APPLIED GENERAL TOPOLOGY

Appl. Gen. Topol. 18, no. 1 (2017), 53-59
doi:10.4995/agt.2017.4676

On quasi-orbital space

Hattab Hawete

Laboratoire des systèmes dynamiques et combinatoires, Faculté des Sciences de Sfax, Tunisia.
Umm Al Qura University Makkah KSA. (hattab.hawete@yahoo.fr)

Communicated by F. Balibrea

Abstract

Let G be a subgroup of the group Homeo(E) of homeomorphisms of a Hausdorff topological space E. The class of an orbit O of G is the union of all orbits having the same closure as O. We denote by $E/\widetilde{G}$ the space of classes of orbits called quasi-orbit space. A space X is called a quasi-orbital space if it is homeomorphic to $E/\widetilde{G}$ where E is a compact Hausdorff space. In this paper, we show that every infinite second countable quasi-compact T_0-space is the quotient of a quasi-orbital space.

2010 MSC: *54F65; 54H20.*

Keywords: *homeomorphism; group; quasi-orbit space; quasi-orbital space.*

1. Introduction

The standard setting for topological dynamics is a group of homeomorphisms G on a compact Hausdorff space E [6]. This group induces an open equivalence relation defined by the family of orbits ($Gx = \{gx : g \in G\}, x \in E$). We denote by E/G the orbit space equipped with the quotient topology. The study of this space is difficult: just consider the example of a group generated by an irrational rotation on the circle; indeed the orbit space does not verify the weaker separation axioms, as the T_0 separation axiom. For this reason [8, 1, 2, 7] consider an intermediary quotient, called the quasi-orbit space.

The class of the orbit Gx is $\widetilde{G}x = \bigcup_{\overline{O}=\overline{Gx}} O$. The family $(\widetilde{G}x, x \in E)$ determines an open equivalent relation on E [8]. Let $E/\widetilde{G}$ the space of classes of

Received 09 February 2016 – Accepted 21 July 2016

orbits equipped with the quotient topology. The space of classes of orbits is called the *quasi-orbit space*. The space $E/\widetilde{G}$ is a T_0-space and its the universal T_0-space associated to the orbit space E/G as in Bourbaki [3, Exercice 27 page I-104]. Let $p : E \to E/\widetilde{G}$ be the canonical projection. The map p is open. The map $\varphi : E/G \to E/\widetilde{G}$ which associates to each orbit its class is an onto quasi-homeomorphism[1]. Thus $E/\widetilde{G}$ is a good representative of E/G. According to [8, 1], the space $E/\widetilde{G}$ keeps information on the initial dynamical system.

A space X is a *quasi-orbital space* if it is homeomorphic to a quasi-orbit $E/\widetilde{G}$ where E is a compact Hausdorff space and G is a subgroup of homeomorphisms of E.

In [1], the authors asked the following problem: under which conditions a T_0-space is quasi-orbital? In [2] the authors showed that a finite T_0-space is quasi-orbital. Note that, according to [1, Example 3.4], if X is a non quasi-compact space then E is not in general compact.

In this paper we study this problem for an infinite T_0-space. Our main result is the following:

Theorem 1.1. *Every second countable quasi-compact T_0-space is the quotient of a quasi-orbital space.*

If E is a locally compact second countable topological space and G is a subgroup of homeomorphisms of E then, according to [8, 7], $E/\widetilde{G}$ satisfies the following properties:

(1) $E/\widetilde{G}$ is sober[2];
(2) If G has a minimal set then, $E/\widetilde{G}$ is quasi-compact.

In this paper, we show that if E is a locally compact topological space and G is a subgroup of homeomorphisms of E then, if $E/\widetilde{G}$ is quasi-compact then it is quasi-orbital.

The paper consists of three sections. After introduction we will show some properties of the quasi-orbital space. In section 3 we prove the main theorem.

2. Quasi-orbital spaces

In this section we study some properties of the quasi-orbital spaces.

Proposition 2.1. *A closed subspace of a quasi-orbital space is quasi-orbital.*

Proof. Let Y be a closed subset of a quasi-orbital space X. There exist a compact and Haudorff space E and a subgroup G of $Homeo(E)$ such that X is homeomorphic to to the quasi-orbit space $E/\widetilde{G}$; let φ such homeomorphism. $S = p^{-1}(\varphi(Y))$ is an invariant compact subset of E. We denote by $H = G/S$

[1]A continuous map $f : X \to Y$ between two topological spaces is called *a quasi-homeomorphism* if the map which assigns to each open set $V \subset Y$ the open set $f^{-1}(V)$ is a bijective map.

[2]A space X is sober if every irreducible, nonempty, closed subset M of X has a unique generic point m, i.e. $M = \overline{\{m\}}$.

the induced subgroup of G on S. Since S is an invariant subset of E, we have for each $x \in S$, $H(x) = G(x)$.

We will show that $S/\widetilde{H}$ is homeomorphic to $\varphi(Y)$ and so to Y. Let $f : S/\widetilde{H} \to \varphi(Y)$ which maps any class of an orbit Hx to the class of the orbit Gx. We will prove now that the bijective map f is a homeomorphism.

Let V be an open subset of $\varphi(X)$, that means that $V = U \cap \varphi(X)$ where U is an open subset of $E/\widetilde{G}$. So we have

$$p^{-1}(V) = p^{-1}(U) \cap p^{-1}(\varphi(X)) = p^{-1}(U) \cap S$$

since $p^{-1}(U)$ is an open subset of E, $p^{-1}(V)$ is an open subset of S. Thus V is an open subset of $S/\widetilde{H}$ and so f is a continuous map.

Let $p_1 : S \to S/\widetilde{H}$ be the canonical projection and let V be an open subset of $S/\widetilde{H}$, that means that $p_1^{-1}(V)$ is an open subset of S and so there exists an open subset U of E such that $p_1^{-1}(V) = U \cap S$. We have

$$V = p(p_1^{-1}(V)) = p(U \cap S)$$

Since S is invariant, we deduce that

$$V = p(U) \cap p(S) = p(U) \cap \varphi(X)$$

The fact that p is an open map implies that V is an open subset of $\varphi(X)$. Therefore f is an open map.

Thus f is a homeomorphism and so Y is a quasi-orbital space. □

Example 2.2. This example shows that Proposition 2.1 minus the hypothesis that Y is closed is false. Let f be an increasing homeomorphism of $[0, 1]$ without fixed point in $]0, 1[$ such that $f(0) = 0$, $f(1) = 1$ and $f(\frac{1}{2}) = \frac{3}{4}$. Let (a_n) be an increasing sequence such that $a_0 = \frac{1}{2}$ and converges to $\frac{5}{8}$. Let (b_n) be a decreasing sequence such that $b_0 = \frac{3}{4}$ and converges to $\frac{5}{8}$. Let g be a homeomorphism of $[0, 1]$ such that its support is $\bigcup_{n \geq 0} f^n([a_n, b_n])$ and $g(f^n(a_{n+1})) = f^n(a_{n+1})$. Let G be the group of homeomorphisms of $[0, 1]$ generated by f and g. Let $X = [0, 1]/\widetilde{G}$ be the quasi-orbital space. The subspace $Y = X - p(\frac{5}{8})$ is not closed. On the other hand Y can not be a quasi-orbital space because it is irreducible without generic point [8, Lemma 2.2].

Proposition 2.3. *Let X be a quasi-orbital space and R be an equivalence relation on X which have a closed continuous cross-section s*[3]*. Then X/R is quasi-orbital.*

Proof. Since s is closed, $s(X/R)$ is a closed subset of X and so, according to Proposition 2.1, $s(X/R)$ is quasi-orbital. Since s is closed and continuous, it will be an embedding and so X/R is homeomorphic to $s(X/R)$ which implies that X/R is quasi-orbital. □

[3]According to [13], if X/R is a T_1-space and zero-dimensional, then there exists a continuous cross-section for R.

Remark 2.4. If an open equivalence relation R has a closed and continuous cross-section, then X/R is a T_0-space. Indeed, let a and b two elements of X/R such that $\overline{\{a\}} = \overline{\{b\}}$. Since s is continuous and closed, $s(\overline{\{a\}}) = \overline{s(\{a\})} = \overline{\{s(a)\}}$ and $s(\overline{\{b\}}) = \overline{s(\{b\})} = \overline{\{s(b)\}}$ and so $\overline{\{s(a)\}} = \overline{\{s(b)\}}$. The fact that X is a T_0-space implies that $s(a) = s(b)$ and so $a = b$ (s is injective). Therefore X/R is a T_0-space.

Proposition 2.5. *Let $(X_i, i \in I)$ be a family of quasi-orbital spaces. Then the product $\prod_{i\in I} X_i$ is quasi-orbital.*

Proof. For every $i \in I$, X_i is quasi-orbital, then there exist a compact space E_i and a subgroup G_i of $Homeo(E_i)$ such that X_i is homeomorphic to the quasi-orbits space $E_i/\widetilde{G_i}$. Let $E = \prod_{i\in I} E_i$ be the product space and $G = \prod_{i\in I} G_i$ be the product group. By applying [3, Proposition 7 TG I.27], we have, for each $x = (x_i, i \in I)$, $\overline{G(x)} = \prod_{i\in I} \overline{G_i(x_i)}$ and so $\widetilde{G} = \widetilde{\prod_{i\in I} G_i} = \prod_{i\in I} \widetilde{G_i}$. By applying [3, Corollaire p.TG I.34] it follows that $\prod_{i\in I} X_i$ is homeomorphic to $E/\widetilde{G}$. Since E is compact, $\prod_{i\in I} X_i$ is quasi-orbital. □

Proposition 2.6. *If E is a locally compact space and G is a subgroup of homeomorphisms of E, then if $E/\widetilde{G}$ is quasi-compact then it is a quasi-orbital space.*

Proof. Since $E/\widetilde{G}$ is a quasi-compact space, according to [7, Proposition 2.1], G has a minimal set M. The fact that $E - M$ is an open set of a locally compact set implies that $E - M$ is a locally compact space [3, Proposition 13 TG I.66]. we denote by $H = G/E - M$ the induced subgroup of G on $E - M$. Since $E - M$ is invariant, we have for each $x \in E - M$, $H(x) = G(x)$. Let $\widehat{E} = (E - M) \cup \{\omega\}$ be the one point compactification of $E - M$. We can suppose that H is a group of homeomorphisms of $\widehat{E}$ by putting $H(\omega) = \{\omega\}$.

It is easy to see that the bijection $f : \widehat{E}/\widetilde{H} \to E/\widetilde{G}$ which maps any class of an orbit Hx to the class of the orbit Gx for all $x \in E - M$ and $f(\omega) = p(M)$ is a homeomorphism. Thus $E/\widetilde{G}$ is homeomorphic to $\widehat{E}/\widetilde{H}$. □

3. Proof of Main Theorem

Recall that, a topological space X is a k-space (compactly generated) if the following holds: a subset $A \subset X$ is closed in X if and only if $A \cap K$ is closed in K for every compact subset $K \subset X$ [10]. It is easy to see that the family of closed compact sets determines the topology of a k-space. Any locally compact space is a k-space and any first countable topological space (in particular a metric space) is a k-space. According to [4, p. 248], X is a k-space if and only if it is a quotient space of a locally compact space Z. The space Z is a disjoint sum of all compact subsets $(K_i, i \in I)$ of X: $Z = \coprod_{i\in I} K_i = \{(x,i) : i \in I \text{ and } x \in K_i\}$. The equivalence relation R on Z is defined by: $(x,i)R(y,j)$ if $x = y$. Note that Z is equipped with the disjoint sum topology defined by: U is an open set of Z if $\varphi_j^{-1}(U)$ is an open set of K_j where the map $\varphi_j : K_j \to Z$ is defined by $\varphi_j(x) = (x,j)$. Recall that, for all j, the map φ_j is continuous closed and open and $f : Z \to Y$ is continuous if and only if $f \circ \varphi_j$ is continuous.

Remark 3.1. The set $S = \{0,1\}$ equipped with the topology $\{\varnothing, S, \{1\}\}$ is called the Sierpinski space; it is a connected T_0-space but it is not a T_1-space. If G_1 is a finitely generated abelian subgroup of $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ of finite rank $k \geq 2$ having only a one fixed point $e \in \mathbb{S}^1$, then all other orbits are everywhere dense (N. Kopell, G. Reeb [11], [12]). Thus the quasi-orbits space $\mathbb{S}^1/\widetilde{G_1}$ is homeomorphic to the Sierpinski space S.

Proof (Main Theorem). Since X is a T_0-space, by applying [5, Theorem 2.3.26 p.84], there exists an embedding $\psi : X \to \prod_{i\in I} S_i$ (where S_i is the Sierpinski space $\{0,1\}$). We can suppose that $I \subset \mathbb{N}$; indeed, X is second countable. We know that for each $i \in I$ there is a homeomorphism $f_i : S_i \to \mathbb{S}^1_i/\widetilde{G_i}$ where $\mathbb{S}^1_i$ is the unit circle $\mathbb{S}^1$ and G_i is the group G_1 defined in Remark 3.1. The product map $\prod_{i\in I} f_i : \prod_{i\in I} S_i \to \prod_{i\in I} \mathbb{S}^1_i/\widetilde{G_i}$ is also a homeomorphism. According to [3, Corollaire p.TG I.34], $\prod_{i\in I} \mathbb{S}^1_i/\widetilde{G_i}$ is homeomorphic to $\prod_{i\in I} \mathbb{S}^1_i/\prod_{i\in I} \widetilde{G_i}$. The space $\mathbb{T}^I = \prod_{i\in I} \mathbb{S}^1_i$ is a compact second countable metric space. We put $G^I = \prod_{i\in I} G_i$. The group G^I is abelian. Then we conclude that there exists an embedding $\varphi : X \to \mathbb{T}^I/\widetilde{G^I}$. Let $p : \mathbb{T}^I \to \mathbb{T}^I/\widetilde{G^I}$ be the canonical projection. We denote by $E = p^{-1}(\varphi(X))$ and we denote by $G = G^I/E$ the induced subgroup of G^I on E. Since E is a saturated subset of $\mathbb{T}^I$. We have for each $x \in E$, $G(x) = G^I(x)$.

We will show that $E/\widetilde{G}$ is homeomorphic to $\varphi(X)$ and so to X. Let $f : E/\widetilde{G} \to \varphi(X) \subset \mathbb{T}^I/\widetilde{G^I}$ which maps any class of an orbit $G(x)$ to the class of the orbit $G^I(x)$. We will prove now that this bijective map f is a homeomorphism:

Let V be an open subset of $\varphi(X)$, that means that $V = U \cap \varphi(X)$ where U is an open subset of $\mathbb{T}^I/\widetilde{G^I}$. So we have

$$p^{-1}(V) = p^{-1}(U) \cap p^{-1}(\varphi(X)) = p^{-1}(U) \cap E$$

since $p^{-1}(U)$ is an open subset of $\mathbb{T}^I$, $p^{-1}(V)$ is an open subset of E. Thus V is an open subset of $E/\widetilde{G}$ and so f is a continuous map.

Let $p_1 : E \to E/\widetilde{G}$ be the canonical projection and let V be an open subset of $E/\widetilde{G}$, that means that $p_1^{-1}(V)$ is an open subset of E and so there exists an open subset U of $\mathbb{T}^I$ such that $p_1^{-1}(V) = U \cap E$. We have

$$V = p(p_1^{-1}(V)) = p(U \cap E)$$

Since E is saturated, we deduce that

$$V = p(U) \cap p(E) = p(U) \cap \varphi(X)$$

The fact that p is an open map implies that V is an open subset of $\varphi(X)$. Therefore f is an open map. We conclude that f is a homeomorphism.

Since E is a metric space, it is first countable and so E is a k-space. Thus E is the quotient of a locally compact metric space F by the relation R. Note that F is the disjoint union of all compact subsets of E. Let $q : F \to F/R = E$ be the canonical projection.

Let g be an element of G. We define on F the map $\mathbf{g} : F \to F$ by $\mathbf{g}(x,i) = (g(x), j)$ where $g(K_i)$ is the compact K_j. It is easy to see that $\mathbf{g}$ is a well defined bijection. Let U be an open set of F, then $U = \coprod_{i\in I} U_i \cap K_i$ where U_i is an open set of E. $\mathbf{g}^{-1}(U) = \coprod_{i\in I} g^{-1}(U_i) \cap g^{-1}(K_i)$ and $\mathbf{g}(U) = \coprod_{i\in I} g(U_i) \cap g(K_i)$ and since g is a homeomorphism $g(U_i)$ and $g^{-1}(U_i)$ are open sets of E and g is a permutation of the set of all compact subsets. Then $\mathbf{g}^{-1}(U) = \coprod_{i\in I} g^{-1}(U_i) \cap K_i$ and $\mathbf{g}(U) = \coprod_{i\in I} g(U_i) \cap K_i$ are open sets of F. Therefore $\mathbf{g}$ is a homeomorphism of F. The set $\mathbf{G} = \{\mathbf{g} : g \in G\}$ is a subgroup of homeomorphisms of F.

Since $E/\widetilde{G}$ is quasi-compact, we show Now that $\mathbf{G}$ has a minimal set. We start by showing that $E/\widetilde{G}$ contains a point a such that $\{a\}$ is closed. Since $E/\widetilde{G}$ is quasi-compact, by Zorn's lemma, it contains a minimal set M. Therefore for all $z \in M$ we have $\overline{\{z\}} = M$. From the fact that $E/\widetilde{G}$ is a T_0-space, it follows that M is a single point set $\{a\}$ (indeed $\overline{\{a\}} = \overline{\{b\}} \Rightarrow a = b$). Let x be an element of E such that $p(x) = a$. The fact that $\{a\}$ is closed implies that $p^{-1}(\{a\}) = \widetilde{G}x$ is a closed invariant set of E such that if $y \in \widetilde{G}x$ then $\overline{Gy} = \overline{Gx}$ and so $\widetilde{G}x$ is a minimal set of G. $q^{-1}(\widetilde{G}x)$ is a closed subset of F. If there exist $(x,i) \in q^{-1}(\widetilde{G}x)$ and $\mathbf{g} \in \mathbf{G}$ such that $\mathbf{g}(x,i) = (g(x), j)$ is not in $q^{-1}(\widetilde{G}x)$, then $q(g(x), j))$ is not in $\widetilde{G}x$ and so $g(x)$ is not in $\widetilde{G}x$ which contradicts the fact that $\widetilde{G}x$ is an invariant set. We conclude that $q^{-1}(\widetilde{G}x)$ is a minimal set of $\mathbf{G}$.

The fact that F is locally compact, according to [7, Proposition 2.1], implies that $F/\widetilde{\mathbf{G}}$ is quasi-compact. Then, by applying Proposition 2.6, we have $F/\widetilde{\mathbf{G}}$ is a quasi-orbital space $\widehat{E}/\widetilde{H}$. Let h be the homeomorphism of $\widehat{E}/\widetilde{H}$ and $F/\widetilde{\mathbf{G}}$.

Let $p_2 : F \to F/\widetilde{\mathbf{G}}$ and $p_3 : E \to E/\widetilde{G}$ be the canonical projections. Let $\widetilde{q} : F/\widetilde{\mathbf{G}} \to E/\widetilde{G}$ be the map defined by $\widetilde{q} \circ p_2 = p_3 \circ q$. $\widetilde{q}$ is a continuous and onto map. The map $\widehat{q} = \varphi^{-1} \circ f \circ \widetilde{q} \circ h$ is a continuous and onto map of $\widehat{E}/\widetilde{H}$ to X which implies that X is a quotient of a quasi-orbital space. $\square$

References

[1] C. Bonatti, H. Hattab and E. Salhi, Quasi-orbits spaces associated to T_0-spaces, Fund. Math. 211 (2011), 267–291.

[2] C. Bonatti, H. Hattab, E. Salhi and G. Vago, Hasse diagrams and orbit class spaces, Topology Appl. 158 (2011), 729–740.

[3] N. Bourbaki, Topologie générale chapitre 1 à 4, Masson, 1990.

[4] J. Dugundji, Topology, Allyn and Bacon, Inc., Boston (1966).

[5] R. Engelking, General Topology, 2nd ed., Helderman Verlag, Berlin, 1989.

[6] W. H. Gottschalk and G. A. Hedlund, Topological Dynamics, AMS Colloquium Publications, Vol. 36, 1955.

[7] H. Hattab, Characterization of quasi-orbit spaces, Qualitative theory of dynamical systems (2012).

[8] H. Hattab and E. Salhi, Groups of homeomorphisms and spectral topology, Topology Proc. 28, no. 2 (2004), 503–526.

[9] J. G. Hocking and G. S. Young, Topology, (1969).

[10] J. L. Kelley, General Topology, Van Nostrand, New Work (1955).

[11] N. Kopell, Commuting diffeomorphisms, Proc. Sympos. Pure Math. 14 (1970), 165–184.

[12] G. Reeb, Sur les structures feuilletées de codimension un et sur un théorème de A. Denjoy, Ann. Inst. Fourier 11 (1961), 185–200.

[13] D. E. Miller, A selector for equivalence relations with G_δ orbits, Proc. Amer. Math. Soc. 72, no. 2 (1978), 365–369.