Applied General Topology

Appl. Gen. Topol. 18, no. 1 (2017), 45-52
doi:10.4995/agt.2017.4573

Common fixed point theorems for mappings satisfying (E.A)-property via C-class functions in b-metric spaces

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Communicated by M. Abbas

Abstract

In this paper, we consider and generalize recent b-(E.A)-property results in [11] via the concepts of C-class functions in b- metric spaces. A example is given to support the result.

2010 MSC: *47H10; 54H25.*

Keywords: *Common fixed point; (E.A)-property; b-metric space; C-class function.*

1. Introduction and preliminaries

Bakhtin in [5] introduced the consept of $b-$metric space and prove the Banach fixed point theorem in the setting of $b-$metric spaces. Since then many authors have obtain various generalizations of fixed point theorems in $b-$metric spaces.

On the other hand, Aamri and Moutaawakil in [1] introduced the idea of $(E.A)$ $-$property in metric spaces. Later on some authors employed this concept to obtain some new fixed point results. See ([6, 10]).

In this paper, we prove common fixed point results for two pairs of mappings which satisfy the $b-(E.A)$-property using the concept of C-class functions in $b-$metric spaces.

Received 19 January 2016 – Accepted 28 January 2017

Definition 1.1 ([5]). Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b1) $d(x, y) = 0$ if and only if $x = y$,
(b2) $d(x, y) = d(y, x)$,
(b3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.

It should be noted that, the class of b-metric spaces is effectively larger than that of metric spaces, every metric is a b-metric with $s = 1$.

However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

Definition 1.2 ([7]). Let $\{x_n\}$ be a sequence in a b-metric space (X, d).

(a) $\{x_n\}$ is called $b-$convergent if and only if there is $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.
(b) $\{x_n\}$ is a $b-$Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

A b-metric space is said to be complete if and only if each $b-$Cauchy sequence in this space is $b-$convergent.

Proposition 1.3 ([7]). *In a $b-$metric space (X, d), the following assertions hold:*

(p1) *A $b-$convergent sequence has a unique limit.*
(p2) *Each $b-$convergent sequence is $b-$Cauchy.*
(p3) *In general, a $b-$metric is not continuous.*

Definition 1.4 ([7]). Let (X, d) be a $b-$metric space. A subset $Y \subset X$ is called closed if and only if for each sequence $\{x_n\}$ in Y is $b-$convergent and converges to an element x.

Definition 1.5 ([11]). Let (X, d) be a $b-$metric space and f and g be self-mappings on X.

(i) f and g are said to compatible if whenever a sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are $b-$convergent to some $t \in X$, then

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0.$$

(ii) f and g are said to noncompatible if there exists at least one sequence $\{x_n\}$ in X is such that $\{fx_n\}$ and $\{gx_n\}$ are $b-$convergent to some $t \in X$, but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ does not exist.
(iii) f and g are said to satisfy the $b - (E.A)$-property if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t,$$

for some $t \in X$.

Remark 1.6 ([11]). Noncompatibility implies property $(E.A)$.

Example 1.7 ([11])**.** $X=[0,2]$ and define $d:X\times X\to[0,\infty)$ as follows

$$d(x,y)=(x-y)^2.$$

Let $f,g:X\to X$ be defined by

$$f(x)=\begin{cases}1, x\in[0,1]\\ \frac{x+1}{8}, x\in(1,2]\end{cases}\qquad g(x)=\begin{cases}\frac{3-x}{2}, x\in[0,1]\\ \frac{x}{4}, x\in(1,2]\end{cases}$$

For a sequence $\{x_n\}$ in X such that $x_n=1+\frac{1}{n+2}$, $n=0,1,2,...,$

$$\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n=\frac{1}{4}.$$

So f and g are satisfy the $b-(E.A)$-property. But

$$\lim_{n\to\infty} d(fgx_n,gfx_n)\neq 0.$$

Thus f and g are noncompatible.

Definition 1.8 ([8])**.** Let f and g be given self-mappings on a set X. The pair (f,g) is said to be weakly compatible if f and g commute at their coincidence points (i.e. $fgx=gfx$ whenever $fx=gx$).

In 2014, Ansari [3] introduced the concept of C-class functions. See also [4]

Definition 1.9. A mapping $F:[0,\infty)^2\to\mathbb{R}$ is called C-*class* function if it is continuous and satisfies following axioms:

(i) $F(s,t)\leq s$;
(ii) $F(s,t)=s$ implies that either $s=0$ or $t=0$; for all $s,t\in[0,\infty)$.

Note for some F we have that $F(0,0)=0$.
We denote C-class functions as $\mathcal{C}$.

Example 1.10. The following functions $F:[0,\infty)^2\to\mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t\in[0,\infty)$:

(1) $F(s,t)=s-t$, $F(s,t)=s\Rightarrow t=0$;
(2) $F(s,t)=ms$, $0<m<1$, $F(s,t)=s\Rightarrow s=0$;
(3) $F(s,t)=\frac{s}{(1+t)^r}$; $r\in(0,\infty)$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(4) $F(s,t)=\log(t+a^s)/(1+t)$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(5) $F(s,t)=\ln(1+a^s)/2$, $a>e$, $F(s,1)=s\Rightarrow s=0$;
(6) $F(s,t)=(s+l)^{(1/(1+t)^r)}-l$, $l>1, r\in(0,\infty)$, $F(s,t)=s\Rightarrow t=0$;
(7) $F(s,t)=s\log_{t+a}a$, $a>1$, $F(s,t)=s\Rightarrow s=0$ or $t=0$;
(8) $F(s,t)=s-(\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s,t)=s\Rightarrow t=0$;
(9) $F(s,t)=s\beta(s)$, $\beta:[0,\infty)\to(0,1)$, and is continuous, $F(s,t)=s\Rightarrow s=0$;
(10) $F(s,t)=s-\frac{t}{k+t}, F(s,t)=s\Rightarrow t=0$;
(11) $F(s,t)=s-\varphi(s), F(s,t)=s\Rightarrow s=0$, here $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0\Leftrightarrow t=0$;
(12) $F(s,t)=sh(s,t), F(s,t)=s\Rightarrow s=0$,here $h:[0,\infty)\times[0,\infty)\to[0,\infty)$is a continuous function such that $h(t,s)<1$ for all $t,s>0$;
(13) $F(s,t)=s-(\frac{2+t}{1+t})t$, $F(s,t)=s\Rightarrow t=0$.

(14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$.
(15) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$,here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,
(16) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$.

Definition 1.11 ([9]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

See also [2] and [12].

Definition 1.12 ([3]). An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0 \quad, t > 0$ and $\varphi(0) \geq 0$

2. Main results

Through out this section, we assume ψ is altering distance function, φ is ultra altering distance function and F is a C-class function. We shall start the following theorem.

Theorem 2.1. *Let (X,d) be a $b-$metric space and $f,g,S,T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that*

$$\psi(d(fx,gy)) \leq F(\psi(M_s(x,y)),\varphi(M_s(x,y))), \quad \text{for all } x,y \in X \tag{2.1}$$

where,

$$M_s(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)+d(Sx,gy)}{2s}\right\}.$$

Suppose that one of the pairs (f,S) and (g,T) satisfy the $b-(E.A)$-property and that one of the subspaces $f(X), g(X), S(X)$ and $T(X)$ is closed in X. Then the pairs (f,S) and (g,T) have a point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f,g,S and T have a unique common fixed point.

Proof. If the pairs (f,S) satisfies the $b-(E.A)$-property, then there exists a sequence $\{x_n\}$ in X satisfying

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = q,$$

for some $q \in X$. As $f(X) \subseteq T(X)$ there exists a sequence $\{y_n\}$ in X such that $fx_n = Ty_n$. Hence $\lim_{n\to\infty} Ty_n = q$. Let us show that $\lim_{n\to\infty} gy_n = q$. By (2.1),

$$\psi(d(fx_n,gy_n)) \leq F(\psi(M_s(x_n,y_n)),\varphi(M_s(x_n,y_n))) \leq \psi(M_s(x_n,y_n)) \tag{2.2}$$

where

$$\begin{aligned} M_s(x_n, y_n) &= \max\left\{ \begin{array}{c} d(Sx_n, Ty_n), d(fx_n, Sx_n), d(Ty_n, gy_n), \\ \frac{d(Sx_n, gy_n) + d(fx_n, Ty_n)}{2s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(Sx_n, fx_n), d(fx_n, gy_n), \\ \frac{d(Sx_n, gy_n) + d(fx_n, fx_n)}{2s} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d(Sx_n, fx_n), d(fx_n, gy_n), \\ \frac{s[d(Sx_n, fx_n), d(fx_n, gy_n)]}{2s} \end{array} \right\}. \end{aligned}$$

In (2.2), on taking limit,

$$\psi(\lim_{n\to\infty} d(q, gy_n)) \leq F(\psi(\lim_{n\to\infty} d(q, gy_n)), \varphi(\lim_{n\to\infty} d(q, gy_n))).$$

So, $\psi(\lim_{n\to\infty} d(q, gy_n)) = 0$, or $,\varphi(\lim_{n\to\infty} d(q, gy_n)) = 0$. Thus

$$\lim_{n\to\infty} d(q, gy_n) = 0.$$

Hence $\lim_{n\to\infty} gy_n = q$.

If $T(X)$ is closed subspace of X, then there exists a $r \in X$, such that $Tr = q$. By (2.1),

$$\psi(d(fx_n, gr)) \leq F(\psi(M_s(x_n, r)), \varphi(M_s(x_n, r))) \tag{2.3}$$

where

$$\begin{aligned} M_s(x_n, r) &= \max\left\{ \begin{array}{c} d(Sx_n, Tr), d(fx_n, Sx_n), d(Tr, gr), \\ \frac{d(fx_n, Tr) + d(Sx_n, gr)}{2s} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(Sx_n, q), d(fx_n, Sx_n), d(q, gr), \\ \frac{d(fx_n, q) + d(Sx_n, gr)}{2s} \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$,

$$\begin{aligned} \lim_{n\to\infty} M_s(x_n, r) &= \max\left\{ d(q, q), d(q, q), d(q, gr), \frac{d(q, q) + d(q, gr)}{2s} \right\} \\ &= d(q, gr). \end{aligned}$$

Now, (2.3) and definition of ψ and φ, as $n \to \infty$,

$$\psi(d(q, gr) \leq F(\psi(d(q, gr)), \varphi(d(q, gr)))$$

which implies $\psi(d(q, gr)) = 0$ or $\varphi(d(q, gr)) = 0$ gives $gr = q$. Thus r is a coincidence point of the pair (g, T). As $g(X) \subseteq S(X)$, there exists a point $z \in X$ such that $q = Sz$. We claim that $Sz = fz$. By (2.1), we have

$$\psi(d(fz, gr)) \leq F(\psi(M_s(z, r)), \varphi(M_s(z, r))) \tag{2.4}$$

where

$$\begin{aligned}
M_s(z,r) &= \max\left\{d(Sz,Tr), d(fz,Sz), d(Tr,gr), \frac{d(fz,Tr)+d(Sz,gr)}{2s}\right\} \\
&= \max\left\{d(q,q), d(fz,q), d(q,q), \frac{d(fz,q)+d(q,q)}{2s}\right\} \\
&\leq \max\left\{d(fz,q), \frac{d(fz,q)}{2s}\right\} \\
&= d(fz,q).
\end{aligned}$$

Thus from (2.4),

$$\psi(d(fz,gr)) = \psi(d(fz,q)) \leq F(\psi(d(fz,q)), \varphi(d(fz,q)))$$

implies that $\psi(d(fz,q)) = 0$, or $,\varphi(d(fz,q)) = 0$. Therefore $Sz = fz = q$. Hence z is a coincidence point of the pair (f,S). Thus $fz = Sz = gr = Tr = q$. By weak compatibility of the pairs (f,S) and (g,T), we deduce that$fq = Sq$ and $gq = Tq$. We will show that q is a common fixed point of f, g, S and T. From (2.1) ,

$$\psi(d(fq,q)) = \psi(d(fq,gr)) \leq F(\psi(M_s(q,r)), \varphi(M_s(q,r))) \tag{2.5}$$

where,

$$\begin{aligned}
M_s(q,r) &= \max\left\{d(Sq,Tr), d(fq,Sq), d(Tr,gr), \frac{d(fq,Tr)+d(Sq,gr)}{2s}\right\} \\
&= \max\left\{d(fq,q), d(fq,fq), d(q,q), \frac{d(fq,q)+d(fq,q)}{2s}\right\} \\
&= d(fq,q).
\end{aligned}$$

By (2.5)

$$\psi(d(fq,q)) \leq F(\psi(d(fq,q)), \varphi(d(fq,q))).$$

So $fq = Sq = q$. Similarly, it can be shown $gq = Tq = q$.

To prove the uniqueness of the fixed point of f, g, S and T. Suppose for contradiction that p is another fixed point of f, g, S and T. By (2.1), we obtain

$$\psi(d(q,p)) = \psi(d(fq,gp)) \leq F(\psi(M_s(q,p)), \varphi(M_s(q,p)))$$

and

$$\begin{aligned}
M_s(q,p) &= \max\left\{d(Sq,Tp), d(fq,Sq), d(Tp,gp), \frac{d(fq,Tp)+d(Sq,gp)}{2s}\right\} \\
&= \max\left\{d(q,p), d(q,q), d(p,p), \frac{d(q,p)+d(q,p)}{2s}\right\} \\
&= d(q,p).
\end{aligned}$$

Hence we have

$$\psi(d(q,p)) \leq F(\psi(d(q,p)), \varphi(d(q,p))),$$

which implies that $\psi(d(q,p)) = 0$ or $\varphi(d(q,p)) = 0$. So $q = p$. □

Corollary 2.2. *Let (X,d) be a $b-$metric space and $f,g,S,T : X \to X$ be mappings with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$ such that*

$$d(fx,gy) \leq F(M_s(x,y), \varphi(M_s(x,y))), \text{ for all } x,y \in X,$$

where

$$M_s(x,y) = \max\left\{d(Sx,Ty), d(fx,Sx), d(gy,Ty), \frac{d(fx,Ty)+d(Sx,gy)}{2s}\right\}.$$

Suppose that one of the pairs (f,S) and (g,T) satisfy the $b-(E.A)$-property and that one of the subspaces $f(X), g(X), S(X)$ and $T(X)$ is closed in X. Then the pairs (f,S) and (g,T) have a point of coincidence in X. Moreover, if the pairs (f,S) and (g,T) are weakly compatible, then f,g,S and T have a unique common fixed point.

Corollary 2.3. *Let (X,d) be a $b-$metric space and $f,T : X \to X$ be mappings such that*

$$\psi(d(fx,fy)) \leq F(\psi(M_s(x,y)), \varphi(M_s(x,y))), \text{ for all } x,y \in X,$$

where

$$M_s(x,y) = \max\left\{d(Tx,Ty), d(fx,Tx), d(fy,Ty), \frac{d(fx,Ty)+d(Tx,fy)}{2s}\right\}.$$

Suppose that the pair (f,T) satisfies the $b-(E.A)$-property and $T(X)$ is closed in X. Then the pair (f,T) has a common point of coincidence in X. Moreover, if the pair (f,T) is weakly compatible, then f and T have a unique common fixed point.

Example 2.4. Let $F(s,t) = \frac{99}{100}s$, $X = [0,1]$ and define $d : X \times X \to [0,\infty)$ as follows

$$d(x,y) = \begin{cases} 0, x = y \\ (x+y)^2, x \neq y \end{cases}$$

Then (X,d) is a $b-$metric space with constant $s = 2$. Let $f,g,S,T : X \to X$ be defined by

$$\begin{aligned} f(x) &= \frac{x}{4} \quad , \; g(x) = \begin{cases} 0, x \neq \frac{1}{2} \\ \frac{1}{8}, x = \frac{1}{2} \end{cases} , \quad S(x) = \begin{cases} 2x, 0 \leq x < \frac{1}{2} \\ \frac{1}{8}, \frac{1}{2} \leq x \leq 1 \end{cases} \text{ and} \\ T(x) &= \begin{cases} x, 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \leq x \leq 1 \end{cases} . \end{aligned}$$

Clearly, $f(X)$ is closed and $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$. The sequence $\{x_n\}$, $x_n = \frac{1}{2} + \frac{1}{n}$, is in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Sx_n = \frac{1}{8}$. So that the pair (f,S) satisfies the $b-(E.A)-$property. But the pair (f,S) is noncompatible for $\lim_{n\to\infty} d(fSx_n, Sfx_n) \neq 0$. The altering functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ are defined by $\psi(t) = \sqrt{t}$. To check the contractive condition (2.1), for all $x,y \in X$,

if $x = 0$ or $x = \frac{1}{2}$, then (2.1) is satisfied.

if $x \in \left(0, \frac{1}{2}\right)$, then

$$\psi\left(d\left(fx,gy\right)\right)=\frac{x}{4}\leq\frac{99}{100}\frac{9x}{4}=\frac{99}{100}d\left(fx,Sx\right)\leq\frac{99}{100}\psi(M_s\left(x,y\right)).$$

If $x\in\left(\frac{1}{2},1\right]$, then

$$\psi\left(d\left(fx,gy\right)\right)=\frac{x}{4}\leq\frac{99}{100}\left(\frac{x}{4}+\frac{1}{8}\right)=\frac{99}{100}d\left(fx,Sx\right)\leq\frac{99}{100}\psi(M_s\left(x,y\right)).$$

Then (2.1) is satisfied for all $x,y\in X$. The pairs (f,S) and (g,T) are weakly compatible. Hence, all of the conditions of Theorem 2.1 are satisfied. Moreover 0 is the unique common fixed point of f,g,S and T.

Acknowledgements. *The authors would like to thank the referee for useful comments.*

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