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A note on uniform entropy for maps having topological specification property

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Abstract

We prove that if a uniformly continuous self-map f of a uniform space has topological specification property then the map f has positive uniform entropy, which extends the similar known result for homeomorphisms on compact metric spaces having specification property. An example is also provided to justify that the converse is not true.

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1. Introduction

In [1], authors have defined and studied the notion of topological entropy of a continuous self-map f of a compact topological space as an analogue of the measure theoretic entropy. The relation between topological entropy and measure theoretic entropy is established by the variational principle, which asserts that $h(T) = \sup\{h_\mu(T) | \mu \in \mathcal{P}_T(X)\}$, i.e., topological entropy equals the supremum of the measure theoretic entropies $h_\mu(T)$, where μ ranges over all T-invariant Borel probability measures on X [10].

A dynamical system is called deterministic if its topological entropy vanishes [9]. One may argue that the future of a deterministic dynamical system can be predicted if its past is known [16]. In a similar way positive entropy may

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be related to randomness and chaos. Topological entropy also plays an important role as an invariant for classification of continuous maps up to conjugacy. A remarkable contribution of Bowen is the definition of entropy for uniformly continuous self-maps on metric spaces [4]. Authors in ([11], [12], [14]) have extended Bowen's definition of entropy to uniformly continuous self-maps on uniform spaces. The uniform entropy considered in this paper is the one introduced in [12]. For details on uniform entropy on a uniform space one can also refer [8]. The survey [8] is also devoted to the study of entropy for topological groups wherein the uniform entropy is used to define topological entropy for continuous endomorphisms of locally compact groups. For a review of entropy in various fields of mathematics and science one can refer [2].

Recently in [7], authors have extended Smale's spectral decomposition theorem for Anosov diffeomorphisms of compact manifolds to general topological spaces wherein they have extended the notions of expansivity and shadowing for general topological spaces. Sensitivity and Devaney chaos are also studied for continuous group actions on uniform spaces in [6].

Let X be a non-empty set and let $\triangle_X = \{(x,x)|x \in X\}$, called as the *diagonal* of $X \times X$. A subset M of $X \times X$ is said to be *symmetric* if $M = M^T$, where $M^T = \{(y,x)|(x,y) \in M\}$. The *composite* $U \circ V$ of two subsets U and V of $X \times X$ is defined to be the set $\{(x,y) \in X \times X|$ there exists $z \in X$ satisfying $(x,z) \in U$ and $(z,y) \in V\}$.

Definition 1.1 ([13])**.** Let X be a non-empty set. A uniform structure on X is a non-empty set $\mathcal{U}$ of subsets of $X \times X$ satisfying the following conditions:

i: if $U \in \mathcal{U}$ then $\triangle_X \subset U$,
ii: if $U \in \mathcal{U}$ and $U \subset V \subset X \times X$ then $V \in \mathcal{U}$,
iii: if $U \in \mathcal{U}$ and $V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$,
iv: if $U \in \mathcal{U}$ then $U^T \in \mathcal{U}$,
v: if $U \in \mathcal{U}$ then there exists $V \in \mathcal{U}$ such that $V \circ V \subset U$.

The elements of $\mathcal{U}$ are then called the *entourages* of the uniform structure and the pair $(X,\mathcal{U})$ is called a *uniform space*.

We work here with uniform space $(X,\mathcal{U})$. Note that the fact that points x and y are close (in terms of distance) in a metric space X is equivalent to the fact that point (x,y) is close to the diagonal $\triangle_X$ of $X \times X$ in a uniform space $(X,\mathcal{U})$. If $(X,\mathcal{U})$ is a uniform space, then there is an induced topology on X characterized by the fact that the neighborhoods of an arbitrary point $x \in X$ consists of the sets $U[x]$, where U varies over all entourages of X. The set $U[x] = \{y \in X|(x,y) \in U\}$ is called the *cross section* of U at $x \in X$. We denote the subsets of $X \times X$ by U and the subsets of X by $\dot{U}$.

The specification property for homeomorphisms on a compact metric space has turned out to be an important notion in the study of dynamical systems. It was first introduced by Bowen to give the distribution of periodic points for Axiom A diffeomorphisms [5]. Informally the specification property means that it is possible to shadow two distinct pieces of orbits which are sufficiently apart in time by a single orbit. Let f be a self homeomorphism on a compact metric space (X, d). The map f is said to satisfy *specification property* if for every $\epsilon > 0$ there exists a positive integer $M(\epsilon)$ such that for any finite sequence $x_1, x_2, ..., x_k$ in X, any integers $a_1 \leq b_1 < a_2 \leq b_2 < ... < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$ $(2 \leq j \leq k)$ and $p > M + (b_k - a_1)$, there exists $x \in X$ such that $f^p(x) = x$ and $d(f^i(x), f^i(x_j)) < \epsilon$ $(a_j \leq i \leq b_j, 1 \leq j \leq k)$.

In [3], it has been proved that on a compact metric space X if $f : X \to X$ is a homeomorphism satisfying specification property then the topological entropy of f is positive. In section 2, we extend this result to uniformly continuous self-maps defined on uniform spaces. In particular, we prove that a uniformly continuous map defined on a uniform space having topological specification property has positive uniform entropy.

2. Main Result

In [15], we have introduced the notion of topological specification property for a homeomorphism on a uniform space however the notion of topological specification property can be defined for continuous maps as follows:

Definition 2.1. A continuous self-map f is said to have topological specification property if for every symmetric neighborhood U of the diagonal $\triangle_X$ there exists a positive integer M such that for any finite sequence of points $x_1, x_2, ..., x_k$ in X, any integers $0 = a_1 \leq b_1 < a_2 \leq b_2 < ... < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$ $(2 \leq j \leq k)$ and any $p > M + (b_k - a_1)$, there exists $x \in X$ such that $f^p(x) = x$ and $F^i(x, x_j) \in U$, $a_j \leq i \leq b_j$, $1 \leq j \leq k$.

Remark 2.2. If (X, d) is a compact metric space, then for any neighborhood U of $\triangle_X$, we can find $\epsilon > 0$ such that $U_\epsilon = d^{-1}(0, \epsilon) \subset U$. On the other hand, every U_ϵ is a neighborhood of $\triangle_X$. Thus for compact metric spaces, definitions of topological specification property and of specification property coincide.

Let $f : X \to X$ be a uniformly continuous map, $\mathcal{U}^s$ denote the set of all symmetric elements of $\mathcal{U}$, $\mathcal{U}^o$ denote the set of open elements of $\mathcal{U}$, n be a positive integer and let $U \in \mathcal{U}^s$. A subset $\dot{E}$ of X is said to be (n, U)*-separated* with respect to f if for each pair of distinct points x, y in $\dot{E}$ there exists j such that $0 \leq j < n$ and $F^j(x, y) \notin U$, where $F = f \times f$. For a subset $\dot{Z}$ of X, a subset $\dot{A}$ of X is said to be an (n, U)*-spanning* set for $\dot{Z}$ with respect to f if for each $x \in \dot{Z}$ there exists $y \in \dot{A}$ such that for all j with $0 \leq j < n$, we have $F^j(x, y) \in U$. Let $\mathcal{K}(X)$ denote the set of all compact subsets of X. For $\dot{K} \in \mathcal{K}(X)$, let $s_n(U, \dot{K}, f)$ denote the maximal cardinality of (n, U)-separated sets contained in $\dot{K}$ and let $r_n(U, \dot{K}, f)$ denote the minimal cardinality of (n, U)-spanning sets for $\dot{K}$. Note that a maximal (n, U)-separated subset of $\dot{K}$ is an (n, U)-spanning set for $\dot{K}$. Define

$$\overline{r}_f(U, \dot{K}) = \limsup_{n\to\infty} \tfrac{1}{n} log r_n(U, \dot{K}, f)$$
$$\overline{s}_f(U, \dot{K}) = \limsup_{n\to\infty} \tfrac{1}{n} log s_n(U, \dot{K}, f)$$

and

$$\begin{aligned} h(f,\dot{K},\mathcal{U}) &= lim\{\overline{r}_f(U,\dot{K})|U\in\mathcal{U}^s\} \\ &= lim\{\overline{r}_f(U,\dot{K})|U\in\mathcal{U}^o\} \\ &= lim\{\overline{s}_f(U,\dot{K})|U\in\mathcal{U}^s\} \\ &= lim\{\overline{s}_f(U,\dot{K})|U\in\mathcal{U}^o\}\ . \end{aligned}$$

For the proofs of above equalities one can refer Lemma 1 in [12].

Definition 2.3 ([12])**.** The number $h(f,\mathcal{U})$ defined by $sup\{h(f,\dot{K},\mathcal{U})|\dot{K}\in\mathcal{K}(X)\}$ is called the *uniform entropy* of f with respect to the uniformity $\mathcal{U}$.

Remark 2.4. For a uniformly continuous self-map of a complete metric space, the uniform entropy is equal to the Bowen's entropy [14].

Theorem 2.5. *Let $(X,\mathcal{U})$ be a uniform space and let $f : X \to X$ be a uniformly continuous map having topological specification property then $h(f,\mathcal{U})$ is positive.*

Proof. Let $x,y\in X$, $x\neq y$, U be a symmetric neighborhood of $\triangle_X$ such that $(x,y)\notin U^2 = U\circ U$ and M be a number as in the definition of topological specification property.

Consider two distinct $(n+1)$-tuples, $(z_0,z_1,z_2,...,z_n)$ and $(z'_0,z'_1,z'_2,...,z'_n)$ with $z_0 = x$, $z'_0 = y$, $z_i,z'_i\in\{x,y\}(1\leq i\leq n)$ and integers $a_0 = b_0 = 0$, $a_1 = b_1 = M$, $a_2 = b_2 = 2M$, ..., $a_n = b_n = nM$. Since f has topological specification property, there exist $z,z'\in X$ satisfying the definition of topological specification property. Note that $z\neq z'$. For if $z = z'$ then

$$\begin{gathered} F^i(z,z_i)\in U,\ a_j\leq i\leq b_j \text{ and } 0\leq i,j\leq n \text{ and} \\ F^i(z,z'_i)\in U,\ a_j\leq i\leq b_j \text{ and } 0\leq i,j\leq n. \end{gathered}$$

Therefore $F^{iM}(z,z_i)\in U$, $0\leq i\leq n$ and $F^{iM}(z,z'_i)\in U$, $0\leq i\leq n$. For $i=0$, $(z,z_0)\in U$ and $(z,z'_0)\in U$ implying that $(x,y)\in U^2$, which contradicts our choice of U.

Using similar arguments, one can prove that for distinct $(n+1)$-tuples $(z_0,z_1,z_2,...,z_n)$ with $z_i\in\{x,y\}(0\leq i\leq n)$ associated z are different. Thus there are at least 2^{n+1} points which are (nM,U)-separated which implies

$$\begin{aligned} h(f,\mathcal{U}) &= sup\{h(f,\dot{K},\mathcal{U})|\dot{K}\in\mathcal{K}(X)\} \\ &\geq \lim_{U\in\mathcal{U}^s}\overline{s}_f(U,\dot{K}) \\ &\quad= \lim_{n\to\infty} sup\tfrac{1}{n}log s_n(U,\dot{K},f) \\ &\quad= \lim_{n\to\infty} sup\tfrac{1}{nM}log 2^{n+1} \\ &\quad= \tfrac{log2}{M} > 0. \end{aligned}$$

□

The following example justifies that the converse of the above result is not true.

Example 2.6. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x) = 2x$, for all $x\in\mathbb{R}$, where the set of real numbers $\mathbb{R}$ has the usual uniformity $\mathcal{U}$. Note that the Bowen's entropy of f is $log2$ [11]. By Remark 2.4, $h(f,\mathcal{U}) = log2$. It is known that a map f having topological specification property has the dense set of periodic points [15]. Since $f(x) = 2x$ does not have dense set of periodic points therefore it does not have the topological specification property.

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