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Digital fixed points, approximate fixed points, and universal functions

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Abstract

A. Rosenfeld [23] introduced the notion of a digitally continuous function between digital images, and showed that although digital images need not have fixed point properties analogous to those of the Euclidean spaces modeled by the images, there often are approximate fixed point properties of such images. In the current paper, we obtain additional results concerning fixed points and approximate fixed points of digitally continuous functions. Among these are several results concerning the relationship between universal functions and the approximate fixed point property (AFPP).

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1. Introduction

In *digital topology*, we study geometric and topological properties of digital images via tools adapted from geometric and algebraic topology. Prominent among these tools is a digital version of continuous functions. In the current

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paper, we study fixed points and approximate fixed points of digitally continuous functions. We present a number of original results and some corrections of previously published assertions.

The paper is organized as follows. Section 2 reviews background material. In section 3, we show that a digital image X has the Fixed Point Property (FPP) if and only if X has a single point. In section 4 we introduce approximate fixed points and the Approximate Fixed Point Property (AFPP). We give examples of digital images that have, and that don't have, this property. In section 5 we study universal functions on digital images and their relation to the AFPP. In section 6 we correct errors that appeared in earlier papers. Concluding remarks appear in section 7.

2. Preliminaries

2.1. **General Properties.** A *fixed point* of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$.

For a finite set X, we denote by $|X|$ the number of distinct members of X.

Let $\mathbf{N}$ be the set of natural numbers and let $\mathbf{Z}$ denote the set of integers. Then $\mathbf{Z}^n$ is the set of lattice points in Euclidean $n-$dimensional space.

A *digital image* is a pair (X, κ), where $\varnothing \neq X \subset \mathbf{Z}^n$ for some positive integer n and κ is an adjacency relation on X. Technically, then, a digital image (X, κ) is an undirected graph whose vertex set is the set of members of X and whose edge set is the set of unordered pairs $\{x_0, x_1\} \subset X$ such that $x_0 \neq x_1$ and x_0 and x_1 are κ-adjacent.

Adjacency relations commonly used for digital images include the following [22]. Two points p and q in $\mathbf{Z}^2$ are $8-$*adjacent* if they are distinct and differ by at most 1 in each coordinate; p and q in $\mathbf{Z}^2$ are $4-$*adjacent* if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathbf{Z}^3$ are $26-$*adjacent* if they are distinct and differ by at most 1 in each coordinate; they are $18-$*adjacent* if they are 26-adjacent and differ in at most two coordinates; they are $6-$*adjacent* if they are 18-adjacent and differ in exactly one coordinate. For $k \in \{4, 8, 6, 18, 26\}$, a $k-$*neighbor* of a lattice point p is a point that is $k-$adjacent to p.

The adjacencies discussed above are generalized as follows. Let u, n be positive integers, $1 \leq u \leq n$. Distinct points $p, q \in \mathbf{Z}^n$ are called c_u-adjacent if there are at most u distinct coordinates j for which $|p_j - q_j| = 1$, and for all other coordinates j, $p_j = q_j$. The notation c_u represents the number of points $q \in \mathbf{Z}^n$ that are adjacent to a given point $p \in \mathbf{Z}^n$ in this sense. Thus the values mentioned above: if $n = 1$ we have $c_1 = 2$; if $n = 2$ we have $c_1 = 4$ and $c_2 = 8$; if $n = 3$ we have $c_1 = 6$, $c_2 = 18$, and $c_3 = 26$. Yet more general adjacency relations are discussed in [19].

Let κ be an adjacency relation defined on $\mathbf{Z}^n$. A digital image $X \subset \mathbf{Z}^n$ is $\kappa-$*connected* [19] if and only if for every pair of points $\{x, y\} \subset X$, $x \neq y$, there exists a set $\{x_0, x_1, \ldots, x_c\} \subset X$ such that $x = x_0$, $x_c = y$, and x_i and x_{i+1} are $\kappa-$neighbors, $i \in \{0, 1, \ldots, c-1\}$. A κ-*component* of X is a maximal κ-connected subset of X.

Often, we must assume some adjacency relation for the *white pixels* in $\mathbf{Z}^n$, *i.e.*, the pixels of $\mathbf{Z}^n \setminus X$ (the pixels that belong to X are sometimes referred to as *black pixels*). In this paper, we are not concerned with adjacencies between white pixels.

Definition 2.1 ([3])**.** Let $a, b \in \mathbf{Z}$, $a < b$. A digital interval is a set of the form

$$[a, b]_{\mathbf{Z}} \;=\; \{z \in \mathbf{Z} \mid a \leq z \leq b\}$$

in which $2-$adjacency is assumed.

Definition 2.2 ([4]; see also [23])**.** Let $X \subset \mathbf{Z}^{n_0}$, $Y \subset \mathbf{Z}^{n_1}$. Let $f : X \to Y$ be a function. Let κ_i be an adjacency relation defined on $\mathbf{Z}^{n_i}$, $i \in \{0, 1\}$. We say f is $(\kappa_0, \kappa_1)-$continuous if for every κ_0-connected subset A of X, $f(A)$ is a κ_1-connected subset of Y.

See also [11, 12], where similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings*.

If a and b are members of a digital image (X, κ), we write $a \leftrightarrow_\kappa b$, or $a \leftrightarrow b$ when κ is understood, to indicate that either $a = b$ or a and b are κ-adjacent.

We say a function satisfying Definition 2.2 is *digitally continuous*. This definition implies the following.

Proposition 2.3 ([4]; see also [23])**.** *Let X and Y be digital images. Then the function $f : X \to Y$ is (κ_0, κ_1)-continuous if and only if for every $\{x_0, x_1\} \subset X$ such that x_0 and x_1 are κ_0-adjacent, $f(x_0) \leftrightarrow_{\kappa_1} f(x_1)$.*

For example, if κ is an adjacency relation on a digital image Y, then $f : [a, b]_{\mathbf{Z}} \to Y$ is $(2, \kappa)-$continuous if and only if for every $\{c, c+1\} \subset [a, b]_{\mathbf{Z}}$, $f(c) \leftrightarrow_\kappa f(c+1)$.

We have the following.

Proposition 2.4 ([4])**.** *Composition preserves digital continuity,* i.e.*, if $f : X \to Y$ and $g : Y \to Z$ are, respectively, $(\kappa_0, \kappa_1)-$continuous and $(\kappa_1, \kappa_2)-$continuous functions, then the composite function $g \circ f : X \to Z$ is $(\kappa_0, \kappa_2)-$continuous.*

We say digital images (X, κ) and (Y, λ) are $(\kappa, \lambda) -$ *isomorphic* (called $(\kappa, \lambda) -$ *homeomorphic* in [3, 5]) if there is a bijection $h : X \to Y$ that is (κ, λ)-continuous, such that the function $h^{-1} : Y \to X$ is (λ, κ)-continuous.

Classical notions of topology [2] yielded the concept of digital retraction in [3]. Let (X, κ) be a digital image and let A be a nonempty subset of X. A *retraction* of X onto A is a (κ, κ)-continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$.

A *digital simple closed curve* is a digital image $X = \{x_i\}_{i=0}^{m-1}$, with $m \geq 4$, such that the points of X are *labeled circularly*, i.e., x_i and x_j are adjacent if and only if $j = (i-1)\ (mod\ m)$ or $j = (i+1)\ (mod\ m)$.

2.2. **Digital homotopy.** A homotopy between continuous functions may be thought of as a continuous deformation of one of the functions into the other over a time period.

Definition 2.5 ([4]; see also [21])**.** Let X and Y be digital images. Let $f, g : X \to Y$ be (κ, κ')-continuous functions. Suppose there is a positive integer m and a function $F : X \times [0, m]_{\mathbf{Z}} \to Y$ such that

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \to Y$ defined by
$$F_x(t) \;=\; F(x, t) \text{ for all } t \in [0, m]_{\mathbf{Z}}$$
is $(2, \kappa')-$continuous.
- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \to Y$ defined by
$$F_t(x) \;=\; F(x, t) \text{ for all } x \in X$$
is $(\kappa, \kappa')-$continuous.

Then F is a digital $(\kappa, \kappa')-$homotopy between f and g, and f and g are digitally $(\kappa, \kappa')-$homotopic in Y.

When the adjacency relations κ and κ' are understood in context, we say f and g are *digitally homotopic* to abbreviate "digitally $(\kappa, \kappa')-$homotopic in Y."

Definition 2.6. A digital image (X, κ) is κ-contractible [21, 3] if its identity map is (κ, κ)-homotopic to a constant function $\overline{p}$ for some $p \in X$.

When κ is understood, we speak of *contractibility* for short.

2.3. **Digital simplicial homology.** Our presentation of digital simplicial homology is taken from that of [16].

A set of $m + 1$ distinct mutually adjacent points is an *m-simplex*.

Definition 2.7. If α_q is the number of (κ, q)-simplices in X and $m = \max\{q \in \mathbf{N}^* \mid \alpha_q \neq 0\}$, then m is the *dimension of* (X, κ), denoted $dim(X, \kappa)$ or $dim(X)$, and the *Euler characteristic of* (X, κ), $\chi(X, \kappa)$, is defined by
$$\chi(X, \kappa) = \sum_{q=0}^{m} (-1)^q \alpha_q. \;\square$$

For $q \in \mathbf{N}^*$, the group of q-chains of (X, κ), denoted $C_q^\kappa(X)$, is the free Abelian group with basis being the set of q-simplices of X.

Let $\delta_q : C_q^\kappa(X) \to C_{q-1}^\kappa(X)$ defined by
$$\delta_q(< p_0, p_1, \ldots, p_q >) = \begin{cases} \sum_{i=0}^{q} (-1)^i < p_0, p_1, \ldots, \hat{p}_i, \ldots, p_q > & \text{if } 0 \le q \le dim(X); \\ 0 & \text{if } q > dim(X), \end{cases}$$
where $\hat{p}_i$ means that p_i is omitted from the vertices of the simplex considered. Then δ_q is a homomorphism, and we have $\delta_{q-1} \circ \delta_q = 0$ [1]. This gives rise to the following groups [9].

- $Z_q^\kappa(X) = Ker\,\delta_q$, the group of digital simplicial q-*cycles* of X.
- $B_q^\kappa(X) = Im\,\delta_{q+1}$, the group of digital simplicial q-*boundaries* of X.
- The quotient group $H_q^\kappa(X) = Z_q^\kappa(X) \,/\, B_q^\kappa(X)$, the q-th digital simplicial homology group of X.

We have the following.

Theorem 2.8 ([9]). *Let (X, κ) be a directed digital simplicial complex of dimension m.*

- *$H_q(X)$ is a finitely generated abelian group for every $q \geq 0$.*
- *$H_q(X)$ is a trivial group for all $q > m$.*
- *$H_m(X)$ is a free abelian group, possibly $\{0\}$.*

3. Fixed point property

We say a digital image (X, κ) has the *fixed point property (FPP)* if every (κ, κ)-continuous function $f : X \to X$ has a fixed point. Some properties of digital images with the FPP were studied in [14]. However, the following shows that for digital images with c_u-adjacencies, the FPP is not very interesting.

Theorem 3.1. *Let (X, κ) be a digital image. Then (X, κ) has the FPP if and only if $|X| = 1$.*

Proof. Clearly, if $|X| = 1$ then (X, κ) has the FPP.

Now suppose $|X| > 1$. If (X, κ) has more than 1 κ-component, then there is a (κ, κ) continuous map $f : X \to X$ such that for all $x \in X$, x and $f(x)$ are in different κ-components of X. Such a map does not have a fixed point.

Therefore, we may assume X is κ-connected. Since $|X| > 1$, there are distinct κ-adjacent points $x_0, x_1 \in X$. Consider the map $f : X \to X$ given by

$$f(x) = \begin{cases} x_0 & \text{if } x \neq x_0; \\ x_1 & \text{if } x = x_0. \end{cases}$$

Consider a pair y_0, y_1 of κ-adjacent members of X.

- If one of these points, say, y_0, coincides with x_0, we have $f(y_0) = f(x_0) = x_1$ and, since $y_1 \neq x_0$, $f(y_1) = x_0$, so $f(y_0)$ and $f(y_1)$ are κ-adjacent.
- If both y_0 and y_1 are distinct from x_0, then $f(y_0) = x_1 = f(y_1)$

Therefore, f is (κ, κ)-continuous. Clearly, f does not have a fixed point. Therefore, (X, κ) does not have the FPP. $\square$

4. Approximate fixed points

Given a digital image (X, κ) and a (κ, κ)-continuous function $f : X \to X$, we say $p \in X$ is an *approximate fixed point of* f if either $f(p) = p$, or p and $f(p)$ are κ-adjacent. We say a digital image (X, κ) has the *approximate fixed point property (AFPP)* if every (κ, κ)-continuous function $f : X \to X$ has an approximate fixed point.

Theorem 4.1 ([23]). *Let $I = \Pi_{i=1}^{n} [a_i, b_i]_{\mathbf{Z}}$. Then (I, c_n) has the AFPP.*

Theorems 3.1 and 4.1 show that it is worthwhile to consider the AFPP, rather than the FPP, for digital images. We have the following.

Theorem 4.2. *Suppose (X, κ) has the AFPP and there is a (κ, λ)-isomorphism $h : X \to Y$. Then (Y, λ) has the AFPP.*

Proof. Let $f : Y \to Y$ be (λ, λ)-continuous. By Proposition 2.4, the function $g = h^{-1} \circ f \circ h : X \to X$ is (κ, κ) continuous, so our hypothesis implies there exists $p \in X$ such that $p \leftrightarrow_\kappa g(p)$.

Then

$$h(p) \leftrightarrow_\lambda h \circ g(p) \ = \ h \circ h^{-1} \circ f \circ h(p) \ = \ f(h(p)),$$

so $h(p)$ is an approximate fixed point of f. □

Proposition 4.3. *A digital simple closed curve of 4 or more points does not have the AFPP.*

Proof. Let $S = (\{s_i\}_{i=0}^{m-1}, \kappa)$ with s_i and s_j κ-adjacent if and only if $i = (j+1) \, mod \, m$ or $i = (j-1) \, mod \, m$. Then the function $f : S \to S$ defined by $f(s_i) = s_{(i+2) \, mod \, m}$ is (κ, κ)-continuous, and, for each i, s_i and $f(s_i)$ are neither equal nor κ-adjacent. □

Next, we show retractions preserve the AFPP.

Theorem 4.4. *Let (X, κ) be a digital image, and let $Y \subset X$ be a (κ, κ)-retract of X. If (X, κ) has the AFPP, then (Y, κ) has the AFPP.*

Proof. Let $r : X \to Y$ be a (κ, κ) retraction. Let $f : Y \to Y$ be a (κ, κ)-continuous function. Let $i : Y \to X$ be the inclusion map. By Proposition 2.4, $g = i \circ f \circ r : X \to X$ is (κ, κ)-continuous. Therefore, g has an approximate fixed point $x_0 \in X$.

Let $x_1 = g(x_0) \in Y$. By choice of x_0, it follows that $x_0 \leftrightarrow x_1$. Then

$$x_1 \ = \ g(x_0) \leftrightarrow g(x_1) \ = \ i \circ f \circ r(x_1) \ = \ i \circ f(x_1) \ = \ f(x_1).$$

Thus x_1 is an approximate fixed point of f. □

Following a classical construction of topology, the *wedge* of two digital images (A, κ) and (B, λ), denoted $A \wedge B$, is defined [17] as the union of the digital images (A', μ) and (B', μ), where

- $A' \cap B'$ has a single point, p;
- If $a \in A'$ and $b \in B'$ are μ-adjacent, then either $a = p$ or $b = p$;
- (A', μ) and (A, κ) are isomorphic; and
- (B', μ) and (B, λ) are isomorphic.

In practice, we often have $\kappa = \lambda = \mu$, $A = A'$, $B = B'$.

We have the following.

Theorem 4.5. *Let A and B be digital images. Then $(A \wedge B, \kappa)$ has the AFPP if and only if both (A, κ) and (B, κ) have the AFPP.*

Proof. Let $A \cap B = \{p\}$. Let $p_A, p_B : A \wedge B \to A \wedge B$ be the functions

$$p_A(x) = \begin{cases} x & \text{if } x \in A; \\ p & \text{if } x \in B. \end{cases} \qquad p_B(x) = \begin{cases} p & \text{if } x \in A; \\ x & \text{if } x \in B. \end{cases}$$

It is easily seen that both of these functions are well defined and (κ, κ)-continuous. Also, let $i_A : A \to A \wedge B$ and $i_B : B \to A \wedge B$ be the inclusion functions, which are clearly (κ, κ)-continuous.

Suppose (A, κ) and (B, κ) have the AFPP. Let $f : A \wedge B \to A \wedge B$ be (κ, κ)-continuous. We must show that there exists a point of $A \wedge B$ that is equal or κ-adjacent to its image under f. If $f(p) = p$, then we have realized that goal. Otherwise, without loss of generality, $f(p) \in A \setminus \{p\}$. By Proposition 2.4, $h = p_A \circ f \circ i_A : A \to A$ is (κ, κ)-continuous. Since A has the AFPP, there exists $a \in A$ such that

$$h(a) \leftrightarrow_\kappa a. \tag{4.1}$$

If $f(a) \in B$, then

$$p = p_A \circ f(a) = p_A \circ f \circ i_A(a) = h(a).$$

It follows from statement (4.1) that

$$p \leftrightarrow_\kappa a. \tag{4.2}$$

If $f(a) \neq p$ then $f(a) \in B \setminus \{p\}$ and $f(p) \in A \setminus \{p\}$, so $f(a)$ and $f(p)$ are distinct, non-adjacent points. This is a contradiction of statement (4.2), since f is continuous. Therefore, we have $f(a) \in A$. Then

$$f(a) = p_A \circ f(a) = p_A \circ f \circ i_A(a) = h(a).$$

It follows from statement (4.1) that $f(a) \leftrightarrow_\kappa a$.

Since f was arbitrarily selected, it follows that $(A \wedge B, \kappa)$ has the AFPP.

Conversely, suppose $(A \wedge B, \kappa)$ has the AFPP. Since the maps p_A and p_B are (κ, κ)-retractions of $(A \wedge B, \kappa)$ onto (A, κ) and (B, κ), respectively, it follows from Theorem 4.4 that (A, κ) and (B, κ) have the AFPP. ☐

5. Universal functions and the AFPP

In this section, we define the notion of a universal function and study its relation to the AFPP.

Definition 5.1. Let (X, κ) and (Y, λ) be digital images. A (κ, λ)-continuous function $f : X \to Y$ is universal for (X, Y) if given a (κ, λ)-continuous function $g : X \to Y$, there exists $x \in X$ such that $f(x) \leftrightarrow_\lambda g(x)$.

The notion of a dominating set in graph theory corresponds to the notion of a dense set in a topological space.

Definition 5.2 ([10])**.** Let (X, κ) be a nonempty digital image. Let Y be a nonempty subset of X. We say Y is κ-dominating in X if for every $x \in X$ there exists $y \in Y$ such that $x \leftrightarrow_\kappa y$.

Theorem 5.3. *Let (X, κ) and (Y, λ) be digital images. Let $f : X \to Y$ be a universal function for (X, Y). Then $f(X)$ is λ-dominating in Y.*

Proof. Let $y \in Y$ and consider the constant function $c_y : X \to Y$ defined by $c_y(x) = y$ for all $x \in X$. This function is clearly (κ, λ) continuous. Since f is universal, there exists $x_y \in X$ such that $f(x_y)$ is either equal to or λ-adjacent to y. Since y was arbitrarily chosen, the assertion follows. ☐

Proposition 5.4. *Let X be a κ-connected digital image of m points. Let (Y, λ) be a digital interval or a digital simple closed curve of n points, with $n > m+2$. Then there is no universal function from X to Y.*

Proof. Let $f : X \to Y$ be a (κ, λ) continuous function. Then $f(X)$ is a λ-connected subset of Y, and $|f(X)| \leq m < n$.

We show that $Y \setminus f(X)$ has a component with at least 2 points, one of which is not λ-adjacent to any member of $f(X)$.

- If Y is a digital interval $[a, a+n-1]_{\mathbf{Z}}$, then, since $f(X)$ is a connected subset of Y, $f(X) = [u, v]_{\mathbf{Z}}$. Consider the following possibilities.
  - $v \leq a+n-3$. Then the endpoint $a+n-1$ of $Y \setminus f(X)$ is not adjacent to any point of $f(X)$.
  - $v > a+n-3$. Therefore, $v \geq a+n-2$. Then

$$u = v - |f(X)| + 1 \geq a + n - 2 - |f(X)| + 1 \geq a + n - (m+2) + 1 > a + 1.$$

    I.e., $u \geq a+2$, so the point a of $Y \setminus f(X)$ is not adjacent to any point of $f(X)$.
- If Y is a digital simple closed curve, we may assume $Y = \{y_j\}_{j=0}^{n-1}$, where y_a and y_b are adjacent if and only if $a = (b+1)$ *mod* n or $a = (b-1)$ *mod* n. Since $f(X)$ is connected, we may assume without loss of generality that $f(X) = \{y_j\}_{j=0}^{r}$ where $0 \leq r < m < n-2$. Then y_{r+2} is a point of $Y \setminus f(X)$ that is not adjacent to any point of $f(X)$.

Thus, $f(X)$ is not λ-dominating in Y. The assertion follows from Theorem 5.3. $\square$

Proposition 5.5. *Let (X, κ) be a digital image. Then (X, κ) has the AFPP if and only if the identity function 1_X is universal for (X, X).*

Proof. The function 1_X is universal if and only if for every (κ, κ)-continuous $f : X \to X$, there exists $x \in X$ such that $f(x) \leftrightarrow_\kappa 1_X(x) = x$, which is true if and only if (X, κ) has the AFPP. $\square$

Theorem 5.6. *Let (X, κ) and (Y, λ) be digital images and let $U \subset X$. If the restriction function $f|_U : (U, \kappa) \to (Y, \lambda)$ is a universal function for (U, Y), then f is a universal function for (X, Y).*

Proof. Let $h : X \to Y$ be (κ, λ)-continuous. Since $f|_U$ is universal, there exists $u \in U \subset X$ such that $h(u) = h|_U(u) \leftrightarrow_\lambda f|_U(u) = f(u)$. Hence f is universal for (X, Y). $\square$

Theorem 5.7. *Let (W, κ), (X, λ), and (Y, μ) be digital images. Let $f : W \to X$ be (κ, λ)-continuous and let $g : X \to Y$ be (λ, μ)-continuous. If $g \circ f$ is universal, then g is also universal.*

Proof. Let $h : X \to Y$ be (λ, μ)-continuous. Since $g \circ f$ is universal, there exists $w \in W$ such that $(g \circ f)(w) \leftrightarrow_\mu (h \circ f)(w)$. I.e., for $x = f(w) \in X$ we have $g(x) \leftrightarrow_\mu h(x)$. Since h was arbitrarily chosen, the assertion follows. $\square$

Theorem 5.8. *If $g : (U, \mu) \to (X, \kappa)$ and $h : (Y, \lambda) \to (V, \nu)$ are digital isomorphisms and $f : X \to Y$ is (κ, λ)-continuous, then the following are equivalent.*

(1) *f is a universal function for (X, Y).*
(2) *$f \circ g$ is universal.*
(3) *$h \circ f$ is universal.*

Proof. (1 implies 2): Let $k : U \to Y$ be (μ, λ)-continuous. Since f is universal, there exists $x \in X$ such that $(k \circ g^{-1})(x) \leftrightarrow f(x)$. By substituting $x = g(g^{-1}(x))$, we have $k(g^{-1}(x)) \leftrightarrow (f \circ g)(g^{-1}(x))$. Since k was arbitrarily chosen and $g^{-1}(x) \in U$, it follows that $f \circ g$ is universal.

(2 implies 1): This follows from Theorem 5.7.

(1 implies 3): Let $m : X \to V$ be (κ, ν)-continuous. Since f is universal, there exists $x \in X$ such that $(h^{-1} \circ m)(x) \leftrightarrow f(x)$. Then $m(x) = h((h^{-1} \circ m)(x)) \leftrightarrow_\nu (h \circ f)(x)$. Since m was arbitrarily chosen, it follows that $h \circ f$ is universal.

(3 implies 1): Suppose $h \circ f$ is universal. Then given a (κ, λ)-continuous $r : X \to Y$, there exists $x \in X$ such that $h \circ f(x) \leftrightarrow_\nu h \circ r(x)$. Therefore, $f(x) = (h^{-1} \circ h \circ f)(x) \leftrightarrow_\lambda (h^{-1} \circ h \circ r)(x) = r(x)$. Since r was arbitrarily chosen, it follows that f is universal. □

Corollary 5.9. *Let $f : (X, \kappa) \to (Y, \lambda)$ be a digital isomorphism. Then f is universal for (X, Y) if and only if (X, κ) has the AFPP.*

Proof. The function f is universal, by Theorem 5.8, if and only if $f \circ f^{-1} = 1_X$ is universal, which, by Proposition 5.5, is true if and only if (X, κ) has the AFPP. □

It may be useful to remind the reader for the following theorem that points that are c_n-adjacent in $\mathbf{Z}^n$ may differ in every coordinate. Concerning products, we have the following.

Theorem 5.10. *Let $(X_i, c_{n_i}) \subset \mathbf{Z}^{n_i}$, $i = 1, 2, \ldots, m$. Let $s = \sum_{i=1}^m n_i$. Consider the digital image $X = \Pi_{i=1}^m X_i \subset \mathbf{Z}^s$. If (X, c_s) has the AFPP then each (X_i, c_{n_i}) has the AFPP.*

Proof. Suppose (X, c_s) has the AFPP. Let $f_i : X_i \to X_i$ be (c_{n_i}, c_{n_i})-continuous. Then the function $f : X \to X$ defined by

$$f(x_1, x_2, \ldots, x_m) = (f_1(x_1), f_2(x_2), \ldots, f_m(x_m))$$

is (c_s, c_s)-continuous. By Proposition 5.5, 1_X is universal for (X, X). Therefore, there is a point $x_* = (x_{1,*}, x_{2,*}, \ldots, x_{m,*}) \in X$ with $x_{i,*} \in X_i$ such that $x_* \leftrightarrow_{c_s} f(x_*)$. Therefore, $x_{i,*} \leftrightarrow_{c_{n_i}} f_i(x_{i,*})$ for all i. Since f_i was arbitrarily chosen, it follows that (X_i, c_{n_i}) has the AFPP. □

6. Corrections of published assertions

In this section, we correct some assertions that appear in [14, 16].

We show below that the function $F : [0,1]_{\mathbf{Z}} \to [0,1]_{\mathbf{Z}}$ defined by $F(x) = 1-x$ (i.e., $F(0) = 1$, $F(1) = 0$) provides a counterexample to several of the assertions of [14]. Clearly this function is $(2,2)$-continuous and does not have a fixed point.

We will need the following.

Definition 6.1 ([15])**.** Let (X, κ) be a digital image whose digital homology groups are finitely generated and vanish above some dimension n. Let $f : X \to X$ be a (κ, κ)-continuous map. The Lefschetz number of f, denoted $\lambda(f)$, is defined as

$$\lambda(f) = \sum_{i=0}^{n} (-1)^i \, tr(f_{i,*}),$$

where $f_{i,*} : H_i^{\kappa}(X) \to H_i^{\kappa}(X)$ is the map induced by f on the i^{th} homology group of (X, κ) and $tr(f_{i,*})$ is the trace of $f_{i,*}$.

In studying digital maps from a sphere to itself, there is a question of how to represent a Euclidean sphere digitally.

- As in [16], we will represent S^1 by the set $S_1 = [-1,1]_{\mathbf{Z}}^2 \setminus \{(0,0)\} \subset \mathbf{Z}^2$ and c_1-adjacency with points $\{x_j\}_{j=0}^{7}$ labeled circularly.
- More generally, as in [16], we will represent S^n by the set $S_n = [-1,1]_{\mathbf{Z}}^{n+1} \setminus \{0_{n+1}\} \subset \mathbf{Z}^{n+1}$ and c_1-adjacency, where 0_{n+1} is the origin in $\mathbf{Z}^{n+1}$.

Definition 6.2 ([16, 24])**.** Suppose a continuous function $f : (S_n, \kappa) \to (S_n, \kappa)$ induces a homomorphism on the n-th homology group, $f_* : H_n^{\kappa}(S_n) \to H_n^{\kappa}(S_n)$, such that $f_*([x]) = m[x]$ for some fixed $m \in \mathbf{Z}$, where $[x]$ is a generator of $H_n^{\kappa}(S_n)$. The value of m is the *degree of* f.

Theorem 6.3 ([8])**.** *Let S be a digital simple closed curve. For an isomorphism of S and a continuous non-surjective self-map of S to be homotopic, we must have $|S| = 4$.*

We state the following corrections.

- Incorrect assertion stated as Theorem 3.3 of [14]: If (X, κ) is a finite digital image and $f : X \to X$ is a (κ, κ)-continuous function with $\lambda(f) \neq 0$, then f has a fixed point.

  In fact, the function F defined above is a counterexample to this assertion, since it is easily seen that $\lambda(F) \neq 0$.
- Incorrect assertion stated as Theorem 3.4 of [14]: Every (c_1, c_1)-continuous function $f : [0,1]_Z \to [0,1]_Z$ has a fixed point.

  In fact, [23] shows that this assertion is false, and the function F defined above is a counterexample.
- Incorrect assertion stated as Theorem 3.5 of [14]: Let $X = \{(0,0), (1,0), (0,1), (1,1)\} \subset Z^2$. Then every (c_1, c_1)-continuous function $f : X \to X$ has a fixed point.

In fact, we can use the function F above to obtain a counterexample. Let $G : X \to X$ be defined by $G(x,y) = (x, F(y))$. Then G is (c_1, c_1)-continuous and has no fixed point. Alternately, it follows from Theorem 3.1 that the assertion is incorrect.

- Incorrect assertion stated as Theorem 3.8 of [14]: Let (X,κ) be a κ-contractible digital image. Then every (κ,κ)-continuous map $f : X \to X$ has a fixed point.

  In fact, since $[0,1]_Z$ is c_1-contractible, the function F above provides a counterexample to this assertion. Alternately, it follows from Theorem 3.1 that the assertion is incorrect.
- Incorrect assertion stated as Example 3.9 of [14]: Let $X = \{(0,0), (0,1), (1,1)\} \subset \mathbf{Z}^2$. Then (X, c_2) has the FPP.

  In fact, the map $f : X \to X$ defined by $f(0,0) = (0,1)$, $f(0,1) = (1,1)$, $f(1,1) = (0,0)$ is (c_2, c_2)-continuous and has no fixed points. Alternately, it follows from Theorem 3.1 that the assertion is incorrect.
- Incorrect assertion stated as Corollary 3.10 of [14]: Any digital image with the same digital homology groups as a single point image has the FPP.

  To show this assertion is incorrect, observe that if $X = \{(0,0), (0,1), (1,1)\} \subset \mathbf{Z}^2$ and Y is a digital image of one point in $\mathbf{Z}^2$, then [1, 9]

$$H_q^8(X) = H_q^8(Y) = \begin{cases} \mathbf{Z} & \text{if } q = 0; \\ 0 & \text{if } q \neq 0. \end{cases}$$

  It follows from Theorem 3.1 that the assertion is incorrect.
- Incorrect assertions stated as Example 3.17 and Corollary 3.18 of [14]: The digital images $(MSS_6', 6)$ and $(P^2, 6)$, each with more than one point, have the FPP.

  It follows from Theorem 3.1 that these assertions are incorrect.
- Incorrect assertion stated as Theorem 3.5 of [16]: If (X,κ) is a finite digital image and $f : X \to X$ is a (κ,κ)-continuous function with $\lambda(f) \neq 0$, then any map homotopic to f has a fixed point.

  In fact, we observed above that the function F, which is homotopic to itself and has $\lambda(F) \neq 0$, does not have a fixed point.
- Incorrect assertion stated as Theorem 3.7 of [16]: If (X,κ) is a digital image such that $\chi(X,\kappa) \neq 0$, then any map homotopic to the identity has a fixed point.

  In fact, we can take $X = [0,1]_{\mathbf{Z}}$, for which $\chi(X, c_1) = (-1)^1(2) + (-1)^2(1) \neq 0$, and the function F discussed above is homotopic to 1_X and does not have a fixed point.
- Incorrect assertion stated as Theorem 3.11 of [16]: Let $(S_n, c_1) \subset \mathbf{Z}^{n+1}$ be a digital n-sphere as described above, where $n \in \{1,2\}$. If $f : S_n \to S_n$ is a continuous map of degree $m \neq 1$, then f has a fixed point.

  In fact, we have the following. Elementary calculations show that $H_1^{c_1}(S_1) \approx \mathbf{Z}$; also, $H_1^{c_1}(S_2) \approx \mathbf{Z}^{23}$ [13]. For $n \in \{1,2\}$, as in the proof of Theorem 3.1, we can choose distinct and adjacent x_0 and x_1

in S_n and let $f : S_n \to S_n$ be given by $f(x) = x_0$ for $x \neq x_0$ and $f(x_0) = x_1$. Clearly, f is continuous and does not have a fixed point. Since $f_* : H_1(S_n) \to H_1(S_n)$ is 0, the degree of f is 0.

- Proposition 3.12 of [16] depends on an unstated assumption that (recall Definition 2.7) $\alpha_q^\kappa(X)$ is finite for all q, a condition that is satisfied if and only if X is finite; after all, one can study infinite digital images (X, κ), as in [7], for which, e.g., $\alpha_0^\kappa(X) = \infty$. E.g., we could take $X = \mathbf{Z}$; according to Definition 2.7, $\chi(\mathbf{Z}, c_1)$ is undefined, since $\alpha_1^{c_1}(\mathbf{Z}) = \alpha_0^{c_1}(\mathbf{Z}) = \infty$. Therefore, the proposition should be stated as follows.

  Let (X, κ) be a finite digital image and suppose $f : (X, \kappa) \to (X, \kappa)$ is continuous. If $f_* : H_*^\kappa(X) \to H_*^\kappa(X)$ is defined by $f_*(z) = kz$ where $k \in \mathbf{Z}$, i.e., if there exists $k \in \mathbf{Z}$ such that in every dimension i we have f inducing the homomorphism $f_{i*} : H_i^\kappa(X) \to H_i^\kappa(X)$ defined by $f_{i*}(z) = kz$, then $\lambda(f) = k\,\chi(X)$.

- A theoretically minor, but possibly confusing, error in Theorem 3.14 of [16]: In discussing an *antipodal map* $f : X \to X$, one needs the property that for every $x \in X$ we have $-x \in X$; this property does not characterize the version of S_2 used in Theorem 3.14 of [16]. In the following, we use $S_2 = [-1, 1]_{\mathbf{Z}}^3 \setminus \{(0, 0, 0)\}$, as described above.

  Theorem 3.14 of [16] asserts that

  If $\alpha_i : (S_i, c_1) \to (S_i, c_1)$ is the antipodal map between two digital i-spheres $S_i \subset Z^{i+1}$, for $i \in \{1, 2\}$, then α_i has degree $(-1)^{i+1}$.

  In fact, we show that this assertion is correct for $i = 1$, although an argument different from that of [16] must be given, as the argument of [16] makes use of Theorem 3.4 of [16] (= Theorem 3.3 of [14]), which, as noted above, is incorrect. For $i = 2$, we show the assertion is not well defined.

  - For $i = 1$, we have the following. Let the points $\{e_j\}_{j=0}^7$ of S_1 be circularly ordered. For notational convenience, let $e_8 = e_0$, and, more generally, index arithmetic is assumed to be modulo 8. The generators of the 1-chains of S_1 are the members of $\{< e_j e_{j+1} >\}_{j=0}^7$. We have

$$0 = \delta(\sum_{j=0}^{7} u_j < e_j e_{j+1} >) = \sum_{j=0}^{7} u_j(e_j - e_{j+1}) = \sum_{j=1}^{8} (u_j - u_{j-1}) e_j$$

    implies $u_0 = u_1 = \cdots = u_7$. Therefore, $Z_1(S_1)$ is generated by $\sum_{j=0}^7 < e_j e_{j+1} >$. Since, clearly, $B_1(S_1) = \{0\}$, we have $H_1(S_1) = Z_1(S_1)/B_1(S_1)$ is isomorphic to $\mathbf{Z}$. Therefore, the homomorphism $(\alpha_1)_* : H_1(S_1) \to H_1(S_1)$ induced by α_1 must satisfy $(\alpha_1)_*(x) = kx$ for some integer k.

Indeed, since the antipode of e_j is e_{j+4}, we have

$$(\alpha_1)_*(\sum_{j=0}^{7} < e_j e_{j+1} >) = \sum_{j=0}^{7} < e_{j+4} e_{j+5} > = \sum_{j=0}^{7} < e_j e_{j+1} >,$$

so $k = 1 = (-1)^{1+1}$, as asserted.

- For i=2, we observe that, using the c_1 adjacency, there is no triple of distinct, mutually adjacent points in S_2. Therefore, $H_2(S_2) = \{0\}$. Therefore, the degree of α_2 is not well defined since for any integer k we have $(\alpha_2)_*(x) = kx$ for all $x = 0 \in H_2(S_2)$.

- Incorrect assertion stated as Theorem 3.15 of [16]: Let S_1 be a digital simple closed curve in $\mathbf{Z}^2$. If $h : (S_1, c_1) \to (S_1, c_1)$ is a continuous function that is homotopic to a constant function in S_1, then h has a fixed point.

  In fact, we can take S_1 as above with its points ordered circularly, $S_1 = \{x_j\}_{j=0}^{7}$ where distinct points x_u, x_v are adjacent if and only if $u + 1 = v \ mod \ 8$ or $u - 1 = v \ mod \ 8$. Then, as in the proof of Theorem 3.1, the function $h : S_1 \to S_1$ given by

$$h(x) = \begin{cases} x_0 & \text{if } x \neq x_0; \\ x_1 & \text{if } x = x_0, \end{cases}$$

  is continuous and homotopic to the constant function $\overline{x_0}$ in S_1 but has no fixed point.

- Correct (for $|S_1| > 4$) assertion incorrectly "proven" as Corollary 3.16 of [16]: Let S_1 be as above. Let $h : (S_1, c_1) \to (S_1, c_1)$ be given by $h(x_i) = x_{(i+1) \ mod \ m}$, where $m = |S_1|$. Then h is not homotopic in S_1 to a constant map.

  The argument given for this assertion in [16] depends on Theorem 3.15 of [16], which, we have shown above, is incorrect. However, since h is easily seen to be an isomorphism, by Theorem 6.3, the current assertion is true if and only if $|S_1| > 4$.

7. Summary

We have shown that only single-point digital images have the fixed point property. However, digital n-cubes have the approximate fixed point property with respect to the c_n-adjacency [23]. We have shown that the approximate fixed point property is preserved by digital isomorphism and by digital retraction, and we have a result concerning preservation of the AFPP by Cartesian products. We have studied relations between universal functions and the AFPP. We have corrected several errors that appeared in previous papers.

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