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The equivalence of two definitions of sequential pseudocompactness

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Abstract

We show that two possible definitions of sequential pseudocompactness *are equivalent, and point out some consequences.*

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1. The equivalence

According to Artico, Marconi, Pelant, Rotter and Tkachenko [1, Definition 1.8], a Tychonoff topological space X is *sequentially pseudocompact* if the following condition holds.

(1) For any family $(O_n)_{n\in\omega}$ of pairwise disjoint nonempty open sets of X, there are an infinite set $J \subseteq \omega$ and a point $x \in X$ such that every neighborhood of x intersects all but finitely many elements of $(O_n)_{n\in J}$.

Notice that in [1] X is assumed to be a Tychonoff space, but the above definition makes sense for an arbitrary topological space.

According to Dow, Porter, Stephenson, and Woods [2, Definition 1.4], a topological space is *sequentially feebly compact* if the following condition holds.

(2) For any sequence $(O_n)_{n\in\omega}$ of nonempty open subsets of X, there are an infinite set $J \subseteq \omega$ and a point $x \in X$ such that every neighborhood of x intersects all but finitely many elements of $(O_n)_{n\in J}$.

(the difference is that in Condition (1) the O_n's are assumed to be pairwise disjoint, while they are arbitrary in Condition (2))

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The above two notions have been rather thoroughly studied by the mentioned authors. In this note we show their equivalence. Putting together the results from [1] and [2] shows that the class of sequentially pseudocompact Tychonoff topological spaces is closed under (possibly infinite) products and contains significant classes of pseudocompact spaces.

Unless otherwise specified, we shall assume no separation axiom.

Theorem 1.1. *For every topological space X, Conditions (1) and (2) above are equivalent.*

Proof. Condition (2) trivially implies Condition (1).

For the converse, suppose that X satisfies Condition (1), and let $(O_n)_{n\in\omega}$ be a sequence of nonempty open sets of X. Suppose by contradiction that

(*) for every infinite set $J \subseteq \omega$ and every point $x \in X$ there is some neighborhood $U(J,x)$ of x such that $N(J,x) = \{n \in J \mid U(J,x) \cap O_n = \varnothing\}$ is infinite.

Without loss of generality, we can assume that $U(J,x)$ is open. We shall construct by simultaneous induction a sequence $(m_i)_{i\in\omega}$ of distinct natural numbers, a sequence $(J_i)_{i\in\omega}$ of infinite subsets of ω, and a sequence of pairwise disjoint nonempty open sets $(U_i)_{i\in\omega}$ such that

(a) $U_i \subseteq O_{m_i}$ for every $i \in \omega$,
(b) $U_i \cap O_n = \varnothing$, for every $i \in \omega$, and $n \in J_i$, and
(c) $J_i \supseteq J_h$, whenever $i \leq h \in \omega$.

Put $m_0 = 0$ and pick $x_0 \in O_0$ (this is possible, since O_0 in nonempty). Apply (*) with $J = \omega$ and $x = x_0$, and let $U_0 = U(\omega, x_0) \cap O_0 \subseteq O_0 = O_{m_0}$ and $J_0 = N(\omega, x_0)$. U_0 is nonempty, since $x_0 \in U(\omega, x_0) \cap O_0$. By (*), J_0 is infinite, and Clause (b) is satisfied for $i = 0$. The basis of the induction is completed.

Suppose now that $0 \neq i \in \omega$, and that we have constructed finite sequences $(m_k)_{k<i}$, $(J_k)_{k<i}$, and $(U_k)_{k<i}$ satisfying the desired properties. Let m_i be any element of J_{i-1}. Since J_{i-1} is infinite, we can choose m_i distinct from all the m_k's, for $k < i$ (however, this follows automatically from (a) - (c)). Let x_i be any element of the nonempty O_{m_i}. Apply (*) with $J = J_{i-1}$ and $x = x_i$, and let $U_i = U(J_{i-1}, x_i) \cap O_{m_i} \subseteq O_{m_i}$ and $J_i = N(J_{i-1}, x_i)$. As above, U_i is nonempty, since $x_i \in U(J_{i-1}, x_i) \cap O_{m_i}$. By the definition of $N(J_{i-1}, x_i)$, we have that $J_i \subseteq J_{i-1}$, hence Clause (c) holds, by the inductive hypothesis. By (*), J_i is infinite, and moreover Clause (b) is satisfied for i. It remains to show that U_i is disjoint from U_k, for $k < i$. Since, by construction, $m_i \in J_{i-1}$, then, by (c) of the inductive hypothesis, for every $k < i$, we have that $m_i \in J_k$, hence, by (b), $U_k \cap O_{m_i} = \varnothing$, hence also $U_k \cap U_i = \varnothing$, since by construction $U_i \subseteq O_{m_i}$. The induction step is thus complete.

Having constructed sequences satisfying the above properties, we can apply Condition (1) to the sequence $(U_i)_{i\in\omega}$ of nonempty pairwise disjoint open sets, getting some $J \subseteq \omega$ and some $x \in X$ such that every neighborhood of x intersects all but finitely many elements of $(U_i)_{i\in J}$. If we put $J^* = \{m_i \mid i \in J\}$, then every neighborhood of x intersects all but finitely many elements of

$(O_n)_{n\in J^*}$, because of Clause (a). We have reached a contradiction, thus the theorem is proved. □

2. Consequences

Putting together results from [1, 2], and using Theorem 1.1, we get some nice results about sequential pseudocompactness. First, we recall the following results from [2].

Theorem 2.1.

[2, Theorems 4.1 and 4.4] *A product of nonempty topological spaces is sequentially feebly compact if and only if each factor is sequentially feebly compact.*

[2, Theorem 4.3] *Every product of feebly compact spaces, all but one of which are sequentially feebly compact, is feebly compact.*

Recall that a topological space is *feebly compact* if, for any sequence $(O_n)_{n\in\omega}$ of nonempty open sets of X, there is a point $x \in X$ such that $\{n \in \omega \mid U \cap O_n \neq \varnothing\}$ is infinite, for every neighborhood U of x. Clearly, Condition (2) implies feeble compactness. It is well known that a Tychonoff space is feebly compact if and only if it is pseudocompact.

From Theorem 2.1 we immediately get:

Corollary 2.2. *Let $\mathcal{P}$ be a family of properties of topological spaces such that every feebly compact topological space satisfying at least one $P \in \mathcal{P}$ is sequentially feebly compact. Then every product of feebly compact spaces, all but one of which satisfy some $P \in \mathcal{P}$, is feebly compact (pseudocompact, if the product is Tychonoff).*

Corollary 2.2 is interesting since, restricted to Tychonoff spaces, and using Condition (1), [1] found many properties P which satisfy the assumption in Corollary 2.2; among them, the properties of being a topological group, or a scattered space, or a first countable space, or a ψ-ω-scattered space. From Theorem 1.1 and Corollary 2.2 we get a proof that any product of pseudocompact Tychonoff spaces satisfying one of the above properties is pseudocompact. In particular, from Theorems 1.1, 2.1 and [1, Proposition 1.10], we get another proof of the classical result by Comfort and Ross that any product of Tychonoff pseudocompact topological groups is pseudocompact. However, it is not clear whether this is a real simplification: it might be the case that any proof that every Tychonoff pseudocompact topological group is sequentially pseudocompact already contains enough sophistication to be easily converted into a direct proof of Comfort and Ross Theorem.

The above considerations lead to the following problem.

Problem 2.3. *Is there some significant part of the (topological) theory of pseudocompact topological groups which follows already from the assumption of sequential pseudocompactness?*

More precisely, are there other theorems holding for pseudocompact topological groups which can be generalized to sequentially pseudocompact topological spaces (with not necessarily some algebraic structure on them)?

3. Further remarks

The relationship between sequential feeble compactness and sequential compactness does not always parallel the relationship between feeble compactness and countable compactness. Both $\beta\omega$ and $D^{\mathfrak{c}}$ are classical examples of compact not sequentially compact spaces. As noticed on [1, p. 7] and [2, Example 2.9], $\beta\omega$ is not sequentially pseudocompact. On the other hand, $D^{\mathfrak{c}}$ is sequentially pseudocompact, by Theorem 2.1. Thus, compactness together with sequential pseudocompactness do not necessarily imply sequential compactness. In particular, normality and sequential pseudocompactness do not imply sequential compactness (thus the result that normality and pseudocompactness imply countable compactness cannot be generalized in the obvious way). Also, a compact subspace of a compact sequentially pseudocompact space is not necessarily sequentially pseudocompact, since $\beta\omega$ can be embedded in $D^{\mathfrak{c}}$. In particular, a closed subspace of a sequentially pseudocompact space is not necessarily sequentially pseudocompact.

The proof of [2, Theorem 4.1] actually shows a little more. If $(X_h)_{h\in H}$ is a family of topological spaces, the *ω-box topology* on $\prod_{h\in H} X_h$ is defined as the topology a base of which consists of the sets of the form $\prod_{h\in H} O_h$, where each O_h is an open set of X_h, and $|\{h \in H \mid O_h \neq X_h\}| \leq \omega$.

Proposition 3.1. *Suppose that $(X_h)_{h\in H}$ is a family of sequentially feebly compact topological spaces. If $(O_n)_{n\in\omega}$ is a sequence of nonempty open sets in the ω-box topology on $\prod_{h\in H} X_h$, then there are an infinite set $J \subseteq \omega$ and a point $x \in \prod_{h\in H} X_h$ such that $\{n \in J \mid U \cap O_n = \varnothing\}$ is finite, for every neighborhood U of x in the Tychonoff product topology on $\prod_{h\in H} X_h$.*

Proof. Same as the proof of [2, Theorem 4.1], since it is no loss of generality to deal with elements of the standard base of the ω-box topology, and since the set R defined as in the proof of [2, Theorem 4.1] is countable in this case, too, being the countable union of a family of countable sets. See [3] for full details. □

In the statement of Proposition 3.1, the neighborhoods U of x have to be considered in the Tychonoff product topology. The statement would turn out to be false allowing U vary among the neighborhoods of x in the ω-box topology. Indeed, for example, a discrete two-element space is vacuously sequentially pseudocompact; however, its ω^{th} power in the ω-box topology is a discrete space of cardinality $\mathfrak{c}$, hence the conclusion of Proposition 3.1 would fail.

The following notions might deserve some study, in particular when α is a cardinal.

Definition 3.2. For α an infinite limit ordinal, we say that a topological space X is *sequentially α-feebly compact* if, for any sequence $(O_\beta)_{\beta\in\alpha}$ of nonempty open sets of X, there are some $x \in X$ and a subset Z of α such that Z has order type α, and, for every neighborhood U of x, there is $\beta < \alpha$ such that $U \cap O_\gamma \neq \varnothing$, for every $\gamma \in Z$ such that $\gamma > \beta$.

If we modify the above definition by further requesting that the (O_β)'s are pairwise disjoint, we say that X is *d-sequentially α-feebly compact*. Clearly, for every α, sequential α-feeble compactness implies d-sequential α-feeble compactness, and, for $\alpha = \omega$, both notions are equivalent (and equivalent to sequential feeble compactness), by Theorem 1.1.

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