\}$ in Theorem 3.9 and 3.10, then we have the following corollaries:

Corollary 3.11. *Let (X,p) be a complete partial metric space and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. Suppose that $S : X \times X$ is a continuous mapping satisfying the following conditions:*

(i) *S is a generalized (α,η)-Θ-Geraghty type contraction;*
(ii) *S is triangular α-admissible with respect to η;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$.*

Then S has a unique fixed point $z \in X$.

Corollary 3.12. *Let (X,p) be a complete partial metric space and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. Suppose that $S : X \times X$ is a mapping satisfying the following conditions:*

(i) *S is a generalized (α,η)-Θ-Geraghty type contraction;*
(ii) *S is triangular α-admissible with respect to η;*
(iii) *there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$;*
(iv) (B) *holds.*

Then S has a unique fixed point $z \in X$.

If $M(x,y) = \max\{p(x,y), p(x,Sx), p(y,Sy)\}$ and $p(x,x) = 0$ for all $x \in X$ in Theorem 3.9 and 3.10, then we have the following corollaries:

Corollary 3.13. *Let (X,p) be a complete metric space and $\alpha, \eta : X \times X \to [0,\infty)$ be two functions. Suppose that $S : X \times X$ is a continuous mapping satisfying the following conditions:*

(i) *S is a generalized (α,η)-Θ-Geraghty type contraction;*
(ii) *S is triangular α-admissible with respect to η;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$.

Then S *has a unique fixed point* $z \in X$.

Corollary 3.14. *Let* (X, p) *be a complete metric space and* $\alpha, \eta : X \times X \to [0, \infty)$ *be two functions. Suppose that* $S : X \times X$ *is a mapping satisfying the following conditions:*

(i) S *is a generalized* (α, η)-Θ-*Geraghty type contraction;*

(ii) S *is triangular* α-*admissible with respect to* η*;*

(iii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$*;*

(iv) (B) *holds.*

Then S *has a unique fixed point* $z \in X$.

Now, we give an example to illustrate Theorem 3.2 as follows:

Example 3.15. Let $X = \{1, 2, 3\}$ and define a mapping $p : X \times X \to [0, \infty)$ by

$$p(1,2) = p(2,1) = \frac{3}{7}, \ \ p(2,3) = p(3,2) = \frac{4}{7},$$
$$p(1,3) = p(3,1) = \frac{5}{7}, \ \ p(1,1) = \frac{1}{10}, \ \ p(2,2) = \frac{2}{10}, \ \ p(3,3) = \frac{3}{10}.$$

Define a function $\theta : (0, \infty) \to (1, \infty)$ by

$$\theta(x) = 1 + x$$

for all $x \in X$. It is easy to check that p is a partial metric. Define a function $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in X, \\ 0, & \text{otherwise,} \end{cases}$$

define two mappings $S, T : X \to X$ by

$$S(x) = 1, \quad T(1) = T(3) = 1, \ \ T(2) = 3$$

for all $x \in X$ and define a function $\beta : [0, \infty) \to [0, 1)$ by

$$\beta(M(x, y)) = \frac{9}{10}$$

for all $x, y \in X$. Since $\alpha(x, y) = 1$ and $\alpha(Sx, Ty) = 1$ for all $x, y \in X$, the pair (S, T) is α-admissible.

Now, we show that the condition (3.1) holds. If $x = 2$ and $y = 3$, then $\alpha(2, 3) = 1$ and

$$\begin{aligned} M(2,3) &= \max\{p(2,3), p(2, S(2)), p(3, T(3))\} \\ &= \max\{p(2,3), p(2,1)), p(3,1)\} \\ &= \max\left\{\frac{4}{7}, \frac{3}{7}, \frac{5}{7}\right\} \\ &= \frac{5}{7} \end{aligned}$$

and so

$$\alpha(2,3)\theta(p(S(2),T(3)) = 1 \cdot \theta(p(1,1)) = \theta\left(\frac{1}{10}\right) = 1 + \frac{1}{10} = \frac{11}{10}.$$

Now, if we choose $k = \frac{1}{2} \in (0,1)$, then we have

$$\begin{aligned}[\theta(\beta(M(2,3))M(2,3)]^k &= \left[\theta\left(\frac{9}{10}\cdot\frac{5}{7}\right)\right]^{1/2} = \left[\theta\left(\frac{9}{14}\right)\right]^{1/2} \\ &= \left(1 + \frac{9}{14}\right)^{1/2} = \left(\frac{23}{14}\right)^{1/2}.\end{aligned}$$

Therefore, we have

$$\frac{11}{10} = \alpha(2,3)\theta(p(S(2),T(3)) \leq [\theta(\beta(M(2,3))M(2,3)]^k = \left(\frac{23}{14}\right)^{1/2}.$$

Similarly, for other cases, it is easy to check that the condition (3.1) holds. Therefore, all the conditions (i)-(iii) of Theorem 3.2) are satisfied. Further, S and T have a unique common fixed point and 1 is a unique common fixed point of S and T.

4. Applications

Following the results of Jachymski [15], let (X,p) be a partial metric space and Δ denotes the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that $V(G)$ the set of vertices coincides with X and $E(G)$ the set of edges contains all loops. Suppose that G has no parallel edges. Then we can analyze G with the pair $(V(G), E(G))$. If x and y are vertices in G, then a *path* in G from x to y of length l is a sequence $\{x_n\}_{i=0}^{l}$ of $(l+1)$ vertices such that $x_0 = x, x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for each $i = 1, 2, \ldots, l$. A graph G is said to be *connected* if there exists a path between any two vertices.

Definition 4.1 ([15])**.** A mapping $T : X \to X$ is called the *Banach G-contraction* or, simply, *G-contraction* if T preserves edge of G, i.e., for all $x, y \in X$,

$$(x,y) \in E(G) \implies (Tx, Ty) \in E(G) \tag{4.1}$$

and T decreases weights of edges of G in the following way: there exists $\alpha \in (0,1)$ such that, for all $x, y \in X$,

$$(x,y) \in E(G) \implies d(Tx,Ty) \leq \alpha d(x,y). \tag{4.2}$$

Definition 4.2 ([15])**.** A mapping $T : X \times X$ is said to be *G-continuous* if, for any $x \in X$ and a sequence $\{x_n\}$ with $x_n \to x$ as $n \to \infty$, $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ implies $Tx_n \to Tx$ as $n \to \infty$.

Definition 4.3. Let (X,p) be a partial metric space endowed with a graph G and $T : X \to X$ be a self-mapping. T is called the Θ-*Geraghty graphic type contraction* if there exist $\theta \in \tilde{\Theta}$, $k \in (0,1)$ and $\beta \in \mathfrak{F}$ such that

$$\theta(p_G(Tx,Ty)) \leq [\theta(\beta(M(x,y))M(x,y)]^k \tag{4.3}$$

for all $x, y \in X$, where,

$$M(x,y) = \max\{p_G(x,y), p_G(x,Tx), p_G(y,Ty)\}.$$

From Theorem 3.2, we have the following:

Theorem 4.4. *Let (X,p) be a complete partial metric space endowed with a graph G. $T : X \to X$ is self-mapping satisfying the following conditions:*

- (i) $(x,y) \in E(G) \Longrightarrow (Tx,Ty) \in E(G)$ *for all* $(x,y) \in X$;
- (ii) *there exists* $x_0 \in X$ *such that* $(x_0, Tx_0) \in E(G)$;
- (iii) T *is* G*-continuous on* (X,p);
- (iv) T *is* Θ*-Geraghty graphic type contraction.*

Then T has a unique fixed point $z \in X$.

Proof. Define a function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ 0, & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Now, we prove that T is α-admissible. Let $x, y \in X$ such that $\alpha(x,y) \geq 1$. Then, by the definition of α and the condition (i), we have $(x,y) \in E(G)$ and $(Tx,Tx) \in E(G)$. So, we have $\alpha(Tx,Ty) \geq 1$. Therefore, T is α-admissible. From the condition (ii), there exists $x_0 \in X$ such that $(x_0, Tx_0) \in E(G)$, that is, $\alpha(x_0, Tx_0) \geq 1$ and, from the condition (iv), T is Θ-Geraghty graphic type contraction. Since $\alpha(x,y) \geq 1$, we have

$$\alpha(x,y)\theta(p_G(Tx,Ty)) \leq [\theta(\beta(M(x,y))M(x,y)]^k.$$

Thus all the conditions of Theorem 3.2 are satisfied and so T has a unique fixed point in X. This completes the proof. □

Now, we give an example to illustrate Theorem 4.4 as follows:

Example 4.5. Let $x = \{1,2,3\}$ be endowed with the function $p : X \times X \to [0,\infty)$ defined by

$$p(1,2) = p(2,1) = \frac{3}{7},\ \ p(2,3) = p(3,2) = \frac{4}{7},$$

$$p(1,3) = p(3,1) = \frac{1}{7},\ \ p(1,1) = \frac{1}{20},\ \ p(2,2) = \frac{2}{20},\ \ p(3,3) = \frac{3}{20}.$$

Define a function $\theta : (0,\infty) \to (1,\infty)$ by $\theta(x) = 1 + x$ for all $x \in X$. It is easy to check that p is a partial metric. Define a mapping $T : X \to X$ by

$$T(1) = T(3) = 1,\ \ T(2) = 3$$

and define a function $\beta : [0,\infty) \to [0,1)$ by $\beta(M(x,y)) = \frac{9}{10}$ for all $x, y \in X$.

Let G be a direct graph such that $V(G) = X$ and $E(G) = \{(x,y) : x, y \in \{1,2,3\}\}$. It is easy to show that T preserves edges in G and T is G-continuous. Also, there exists $x_0 = 1 \in X$ such that $(1, T1) = (1,1) \in E(G)$. With out loss of generality, let $x, y \in X$ such that $x \neq y$.

Now, we show that the condition (4.3) holds. Consider the following cases:

Case I. If $x = 1$ and $y = 2$, then we have

$$\begin{aligned}
\theta(p(T(1), T(2)) &\leq [\theta(\beta(M(1,2))M(1,2)]^{1/2} \\
\theta(p(1,3)) &\leq \left[\theta\left(\frac{9}{10}\cdot\frac{4}{7}\right)\right]^{1/2} \\
\theta\left(\frac{1}{7}\right) &\leq \left[\theta\left(\frac{18}{35}\right)\right]^{1/2} \\
\frac{8}{7} &\leq \left(\frac{53}{35}\right)^{1/2}.
\end{aligned}$$

Case II. If $x = 2$ and $y = 3$, then we have

$$\begin{aligned}
\theta(p(T(2), T(3)) &\leq [\theta(\beta(M(2,3))M(2,3)]^{1/2} \\
\theta(p(3,1)) &\leq \left[\theta\left(\frac{9}{10}\cdot\frac{4}{7}\right)\right]^{1/2} \\
\theta\left(\frac{1}{7}\right) &\leq \left[\theta\left(\frac{18}{35}\right)\right]^{1/2} \\
\frac{8}{7} &\leq \left(\frac{53}{35}\right)^{1/2}.
\end{aligned}$$

Case III. If $x = 3$ and $y = 1$, then we have

$$\begin{aligned}
\theta(p(T(3), T(1)) &\leq [\theta(\beta(M(3,1))M(3,1)]^{1/2} \\
\theta(p(1,1)) &\leq \left[\theta\left(\frac{9}{10}\cdot\frac{1}{7}\right)\right]^{1/2} \\
\theta\left(\frac{1}{20}\right) &\leq \left[\theta\left(\frac{9}{70}\right)\right]^{1/2} \\
\frac{21}{20} &\leq \left(\frac{79}{70}\right)^{1/2}.
\end{aligned}$$

The following figure represents the graph with all the possible cases. Therefore, all the conditions of Theorem 4.4 are satisfied and $z = 1$ is a fixed point of T.

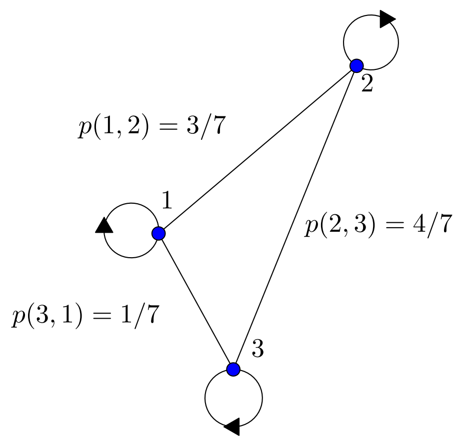

Figure : Graph G defined in Example 4.5

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