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Results about the Alexandroff duplicate space

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Abstract

In this paper, we present some new results about the Alexandroff Duplicate Space. We prove that if a space X has the property P, then its Alexandroff Duplicate space $A(X)$ may not have P, where P is one of the following properties: extremally disconnected, weakly extremally disconnected, quasi-normal, pseudocompact. We prove that if X is α-normal, epinormal, or has property wD, then so is $A(X)$. We prove almost normality is preserved by $A(X)$ under special conditions.

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There are various methods of generating a new topological space from a given one. In 1929, Alexandroff introduced his method by constructing the Double Circumference Space [1]. In 1968, R. Engelking generalized this construction to an arbitrary space as follows [6]: Let X be any topological space. Let $X' = X \times \{1\}$. Note that $X \cap X' = \varnothing$. Let $A(X) = X \cup X'$. For simplicity, for an element $x \in X$, we will denote the element $\langle x, 1\rangle$ in X' by x' and for a subset $B \subseteq X$, let $B' = \{x' : x \in B\} = B \times \{1\} \subseteq X'$. For each $x' \in X'$, let $\mathcal{B}(x') = \{\{x'\}\}$. For each $x \in X$, let $\mathcal{B}(x) = \{U \cup (U' \setminus \{x'\}) : U$ is open in X with $x \in U\}$. Then $\mathcal{B} = \{\mathcal{B}(x) : x \in X\} \cup \{\mathcal{B}(x') : x' \in X'\}$ will generate a unique topology on $A(X)$ such that $\mathcal{B}$ is its neighborhood system. $A(X)$ with this topology is called the *Alexandroff Duplicate of X*. Now, if P is a topological property and X has P, then $A(X)$ may or may not have P. Throughout this paper, we denote an ordered pair by $\langle x, y\rangle$, the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. For a subset A of a space X,

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int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinal α such that $\alpha < \gamma$. The first infinite ordinal is ω and the first uncountable ordinal is ω_1.

A topological space X is called *α-normal* [3] if for any two disjoint closed subsets A and B of X, there exist two open disjoint subsets U and V of X such that $A \cap U$ dense in A and $B \cap V$ dense in B.

Theorem 0.1. *If X is α-normal, then so is its Alexandroff Duplicate $A(X)$.*

Proof. Let E and F be any two disjoint closed sets in $A(X)$. Write $E = E_1 \cup E_2$, where $E_1 = E \cap X$, $E_2 = E \cap X'$ and $F = F_1 \cup F_2$, where $F_1 = F \cap X$, $F_2 = F \cap X'$. So, we have E_1 and F_1 are two disjoint closed sets in X. By α-normality of X, there exist two disjoint open sets U and V of X such that $E_1 \cap U$ is dense in E_1 and $F_1 \cap V$ is dense in F_1. Let $W_1 = (U \cup U' \cup E_2) \setminus F$ and $W_2 = (V \cup V' \cup F_2) \setminus E$. Then W_1 and W_2 are disjoint open sets in $A(X)$. Now, we prove $W_1 \cap E$ is dense in E. Note that $W_1 \cap E = (W_1 \cap E_1) \cup (W_1 \cap E_2) = (U \cap E_1) \cup E_2$, hence $\overline{W_1 \cap E} = \overline{(U \cap E_1) \cup E_2} = \overline{(U \cap E_1)} \cup \overline{E_2} \supset E_1 \cup \overline{E_2} \supset E$. Therefore, $W_1 \cap E$ is dense in E. Similarly, $W_2 \cap F$ is dense in F. Therefore, $A(X)$ is α-normal. $\square$

A space X is called *extremally disconnected* [5] if it is T_1 and the closure of any open set is open. Extremally disconnectedness is not preserved by the Alexandroff Duplicate space and here is a counterexample.

Example 0.2. Consider the Stone-Čech compactification space $\beta\omega$ which is compact Hausdorff, hence Tychonoff. It is well-known that $\beta\omega$ is extremally disconnected. Clearly ω is open in $A(\beta\omega)$ and $\overline{\omega}^{A(\beta\omega)} = \beta\omega$ which is not open in $A(\beta\omega)$.

A space X is called *weakly extremally disconnected* [8] if the closure of any open set is open. Weakly extremally disconnected is not preserved and the above example is a counterexample. The following question is interesting and still open: *"Does there exist a Tychonoff non-discrete space X such that $A(X)$ is extremally disconnected?"*.

A subset B of a space X is called a *closed domain* [5] if $B = \overline{\text{int}B}$. A finite intersection of closed domains is called *π-closed* [9]. A topological space X is called *mildly normal* [7] if for any two disjoint closed domains A and B of X, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. A topological space X is called *quasi-normal* [9] if for any two disjoint π-closed subsets A and B of X, there exist two open disjoint subsets U and V of X such that $A \subseteq U$ and $B \subseteq V$. It is clear from the definitions that every quasi-normal space is mildly normal. Mild normality is not preserved by the Alexandroff Duplicate space [7]. Quasi-normality is not preserved by the Alexandroff Duplicate space and here is a counterexample. We denote the set of all limit points of a set B by B^d and call it *the derived set of B*.

Example 0.3. Consider $\mathbb{R}^{\omega_1}$, which is Tychonoff separable non-normal space ([5], 2.3.15). It is well-known that every closed domain in $\mathbb{R}^{\omega_1}$ depends on a countable set [12]. It follows that every π-closed set in $\mathbb{R}^{\omega_1}$ depends on a countable set. Now, if A and B are disjoint π-closed sets in $\mathbb{R}^{\omega_1}$, then there is a countable $S \subset \omega_1$ such that $A = \pi_S(A) \times \mathbb{R}^{\omega_1 \setminus S}$ and $B = \pi_S(B) \times \mathbb{R}^{\omega_1 \setminus S}$, where π_S is the projection function $\pi_S : \mathbb{R}^{\omega_1} \longrightarrow \mathbb{R}^S$. It follows that $\pi_S(A)$ and $\pi_S(B)$ are disjoint closed sets in $\mathbb{R}^S$. Since S is countable, $\mathbb{R}^S$ is metrizable. So, there exist two open disjoint sets $U_1, V_1 \subset \mathbb{R}^S$ such that $\pi_S(A) \subseteq U_1$ and $\pi_S(B) \subseteq V_1$. Then $A \subseteq U = U_1 \times \mathbb{R}^{\omega_1 \setminus S}$ and $B \subseteq V = V_1 \times \mathbb{R}^{\omega_1 \setminus S}$ where U and V are open in $\mathbb{R}^{\omega_1}$ and disjoint. Therefore, $\mathbb{R}^{\omega_1}$ is quasi-normal.

We show that the Alexandroff Duplicate space $A(\mathbb{R}^{\omega_1})$ is not quasi-normal by showing that it is not mildly normal. Let

$$E = \{\langle n_\xi : \xi < \omega_1 \rangle \in \mathbb{N}^{\omega_1} : \forall\, m \in \mathbb{N} \setminus \{1\}(|\{\xi < \omega_1 : n_\xi = m\}| \leq 1)\}.$$

$$F = \{\langle n_\xi : \xi < \omega_1 \rangle \in \mathbb{N}^{\omega_1} : \forall\, m \in \mathbb{N} \setminus \{2\}(|\{\xi < \omega_1 : n_\xi = m\}| \leq 1)\}.$$

E and F are disjoint closed subsets in $\mathbb{N}^{\omega_1}$, hence closed in $\mathbb{R}^{\omega_1}$. They cannot be separated by disjoint open sets, see [14], and they are perfect, i.e., $E = E^d$ and $F = F^d$. By a theorem from [7] which says: *"If A and B are disjoint subsets of a space X such that A^d and B^d cannot be separated, then $A(X)$ is not mildly normal."*, we conclude that $A(\mathbb{R}^{\omega_1})$ is not mildly normal.

A space X is *pseudocompact* [5] if X is Tychonoff and any continuous real-valued function defined on X is bounded. Equivalently, a Tychonoff space X is pseudocompact if and only if any locally finite family consisting of non-empty open subsets is finite [5]. We will conclude that pseudocompactness is not preserved by the Alexandroff Duplicate space by the following theorem.

Theorem 0.4. *Let X be a Tychonoff space. The Alexandroff Duplicate $A(X)$ is pseudocompact if and only if X is countably compact.*

Proof. If X is not countably compact, then there exists a countably infinite closed discrete subset D of X. Then $D \times \{1\} = D'$ is closed and open set in $A(X)$ which is also countably infinite, hence $A(X)$ is not pseudocompact. Now, assume that X is countably compact. Since for a set $B \subseteq X$ and a point $a \in X$, we have that a is a limit point for B' if and only if a is a limit point for B, we do have that $A(X)$ is also countably compact and hence pseudocompact. $\square$

So, any pseudocompact space which is not countably compact will be an example of a pseudocompact space whose Alexandroff Duplicate space $A(X)$ is not pseudocompact. A Mrówka space $\Psi(\mathcal{A})$, where $\mathcal{A} \subset [\omega]^\omega$ is maximal, is such a space, see [4] and [10].

Arhangel'skii introduced the notions of epinormality and C-normality in 2012, when he was visiting the department of mathematics, King Abdulaziz University, Jeddah, Saudi Arabia. A topological space X is called *C-normal* [2] if there exist a normal space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f_{|C} : C \longrightarrow f(C)$ is a homeomorphism for each compact subspace $C \subseteq X$. It was proved in [2] that if X is C-normal, then so is its Alexandroff Duplicate. A topological space $(X, \mathcal{T})$ is called *epinormal* [2] if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4.

Theorem 0.5. *If X is epinormal, then so is its Alexandroff Duplicate $A(X)$.*

Proof. Let X be any space which is epinormal, let $\mathcal{T}$ be a topology on X, since X is epinormal, then there is a coarser topology $\mathcal{T}^*$ on X such that $(X, \mathcal{T}^*)$ is T_4. Since T_4 is preserved by the Alexandroff Duplicate space, then $A(X, \mathcal{T}^*)$ is also T_4 and it is coarser than $A(X, \mathcal{T})$ by the topology of the Alexandroff Duplicate. Hence, $A(X)$ is epinormal. □

A space X is said to satisfy *Property wD* [11] if for every infinite closed discrete subspace C of X, there exists a discrete family $\{U_n : n \in \omega\}$ of open subsets of X such that each U_n meets C at exactly one point.

Theorem 0.6. *If X satisfies property wD, then so does its Alexandroff Duplicate $A(X)$.*

Proof. Let X be any space which satisfies property wD and consider its Alexandroff Duplicate $A(X)$. To show that $A(X)$ has Property wD, let $C \subseteq A(X)$ be any infinite closed discrete subspace of $A(X)$. Write $C = (C \cap X) \cup (C \cap X')$. For each $x \in C \cap X$, fix an open set U_x in X such that $V_x = U_x \cup (U'_x \setminus \{x'\})$ open in $A(X)$ and

$$(*) \qquad V_x \cap C = \{x\}$$

Case 1: $C \cap X$ is finite. This implies that $C \cap X'$ is infinite. Let $\{x'_n : n \in \omega\} \subseteq C \cap X'$ such that $x'_i \neq x'_j$, for all $i, j \in \omega$ with $i \neq j$. Now, consider the family $\{\{x'_n\} : n \in \omega\}$, then it consists of open sets and each $\{x'_n\}$ meets C at exactly one point. Now, we will show that $\{\{x'_n\} : n \in \omega\}$ is a discrete family in $A(X)$. It is obvious that $\{x'_n : n \in \omega\}$ is discrete and it is closed because if $x \in A(X) \setminus C$, then there is open set U_x containing x such that $U_x \cap C = \varnothing$, and if $x \in C \cap X$ hence, by (*) there is an open set V_x in $A(X)$ containing x such that $V_x \cap C = \{x\}$. Thus, $\{\{x'_n\} : n \in \omega\}$ is discrete family. Therefore, in this case, $A(X)$ satisfies property wD.

Case 2: $C \cap X$ is infinite. Then $C \cap X$ is an infinite closed discrete subspace of X. Since X satisfies the property wD, then there exists a discrete family $\{V_n : n \in \omega\}$ of open subsets of X such that each V_n meets $C \cap X$ at exactly one point $\{x_n\}$. Hence, $\{V_n \cup (V'_n \setminus \{x'_n\}) : n \in \omega\}$ is discrete in $A(X)$ and then meets C in exactly one point $\{x_n\}$. Therefore, also in this case, $A(X)$ satisfies the property wD. □

A space X is called *almost normal* [8] if for any two disjoint closed subsets A and B of X one of which is a closed domain there exist two disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.

Proposition 0.7. *Let X be a topological space. The following are equivalent:*

(1) *X is the only non-empty closed domain in X.*
(2) *Each non-empty open subset is dense in X.*
(3) *The interior of each non-empty proper closed subset of X is empty.*

Proof. (1) $\Longrightarrow$ (2) Let U be any non-empty open set. Suppose U is not dense in X, then $\overline{U}$ is a non-empty proper closed domain in X, which contradicts the hypothesis, hence U is dense in X.
(2) $\Longrightarrow$ (3) Suppose that E is a non-empty proper closed subset of X such that $\mathrm{int}E \neq \varnothing$, then $X = \overline{\mathrm{int}E} \subseteq \overline{E} = E$ which is a contradiction.
(3) $\Longrightarrow$ (1) Suppose that there exists a closed domain B such that $\varnothing \neq B \neq X$, then $\varnothing \neq \overline{\mathrm{int}B} \neq X$, which contradicts the hypothesis, thus X is the only non-empty closed domain in X. □

It is clear that any space that satisfies the conditions of Proposition 7 will be almost normal.

Corollary 0.8. *If X satisfies the conditions of Proposition 7, then its Alexandroff Duplicate $A(X)$ is almost normal.*

Proof. Let E and F be any non-empty disjoint closed subsets of $A(X)$ such that E is a closed domain. Let W be an open set in $A(X)$ such that $\overline{W} = E$. If $W \cap X \neq \varnothing$, then $W \cap X$ is dense in X by Proposition 7, so $X \subset \overline{W} = E$. It follows that $F \subset X'$, so E is closed and open, hence there are two disjoint open sets $U = A(X) \setminus F$ and $V = F$ in $A(X)$ containing E and F respectively.

If $W \cap X = \varnothing$, then $E \subset X'$, so E is closed and open, thus there are two disjoint open sets $U = E$ and $V = A(X) \setminus E$ in $A(X)$ containing E and F respectively. □

Corollary 0.9. *The Alexandroff Duplicate of the countable complement space* [13]*, the finite complement space* [13]*, and the particular point space* [13] *are all almost normal.*

The following problems are still open:

(1) If X is almost normal, is then its Alexandroff Duplicate $A(X)$ almost normal?
(2) If X is β-normal [3], is then its Alexandroff Duplicate $A(X)$ β-normal?

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