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# On quasi-uniform box products

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Abstract

*We revisit the computation of entourage sections of the constant uniformity of the product of countably many copies the Alexandroff one-point compactification called the Fort space. Furthermore, we define the concept of a quasi-uniformity on a product of countably many copies of a quasi-uniform space, where the symmetrised uniformity of our quasi-uniformity coincides with the constant uniformity. We use the concept of Cauchy filter pairs on a quasi-uniform space to discuss the completeness of its quasi-uniform box product.*

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## 1. Introduction

The theory of uniform box products was conveyed for the first time in 2001 by Scott Williams during the ninth Prague International Topological Symposium (Toposym). He proved, for instance, that the box product of equal factors has a compatible complete uniformity whenever its factor does and he showed that the box product of realcompact spaces is realcompact whenever the index set has no subset of measurable cardinality.

Some progress has been made on the concept of uniform box products. For instance in [1] and [3], Bell defined a uniformity on the product of countably many copies of a uniform space which she called the constant uniformity base. It turns out that the topology induced by this uniformity is coarser than the box

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product but finer than the Tychonov product. Her new product was motivated by the idea of the supremum metric on countably many copies of (compact) metric spaces. Moreover, she gave an answer to the question of Scott Williams which asks whether the uniform box product of compact (uniform) spaces is normal. Furthermore, Bell introduced some new ideas on the problem "is the uniform box product of countably many compact spaces collectionwise normal?" that enabled her to prove that the uniform box product of countably many copies of the one-point compactification of a discrete space of cardinality $\aleph_1$ is normal, countably paracompact, and collectionwise Hausdorff in the uniform box topology. In additional to Bell's work, in [8] Hankins modified Bell's proof of the collectionwise Hausdorff property and thereafter, he answered the question"is the uniform box product of denumerably many compact spaces paracompact?"

In this note, we study the concept of a quasi-uniform box product of countably many copies of a quasi-uniform space. We show, for instance, that the quasi-uniformity on a box product of countably many copies of a quasi-uniform space $(X,\mathcal{U})$ is included in the constant uniformity base on the box product of countably many copies of the symmetrized uniform space $(X,\mathcal{U}^s)$ of $(X,\mathcal{U})$. Moreover, we look at the quasi-uniform box product of countably many copies of the one-point compactification of a countable discrete space. We revisit the computation of entourage sections of the constant uniform box product of countably many copies of the Fort space due to Bell [3]. Furthermore, we study $C$-completeness and $D$-completeness in the quasi-uniform box product and in particular, we show that if the factor space of the quasi-uniform box product is quiet, then $C$-completeness implies $D$-completeness in quasi-uniform box products.

## 2. Preliminaries

This section recalls and reviews some well-known results on computation of entourage sections of the constant uniformity of the product of countably many copies of the one-point compactifaction known as the Fort space. For more information on uniform box products we refer the reader to [1, 3, 13].

Let $V$ be an uncountable discrete space, and $X = V \cup \{\infty\}$ be its Fort space, that is, its Alexandroff one-point compactification. Then $(X,\tau)$ is a topological space where $A \in \tau$ if $A \subseteq V$ or if $X \setminus A$ is a finite set and $\infty \in A$. Then the Fort space $X$ can be equipped with the uniformity base $\mathcal{D} = \{D_F : F \subseteq V, F \text{ is finite}\}$ on $X$ compatible with the topology $\tau$ where $D_F = \triangle \cup (X \setminus F)^2$. If $x \in F$, then $D_F(x) = \{x\}$ and if $x \notin F$, then $D_F(x) = X \setminus F \subseteq V$.

Consider the uniform box product $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{D}}\right)$ of the above Fort space where $\overline{\mathcal{D}} = \{\overline{D_F} : D_F \in \mathcal{D}\}$ and

$$\overline{D_F} = \left\{(x,y) \in \prod_{\alpha\in\mathbb{N}} X \times \prod_{\alpha\in\mathbb{N}} X : \text{ for all } \alpha \in \mathbb{N}\ (x(\alpha), y(\alpha)) \in D_F\right\}.$$

It was stated in [1, Remark, p. 2164] that for each $x = (x(\alpha))_{\alpha\in\mathbb{N}} \in \prod_{\alpha\in\mathbb{N}} X$, then

$$\overline{D}_F(x) = \prod_{x(\alpha)\in F} \{x(\alpha)\} \times \prod_{x(\alpha)\notin F} (X \setminus F). \tag{2.1}$$

The above formula has the following explanations: For $x, y \in \prod_{\alpha\in\mathbb{N}} X$, we have that

$$y \in \overline{D}_F(x) \subseteq \prod_{\alpha\in\mathbb{N}} X \quad \text{if and only if} \quad (x, y) \in \overline{D}_F.$$

Furthermore,

$$(x, y) \in \overline{D}_F \quad \text{if and only if} \quad (x(\alpha), y(\alpha)) \in D_F \text{ whenever } \alpha \in \mathbb{N}$$

if and only if $y(\alpha) \in D_F(x(\alpha))$ whenever $\alpha \in \mathbb{N}$.

If $x(\alpha) \in F$ whenever $\alpha \in \mathbb{N}$, then

$$y = (y(\alpha))_{\alpha\in\mathbb{N}} \in \overline{D}_F(x) = \prod_{x(\alpha)\in F} \{x(\alpha)\} \tag{2.2}$$

by Proposition [1, Proposition 3.1].

If $x(\alpha) \notin F$ whenever $\alpha \in \mathbb{N}$, then

$$y = (y(\alpha))_{\alpha\in\mathbb{N}} \in \overline{D}_F(x) = \prod_{x(\alpha)\notin F} (X \setminus F). \tag{2.3}$$

*Remark* 2.1. We point out that the equality in equation (2.1) can be understood to mean "homeomorphic to" under the natural homeomorphism which possibly rearranges the order of factors.

**Example 2.2.** If we equip the Fort space $X$ with the Pervin quasi-uniformity $\mathcal{P}$ with the subbase $\mathcal{S} = \{S_A : A \subseteq V \text{ finite}\}$, where $S_A = [A\times A]\cup[(X\setminus A)\times X]$.

Then $S_A^{-1} = [A \times A] \cup [X \times (X \setminus A)]$ with $A \in \tau$.

Indeed, $(a, b) \in S_A^{-1}$ if and only if $(b, a) \in S_A = [A \times A] \cup [(X \setminus A) \times X]$. Furthermore,

$$S_A \cap S_A^{-1} = \Big([A \times A] \cup [(X \setminus A) \times X]) \cap ([A \times A] \cup [X \times (X \setminus A)]\Big),$$

and thus

$$S_A \cap S_A^{-1} = [A \times A] \cup [(X \setminus A) \times (X \setminus A)].$$

We observe that $S_A \cap S_A^{-1} \supseteq D_A = \triangle \cup (X \setminus A)^2$, and this latter set is an element of the open subbase of the uniformity $\overline{\mathcal{D}}$ on $X$ if $A$ is a finite subset of $V$. For more details on the uniform subbase on $X$, we refer the reader to [1].

It turns out that if $a \in A$, then

$$S_A(a) = \{b \in X : (a, b) \in S_A\} = A.$$

If $a \notin A$, then $S_A(a) = \{b \in X : (a, b) \in S_A\} = \{b \in X\} = X$.

Similarly if $a \in A$, then

$$S_A^{-1}(a) = \{b \in X : (b, a) \in S_A\} = \{b \in X : b \in A \text{ or } b \in X \setminus A\} = X$$

and if $a \in A$, then

$$S_A^{-1}(a) = \{b \in X : (b, a) \in S_A\} = \{b \in X : b \in X \setminus A\} = X \setminus A.$$

*Remark* 2.3. It follows that if $\mathcal{P}$ is the Pervin quasi-uniformity of the Fort space, then $\mathcal{P}^{-1}$ is the conjugate Pervin quasi-uniformity of $\mathcal{P}$ on $X$ with base $\mathcal{S}^{-1}$ where $\mathcal{S}^{-1} = \{S_A^{-1} : A \subseteq V \text{ finite}\}$. Moreover, the uniformity $\mathcal{P}^s = \mathcal{P} \vee \mathcal{P}^{-1}$ is finer than the uniformity subbase $\mathbb{D} = \{D_A : A \subseteq V \text{ finite}\}$ on $X$ (see [1, Proposition 4.2]) compatible with the topology on $X$, where, for any finite subset $A$ of $V$, $D_A = \triangle \cup (X \setminus F) \times (X \setminus F)$.

**Example 2.4.** If we equip the Fort space $X = V \cup \{\infty\}$ with the quasi-uniformity $\mathcal{W}_F$ which has subbase $\{W_F : F \subseteq V,\ F \text{ is finite}\}$, where $W_F = \triangle \cup [X \times (X \setminus F)]$,

we have that

$$W_F \cap W_F^{-1} = (\triangle \cup [X \times (X \setminus F)]) \cap (\triangle \cup [(X \setminus F) \times X]) = \triangle \cup (X \setminus F)^2. \tag{2.4}$$

It follows that if $x \in F$, then

$$W_F(x) = \{y \in X : (x, y) \in W_F\}$$

$$= \{y \in X : x = y \quad \text{or} \quad (x \in X \quad \text{and} \quad y \in X \setminus F)\} = \{x\} \cup (X \setminus F)$$

and

$$W_F^{-1}(x) = \{y \in X : (y, x) \in W_F\}$$

$$= \{y \in X : x = y \quad \text{or} \quad (y \in X \quad \text{and} \quad x \notin F)\} = \{x\}.$$

If $x \notin F$, then we have

$$W_F(x) = \{y \in X : x = y \quad \text{or} \quad (x \in X \quad \text{and} \quad y \in X \backslash F)\} = \{x\} \cup (X \backslash F) = X \backslash F$$

and

$$W_F^{-1}(x) = X.$$

Moreover, it follows that

$$W_F(x) \cap W_F^{-1}(x) = [\{x\} \cup (X \setminus F)] \cap \{x\} = \{x\}$$

whenever $x \in F$. Whenever $x \notin F$, we have

$$W_F(x) \cap W_F^{-1}(x) = X \setminus F \cap X = X \setminus F.$$

*Remark* 2.5. For the Fort space $X = V \cup \{\infty\}$, We observe from (1) that the coarsest uniformity $\mathcal{W}_F^s$ finer than $\mathcal{W}_F$ coincides with the uniformity subbase $\mathbb{D} = \{D_F : F \subseteq V \text{ finite}\}$ on $X$ where $D_F = \triangle \cup (X \setminus F)^2 = W_F^s$. Furthermore, if $F$ is a finite subset of $V$, we have $D_F(x) = W_F(x) \cap W_F^{-1}(x) = W_F^s$ whenever $x \in F$ and whenever $x \notin F$.

## 3. The box product of a quasi-uniform space

In this section we define a quasi-uniformity whose symmetrized quasi-uniformity (uniformity) generates the Tychonov product topology on the product set $\prod_{\alpha\in\mathbb{N}} X$ of countably many copies of a quasi-uniform space $(X,\mathcal{U})$. We also look at a quasi-uniformity whose uniformity generates the box topology on the product set $\prod_{\alpha\in\mathbb{N}} X$.

In [12], Stoltenberg defined the *product topology* on the Cartesian product $\prod_{i\in I} X_i$ of a family $(X_i,\mathcal{U}_i)_{i\in I}$ of quasi-uniform spaces as the topology induced by $\prod_{i\in I}\mathcal{U}_i$, the smallest quasi-uniformity on $\prod_{i\in I} X_i$ such that each projection map $\pi_i : \prod_{i\in I} X_i \to X_i$ is quasi-uniformly continuous. Furthermore, the sets of the form $\{((x_i)_{i\in I},(y_i)_{i\in I}) : (x_i,y_i)\in U_i\}$ whenever $U_i\in\mathcal{U}_i$ and $i\in I$ are sub-base for the quasi-uniformity $\prod_{i\in I}\mathcal{U}_i$. The quasi-uniformity $\prod_{i\in I}\mathcal{U}_i$ is called the *product quasi-uniformity* on $\prod_{i\in I} X_i$.

We are going to omit proof of the following lemma since it is straightforward.

**Lemma 3.1.** *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\prod_{\alpha\in\mathbb{N}} X$ be the product set of countably many copies of $X$. Then $\check{\mathcal{U}}_\alpha = \{\check{U}_\alpha : U\in\mathcal{U} \text{ and } \alpha\in\mathbb{N}\}$ is a filter base generating a quasi-uniformity on $\prod_{\alpha\in\mathbb{N}} X$, where*

$$\check{U}_\alpha = \left\{(x,y)\in\prod_{\beta\in\mathbb{N}} X\times\prod_{\beta\in\mathbb{N}} X : (x(\alpha),y(\alpha))\in U\right\}$$

*whenever $\alpha\in\mathbb{N}$ and $U\in\mathcal{U}$.*

The following has been observed by Bell [3] for uniform spaces.

*Remark* 3.2. Note that $\overline{G}\in\tau(\check{\mathcal{U}}_\alpha)$ if and only if for any $x=(x_\alpha)_{\alpha\in\mathbb{N}}\in\overline{G}$ there exists $\check{U}_\alpha\in\check{\mathcal{U}}_\alpha$ such that $\check{U}_\alpha(x)\subseteq\overline{G}$ whenever $U\in\mathcal{U}$ and $\alpha\in\mathbb{N}$. Thus for any $x,y\in\overline{G}$, we have $(x_\alpha,y_\alpha)\in U$ whenever $U\in\mathcal{U}$ and $\alpha\in\mathbb{N}$. Hence $\overline{G}$ is an open set with respect to the topology induced by the product quasi-uniformity on $\prod_{\alpha\in\mathbb{N}} X$. Observe that the uniformity $(\check{\mathcal{U}}_\alpha)^s$ coincides with the uniformity base on $\prod_{\alpha\in\mathbb{N}} X$ and the topology $\tau((\check{\mathcal{U}}_\alpha)^s)$ induced by the uniformity $(\check{\mathcal{U}}_\alpha)^s$ is the Tychonov product topology on $\prod_{\alpha\in\mathbb{N}} X$.

**Lemma 3.3.** *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\prod_{\alpha\in\mathbb{N}} X$ be the product set of countably many copies of $X$. Then $\widehat{\mathcal{U}}_\psi = \{\widehat{U}_\psi : \psi:\mathbb{N}\to\mathcal{U} \quad \text{is a function}\}$ is a fiter base generating a quasi-uniformity on $\prod_{\alpha\in\mathbb{N}} X$ where*

$$\widehat{U}_\psi = \left\{(x,y)\in\prod_{\alpha\in\mathbb{N}} X\times\prod_{\alpha\in\mathbb{N}} X : \text{ whenever } \alpha\in\mathbb{N}, (x(\alpha),y(\alpha))\in\psi(\alpha)\right\}$$

*whenever $U\in\mathcal{U}$ and $\psi:\mathbb{N}\to\mathcal{U}$ is a function.*

*Remark* 3.4. If $(X,\mathcal{U})$ is a uniform space, then the quasi-uniformity $\widehat{\mathcal{U}}_\psi$ is exactly the uniformity $\mathbb{D}$ in [3, Definition 4.2]. Therefore, for any quasi-uniform space $(X,\mathcal{U})$, the topology $\tau(\widehat{\mathcal{U}}_\psi)^s$ induced by the uniformity base $\widehat{\mathcal{U}}_\psi^s$ is the box topology on $\prod_{\alpha\in\mathbb{N}} X$.

## 4. Quasi-uniform box products

In this section we present the quasi-uniform box product of countably many copies of a quasi-uniform space. The theory of uniform box product of countably many copies of a uniform space was developed by Bell [1]. She proved that the uniform box product has a topology that sits between the Tychonov product and the box product topology.

**Theorem 4.1.** *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\prod_{\alpha\in\mathbb{N}} X$ be the product set of countably many copies of $X$. Then $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$ is a filter base generating a quasi-uniformity on $\prod_{\alpha\in\mathbb{N}} X$ where*

$$\overline{U} = \left\{(x,y) \in \prod_{\alpha\in\mathbb{N}} X \times \prod_{\alpha\in\mathbb{N}} X : \ (x(\alpha), y(\alpha)) \in U \text{ whenever } \alpha \in \mathbb{N}\right\}$$

*whenever $U \in \mathcal{U}$.*

*Proof.* For $U \in \mathcal{U}$ and $x \in \prod_{\alpha\in\mathbb{N}} X$, we have $(x,x) \in \overline{U}$ since for any $\alpha \in \mathbb{N}$, $(x(\alpha), x(\alpha)) \in U$.

Observe that for any $U, V \in \mathcal{U}$ with $U \subseteq V$, it follows that $\overline{U} \subseteq \overline{V}$. Thus $\{\overline{U} : U \in \mathcal{U}\}$ is a filter base on $\prod_{\alpha\in\mathbb{N}} X \times \prod_{\alpha\in\mathbb{N}} X$.

Let $U, V \in \mathcal{U}$ be such that $V^2 \subseteq U$. Suppose that $(x,y) \in \overline{V}^2$. Then there exists $z \in \prod_{\alpha\in\mathbb{N}} X$ such that $(x,z) \in \overline{V}$ and $(z,y) \in \overline{V}$. Hence $(x(\alpha), z(\alpha)) \in V$ and $(z(\alpha), y(\alpha)) \in V$ whenever $\alpha \in \mathbb{N}$.

Moreover, $(x(\alpha), y(\alpha)) \in V^2 \subseteq U$ whenever $\alpha \in \mathbb{N}$. Thus $(x,y) \in \overline{U}$. Therefore, $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$ is a quasi-uniformity on $\prod_{\alpha\in\mathbb{N}} X$. □

Note that if for any given $U \in \mathcal{U}$, the function $\psi$ in Lemma 3.3 is a constant function $\psi(\alpha) = U$ whenever $\alpha \in \mathbb{N}$, then the quasi-uniformity $\widehat{\mathcal{U}}_\psi$ in Lemma 3.3 coincides with the quasi-uniformity $\overline{\mathcal{U}}$ in Theorem 4.1. This is going to motivate the following definition. We point out that this remark was observed by Bell (see [3, p. 15]) for uniform box products.

**Definition 4.2.** Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the quasi-uniformity $\overline{\mathcal{U}}$ is called the *constant quasi-uniformity* on the product $\prod_{\alpha\in\mathbb{N}} X$ and the pair $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is called the *quasi-uniform box product.*

*Remark* 4.3. If a quasi-uniform space $(X,\mathcal{U})$ is such $\mathcal{U} = \mathcal{U}^{-1}$, then $\overline{\mathcal{U}} = \overline{\mathcal{U}}^{-1} = \overline{\mathcal{U}}^{s}$. Therefore, the quasi-uniform box product $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is exactly the constant uniform box product (see [3, Theorem 4.3]).

*Remark* 4.4. For any quasi-uniform box product $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ of a quasi-uniform space $(X,\mathcal{U})$. We have

$$\overline{U}\cap\overline{V}=\overline{U\cap V}$$

whenever $U,V\in\mathcal{U}$.

Indeed $(x,y)\in\overline{U\cap V}$ if and only if $(x(\alpha),y(\alpha))\in U\cap V$ for all $\alpha\in\mathbb{N}$ if and only if $(x(\alpha),y(\alpha))\in U$ and $(x(\alpha),y(\alpha))\in V$ for all $\alpha\in\mathbb{N}$. This is equivalent to $(x,y)\in\overline{U}$ and $(x,y)\in\overline{V}$.

**Lemma 4.5.** *For any quasi-uniform box product $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ of a quasi-uniform space $(X,\mathcal{U})$, the following are true.*

*(1) $\overline{U}^{-1}=\overline{U^{-1}}$ whenever $U\in\mathcal{U}$.*

*(2) $\overline{U^{-1}}\cap\overline{U}=\overline{U^s}=\overline{U}^s$ whenever $U\in\mathcal{U}$.*

*Proof.* We prove (1). Then (2) will follow from (1) and Remark 4.4.
Let $U\in\mathcal{U}$. Then $(x,y)\in\overline{U}^{-1}$ if and only if $(y,x)\in\overline{U}$ if and only if $(y(\alpha),x(\alpha))\in U$ whenever $\alpha\in\mathbb{N}$ if and only if $(x(\alpha),y(\alpha))\in U^{-1}$ whenever $\alpha\in\mathbb{N}$ if and only if $(x,y)\in\overline{U^{-1}}$ □

*Remark* 4.6. If $(X,\mathcal{U})$ is a quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product, then the quasi-uniform space $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}^{-1}}\right)$ is again a quasi-uniform box of $(X,\mathcal{U})$, where $\overline{\mathcal{U}^{-1}}=\{\overline{U^{-1}}:U\in\mathcal{U}\}$ is also a quasi-uniform base on $\prod_{\alpha\in\mathbb{N}} X$. Moreover, $\overline{\mathcal{U}^{-1}}\vee\overline{\mathcal{U}}=\overline{\mathcal{U}}^s$ is a uniformity base on $\prod_{\alpha\in\mathbb{N}} X$ and the pair $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}^s\right)$ is a uniform box product of the uniform space $(X,\mathcal{U}^s)$ which corresponds to the uniform box product in the sense of Bell (see [1, Definition 3.2]).

**Proposition 4.7.** *If $(X,\mathcal{U})$ is a quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is its quasi-uniform box product, then*

$$\overline{U}(x)=\prod_{\alpha\in\mathbb{N}}(U(x(\alpha)))$$

*whenever $U\in\mathcal{U}$ and $x\in\prod_{\alpha\in\mathbb{N}} X$.*

*Proof.* Consider $U\in\mathcal{U}$ and let $y\in\overline{U}(x)$. Then $(x,y)\in\overline{U}$ if and only if $(x(\alpha),y(\alpha))\in U$ whenever $\alpha\in\mathbb{N}$ if and only if $y(\alpha)\in U(x(\alpha))$ whenever $\alpha\in\mathbb{N}$ if and only if $y\in\prod_{\alpha\in\mathbb{N}}(U(x(\alpha)))$. □

**Corollary 4.8.** *If* $(X,\mathcal{U})$ *is a quasi-uniform space and* $\left(\prod_{\alpha\in\mathbb{N}}X,\overline{\mathcal{U}}\right)$ *is its quasi-uniform box product, then*

$$\overline{U}^{-1}(x)=\prod_{\alpha\in\mathbb{N}}(U^{-1}(x(\alpha)))=\overline{U^{-1}}(x)$$

*whenever* $U\in\mathcal{U}$ *and* $x\in\prod_{\alpha\in\mathbb{N}}X$*. Furthermore,*

$$\overline{U}(x)\cap\overline{U}^{-1}(x)=\prod_{\alpha\in\mathbb{N}}(U(x(\alpha)))\cap\prod_{\alpha\in\mathbb{N}}(U^{-1}(x(\alpha)))\supseteq\prod_{\alpha\in\mathbb{N}}(U(x(\alpha))\cap U^{-1}(x(\alpha)))$$

*whenever* $U\in\mathcal{U}$ *and* $x\in\prod_{\alpha\in\mathbb{N}}X$*.*

**Example 4.9.** Let $X$ be the Fort space in Example 2.2 that we equip with its Pervin quasi-uniformity $\mathcal{P}$. Consider the quasi-uniform box product $\left(\prod_{\alpha\in\mathbb{N}}X,\overline{\mathcal{S}}\right)$ of $X$. For any finite subset $A\subseteq V$, if $x\in\prod_{\alpha\in\mathbb{N}}X$, then

$$\overline{S_A}(x)=\left\{y\in\prod_{\alpha\in\mathbb{N}}X:(x(\alpha),y(\alpha))\in S_A \text{ whenever } \alpha\in\mathbb{N}\right\}$$

$$=\left\{y\in\prod_{\alpha\in\mathbb{N}}X:y(\alpha),x(\alpha)\in A \text{ or } x(\alpha)\notin A \text{ and } y(\alpha)\in X \text{ whenever } \alpha\in\mathbb{N}\right\}.$$

Hence

$$\overline{S_A}(x)=\prod_{x(\alpha)\in A}A\times\prod_{x(\alpha)\notin A}X$$

Moreover, if $x\in\prod_{\alpha\in\mathbb{N}}X$, then

$$\overline{S_A{}^{-1}}(x)=\left\{y\in\prod_{\alpha\in\mathbb{N}}X:(y(\alpha),x(\alpha))\in S_A \text{ whenever } \alpha\in\mathbb{N}\right\}$$

$$=\left\{y\in\prod_{\alpha\in\mathbb{N}}X:y(\alpha),x(\alpha)\in A \text{ or } y(\alpha)\in X\backslash A \text{ and } x(\alpha)\in X \text{ whenever } \alpha\in\mathbb{N}\right\}.$$

Hence

$$\overline{S_A{}^{-1}}(x)=\prod_{x(\alpha)\in A}A\times\prod_{x(\alpha)\notin A}(X\setminus A).$$

Therefore

$$\overline{S_A}(x)\cap\overline{S_A{}^{-1}}(x)=\prod_{x(\alpha)\in A}A\times\prod_{x(\alpha)\notin A}(X\setminus A).$$

Observe that

$$\overline{S_A}(x)\cap\overline{S_A{}^{-1}}(x)\supseteq\overline{D_A}(x)=\prod_{x(\alpha)\in A}\{x(\alpha)\}\times\prod_{x(\alpha)\notin A}(X\setminus A),$$

the basic closed and open set of Bell (see [1, p. 2164]).

**Example 4.10.** Consider the quasi-uniformity $\mathcal{W}_F$ on the Fort space, as given in Example 2.4. Let $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{W}_F}\right)$ be the quasi-uniform box product of $X$.

Then whenever $x \in \prod_{\alpha\in\mathbb{N}} X$, we have

$$\overline{W_F}(x) = \prod_{x(\alpha)\in F}\Big(\{x(\alpha)\} \cup X \setminus F\Big) \times \prod_{x(\alpha)\notin F}(X \setminus F)$$

and

$$\overline{W_F^{-1}}(x) = \prod_{x(\alpha)\in F}\{x(\alpha)\} \times \prod_{x(\alpha)\notin F} X.$$

## 5. Properties of filter pairs

In this section, we discuss some properties of filters on quasi-uniform box products. In particular, we prove some properties of filters on a quasi-uniform space that are preserved by filters on their quasi-uniform box products.

We begin by considering one way of defining a filter on a quasi-uniform box product given any filter on its factor space.

**Proposition 5.1.** *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $\mathcal{F}$ is a filter on $(X,\mathcal{U})$, then $\overline{\mathcal{F}}$ defined by $\overline{\mathcal{F}} = \left\{\prod_{\alpha\in\mathbb{N}} F_\alpha : F_\alpha \in \mathcal{F} \text{ and } F_\alpha = X \text{ for all but finitely many } \alpha \in \mathbb{N}\right\}$ is a filter base on $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$.*

In a similar way, we can define a filter on the factor space from any filter on the quasi-uniform box product in the following way:

**Proposition 5.2.** *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $\overline{\mathcal{F}}$ is a filter on $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$, then $\mathcal{F}$, defined by $\mathcal{F} = \left\{F : \prod_{\alpha\in\mathbb{N}} F \in \overline{\mathcal{F}}\right\}$, is a filter on $(X,\mathcal{U})$.*

Suppose $(X,\mathcal{U})$ is a quasi-uniform space and $\mathcal{F}$ and $\mathcal{G}$ are filters on $X$. Then following [10], we say $(\mathcal{F},\mathcal{G})$ is Cauchy filter pair provided that for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. A Cauchy filter pair on a quasi-uniform space $(X,\mathcal{U})$ is called *constant* provided that $\mathcal{F} = \mathcal{G}$.

**Lemma 5.3.** *Let $(X,\mathcal{U})$ be a quasi-uniform space. If $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$, then the filter pair $(\mathcal{F},\mathcal{G})$, where $\mathcal{F} = \left\{F : \prod_{\alpha\in\mathbb{N}} F \in \overline{\mathcal{F}}\right\}$ and $\mathcal{G} = \left\{G : \prod_{\alpha\in\mathbb{N}} G \in \overline{\mathcal{G}}\right\}$, is a Cauchy filter pair on $(X,\mathcal{U})$.*

*Proof.* Consider the Cauchy filter pair $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ on $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$. One sees that $\mathcal{F}$ and $\mathcal{G}$ are filters on $X$ from Proposition 5.2. We need to show that for any $U\in\mathcal{U}$, there are $F\in\mathcal{F}$ and $G\in\mathcal{G}$ such that $F\times G\subseteq U$.

Since $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$, it follows that for any $\overline{U}\in\overline{\mathcal{U}}$, there exists $\overline{F}\in\overline{\mathcal{F}}$ and $\overline{G}\in\overline{\mathcal{G}}$ such that $\overline{F}\times\overline{G}\subseteq\overline{U}$. We choose $F\in\mathcal{F}$ and $G\in\mathcal{G}$ such that $\prod_{\alpha\in\mathbb{N}} F\subseteq\overline{F}$ and $\prod_{\alpha\in\mathbb{N}} G\subseteq\overline{G}$. Let $(x(\alpha),y(\alpha))\in F\times G$ for all $\alpha\in\mathbb{N}$. Then $(x(\alpha))_{\alpha\in\mathbb{N}}\in\overline{F}$ and $(y(\alpha))_{\alpha\in\mathbb{N}}\in\overline{G}$. Thus $((x(\alpha))_{\alpha\in\mathbb{N}},(y(\alpha))_{\alpha\in\mathbb{N}})\in\overline{F}\times\overline{G}\subseteq\overline{U}$. This implies that $((x(\alpha))_{\alpha\in\mathbb{N}},(y(\alpha))_{\alpha\in\mathbb{N}})\in\overline{U}$. Hence for all $\alpha\in\mathbb{N}$, $(x(\alpha),y(\alpha))\in U$. Therefore, $F\times G\subseteq U$. $\square$

A filter $\mathcal{G}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be a *D-Cauchy filter* if there is a filter $\mathcal{F}$ on $X$ such that $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair. We call $\mathcal{F}$ a *cofilter* of $\mathcal{G}$.

**Lemma 5.4.** *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $\mathcal{G}$ is a D-Cauchy filter on $(X,\mathcal{U})$, then the filter $\overline{\mathcal{G}}$, defined by $\overline{\mathcal{G}}=\{\prod_{\alpha\in\mathbb{N}} G : G\in\mathcal{G}\}$, is a D-Cauchy filter on $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$.*

*Proof.* Suppose $\overline{\mathcal{G}}=\{\prod_{\alpha\in\mathbb{N}} G : G\in\mathcal{G}\}$ where $\mathcal{G}$ is a $D$-Cauchy filter on $(X,\mathcal{U})$. Then there exists a filter $\mathcal{F}$ on $X$ such that $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair on $(X,\mathcal{U})$. Define $\overline{\mathcal{F}}$ by $\overline{\mathcal{F}}=\{\prod_{\alpha\in\mathbb{N}} F : F\in\mathcal{F}\}$. Then we need to show that $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$. Suppose $\overline{F}\in\overline{\mathcal{F}}$ and $\overline{G}\in\overline{\mathcal{G}}$. Let $(x(\alpha)_{\alpha\in\mathbb{N}},(y(\alpha))_{\alpha\in\mathbb{N}})\in\overline{F}\times\overline{G}$. Then for all $\alpha\in\mathbb{N}$, $(x(\alpha),y(\alpha))\in F\times G$, where $\overline{F}=\prod_{\alpha\in\mathbb{N}} F$ and $\overline{G}=\prod_{\alpha\in\mathbb{N}} G$. Since $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair, $(x(\alpha),y(\alpha))\in U$ for all $\alpha\in\mathbb{N}$. This implies that $(x(\alpha)_{\alpha\in\mathbb{N}},(y(\alpha))_{\alpha\in\mathbb{N}})\in\overline{U}$ and so $\overline{F}\times\overline{G}\subseteq\overline{U}$. $\square$

## 6. $C$-Completeness and $D$-completness in quasi-uniform box products

In this section, we present some notions of completeness in quasi-uniform spaces that are preserved by their quasi-uniform box products. In particular, we present the notion of $C$-completeness in the quasi-uniform box product of a quasi-uniform space and show the relationship between $D$-completeness and $C$-completeness in the quasi-uniform box product of a quiet quasi-uniform space. Also, since the notion of pair completeness coincides with bicompleteness in quasi-uniform spaces, we show that the quasi-uniform box product of a $D$-complete quiet quasi-uniform space is bicomplete by showing that it is pair complete.

We first consider the notion of quietness in the quasi-uniform box product of a quasi-uniform space. Following [5], we say a quasi-uniform space $(X,\mathcal{U})$ is *quiet* provided that for each $U \in \mathcal{U}$, there is an entourage $V \in \mathcal{U}$ such that if $\mathcal{F}$ and $\mathcal{G}$ are filters on $X$ and $x$ and $y$ are points of $X$ such that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ and $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair on $(X,\mathcal{U})$, then $(x,y) \in U$. If $V$ satisfies the above conditions, we say that $V$ is quiet for $U$.

**Theorem 6.1.** *Let $(X,\mathcal{U})$ be a quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X,\mathcal{U})$ quiet, then $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is quiet.*

*Proof.* Let $\overline{U} \in \overline{\mathcal{U}}$. Suppose there exists $\overline{V} \in \overline{\mathcal{U}}$ such that $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is a Cauchy filter pair on $(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}})$ and $(x(\alpha))_{\alpha\in\mathbb{N}}, (y(\alpha))_{\alpha\in\mathbb{N}} \in \prod_{\alpha\in\mathbb{N}} X$ satisfy $\overline{V}((x(\alpha))_{\alpha\in\mathbb{N}}) \in \overline{\mathcal{G}}$ and $V^{-1}((y(\alpha))_{\alpha\in\mathbb{N}}) \in \overline{\mathcal{F}}$. Then $(\mathcal{F},\mathcal{G})$, where $\mathcal{F} = \{F : \prod_{\alpha\in\mathbb{N}} F \in \overline{\mathcal{F}}\}$ and $\mathcal{G} = \{G : \prod_{\alpha\in\mathbb{N}} G \in \overline{\mathcal{G}}\}$, is a Cauchy filter pair on $(X,\mathcal{U})$ and $V(x(\alpha)) \in \mathcal{G}$ and $V^{-1}(y(\alpha)) \in \mathcal{F}$ whenever $\alpha \in \mathbb{N}$. Since $(X,\mathcal{U})$ is quiet, then $(x(\alpha), y(\alpha)) \in U$ whenever $\alpha \in \mathbb{N}$. Therefore, $((x(\alpha))_{\alpha\in\mathbb{N}}, (y(\alpha))_{\alpha\in\mathbb{N}}) \in \overline{U}$. □

We now look at $C$-completeness and $D$-completeness in quasi-uniform box products. A quasi-uniform space $(X,\mathcal{U})$ is called *$C$-complete* provided that each Cauchy filter pair $(\mathcal{F},\mathcal{G})$ converges. A quasi-uniform space $(X,\mathcal{U})$ is $D$-complete if each $D$-Cauchy filter converges, that is, each second filter of the Cauchy filter pair $(\mathcal{F},\mathcal{G})$ converges with respect to $\tau(\mathcal{U})$.

**Theorem 6.2.** *Let $(X,\mathcal{U})$ be a quiet quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X,\mathcal{U})$ is $C$-complete, then $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is $C$-complete.*

*Proof.* Suppose $(\overline{\mathcal{F}},\overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$. This implies that $(\mathcal{F},\mathcal{G})$, where $\mathcal{F} = \{F : \prod_{\alpha\in\mathbb{N}} F \in \overline{F}\}$ and $\mathcal{G} = \{G : \prod_{\alpha\in\mathbb{N}} G \in \overline{G}\}$, is a Cauchy filter pair on $(X,\mathcal{U})$. Since $(X,\mathcal{U})$ is $C$-complete, then $\mathcal{F}$ converges to $x_0 \in X$ with respect to $\tau(\mathcal{U}^{-1})$. Also, $\mathcal{G}$ converges to $x_0$ (a constant sequence $(x_0, x_0, \cdots)$) with respect to $\tau(\mathcal{U})$. Then for each $U \in \mathcal{U}$, there is $F \in \mathcal{F}$, such that $F \subseteq U^{-1}(x_0)$. Therefore,

$$\prod_{\alpha\in\mathbb{N}} F \subseteq \overline{U}^{-1}((x_0)_{\alpha\in\mathbb{N}}) = \prod_{\alpha\in\mathbb{N}} U^{-1}(x_0)_{\alpha\in\mathbb{N}} \in \overline{\mathcal{F}}.$$

This implies $\overline{\mathcal{F}}$ converges to $(x_0)_{\alpha\in\mathbb{N}}$ with respect to $\tau(\overline{U}^{-1})$. Also, for each $U \in \mathcal{U}$, there is $G \in \mathcal{G}$, such that $G \subseteq U(x_0)$. Therefore,

$$\prod_{\alpha\in\mathbb{N}} G \subseteq \overline{U}((x_0)_{\alpha\in\mathbb{N}}) = \prod_{\alpha\in\mathbb{N}} U(x_0)_{\alpha\in\mathbb{N}} \in \overline{\mathcal{G}}.$$

This implies $\overline{\mathcal{G}}$ converges to $(x_0)_{\alpha\in\mathbb{N}}$ (a constant sequence $(x_0, x_0, \cdots)$) with respect to $\tau(\overline{U}^{-1})$. Therefore, $(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}})$ is $C$-complete. □

**Theorem 6.3.** *Let $(X, \mathcal{U})$ be a quiet quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \mathcal{U})$ is $D$-complete, then $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is $D$-complete.*

*Proof.* Suppose $\overline{\mathcal{G}}$ is a $D$-Cauchy filter on $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$. Then there exists a filter $\overline{\mathcal{F}}$ such that $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$. Thus $(\mathcal{F}, \mathcal{G})$, where $\mathcal{F} = \{F : \prod_{\alpha\in\mathbb{N}} F \in \overline{F}\}$ and $\mathcal{G} = \{G : \prod_{\alpha\in\mathbb{N}} G \in \overline{G}\}$, is a Cauchy filter pair on $(X, \mathcal{U})$. Since $(X, \mathcal{U})$ is $D$-complete, then $\mathcal{G}$ converges to $x_0$ with respect to $\tau(\mathcal{U})$. Then for each $U \in \mathcal{U}$, there is $G \in \mathcal{G}$, such that $G \subseteq U(x_0)$. Therefore,

$$\prod_{\alpha\in\mathbb{N}} G \subseteq \overline{U}((x_0)_{\alpha\in\mathbb{N}}) = \prod_{\alpha\in\mathbb{N}} U(x_0) \in \overline{\mathcal{G}}.$$

This implies $\overline{\mathcal{G}}$ converges to $(x_0)_{\alpha\in\mathbb{N}}$ (a constant sequence $(x_0, x_0, \cdots)$) with respect to $\tau(\overline{U})$. Therefore, $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is $D$-complete. □

*Remark* 6.4. Let $(X, \mathcal{U})$ be a quasi-uniform space. It is not difficult to prove that $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is bicomplete whenever $(X, \mathcal{U})$ is bicomplete. We have seen from our previous results that if $(\overline{\mathcal{F}}, \overline{\mathcal{G}})$ is a Cauchy filter pair on $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$, then $\overline{\mathcal{F}}$ converges with respect to $\tau(\overline{\mathcal{U}^{-1}})$ and $\overline{\mathcal{G}}$ converges with respect to $\tau(\overline{\mathcal{U}})$. Furthermore, one can use the argument that if $(X, \mathcal{U})$ is bicomplete, then $(X, \mathcal{U}^s)$ is complete, therefore $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}^s\right)$ is complete as a uniform box product of $(X, \mathcal{U}^s)$. Hence $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is bicomplete.

We now show the relationship between $C$-completeness and $D$-completeness in the quasi-uniform box product of a uniformly regular quasi-uniform space. Following [10], we say quasi-uniform space $(X, \mathcal{U})$ is uniformly regular if for any $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $cl_{\tau(\mathcal{U})} V(x) \subseteq U(x)$ whenever $x \in X$.

**Lemma 6.5.** *Let $(X, \mathcal{U})$ be a quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. If $(X, \mathcal{U})$ is uniformly regular, then $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is uniformly regular.*

*Proof.* Suppose that $(X,\mathcal{U})$ is uniformly regular. Then for any $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that

$$cl_{\tau(\mathcal{U})} V(t) \subseteq U(t) \text{ whenever } t \in X. \tag{6.1}$$

We need to prove $cl_{\tau(\overline{\mathcal{U}})} \overline{V}(x) \subseteq \overline{U}(x)$ whenever $x \in \prod_{\alpha\in\mathbb{N}} X$.

Let $y \in cl_{\tau(\overline{\mathcal{U}})} \overline{V}(x)$. Then there exists $\overline{W} \in \overline{\mathcal{U}}$ such that $\overline{W}((y(\alpha))_{\alpha\in\mathbb{N}}) \cap \overline{V}((x(\alpha))_{\alpha\in\mathbb{N}}) \neq \varnothing$. This implies $W(y(\alpha)) \cap U(x(\alpha)) \neq \varnothing$ whenever $\alpha \in \mathbb{N}$. Hence $y(\alpha) \in cl_{\tau(\mathcal{U})} V(x(\alpha))$ whenever $\alpha \in \mathbb{N}$.

Furthermore by (6.1), it follows that $y(\alpha) \in U(x(\alpha))$ whenever $\alpha \in \mathbb{N}$.

Therefore, $(y(\alpha))_{\alpha\in\mathbb{N}} \in \overline{U}((x(\alpha))_{\alpha\in\mathbb{N}})$ and this implies that $cl_{\tau(\overline{\mathcal{U}})} \overline{V}(x) \subseteq \overline{U}(x)$. ☐

**Corollary 6.6.** *Let $(X,\mathcal{U})$ be a $D$-complete uniformly regular quiet quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}^{-1}\right)$ is $D$-complete.*

*Proof.* Since $(X,\mathcal{U})$ is $D$-complete and uniformly regular, then by Theorem 6.3 and Lemma 6.5, $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is $D$-complete and uniformly regular. Therefore, by [6, Lemma 2.1], $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}^{-1}\right)$ is $D$-complete. ☐

We recall that a quasi-uniform space is said to be *pair complete* provided that whenever $(\mathcal{F},\mathcal{G})$ is a Cauchy filter pair, there exists a point $p \in X$ such that the filter $\mathcal{G} \underset{\tau(\mathcal{U})}{\longrightarrow} p$ and $\mathcal{F} \underset{\tau(\mathcal{U}^{-1})}{\longrightarrow} p$ (see [6]).

**Corollary 6.7.** *Let $(X,\mathcal{U})$ be a $D$-complete uniformly regular quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is pair complete.*

*Proof.* From Corollary 6.6, we see that $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is $D$-complete and uniformly regular. Then by [6, Proposition 2.2], $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is pair complete. ☐

**Corollary 6.8.** *Let $(X,\mathcal{U})$ be a $D$-complete quiet quasi-uniform space and $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ be its quasi-uniform box product. Then $\left(\prod_{\alpha\in\mathbb{N}} X, \overline{\mathcal{U}}\right)$ is $C$-complete.*

*Proof.* Since $(X,\mathcal{U})$ is quiet and $D$-complete, then by Theorems 6.1 and 6.3, $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is quiet and $D$-complete. Therefore, by Proposition [11, Proposition 3.3.2], $\left(\prod_{\alpha\in\mathbb{N}} X,\overline{\mathcal{U}}\right)$ is $C$-complete. □

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