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A decomposition of normality via a generalization of κ-normality

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Abstract

A simultaneous generalization of κ-normality and weak θ-normality is introduced. Interrelation of this generalization of normality with existing variants of normality is studied. In the process of investigation a new decomposition of normality is obtained.

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Keywords: *regularly open set; regularly closed set; θ-open set; θ-closed set; κ-normal (mildly) normal space; almost normal space; (weakly) (functionally) θ-normal space; weakly κ-normal space; Δ-normal space; strongly seminormal space.*

1. Introduction and Preliminaries

Several generalized notions of normality such as almost normal, κ-normal, Δ-normal, θ-normal, semi-normal, Quasi-normal, π-normal, densely normal etc. exist in the literature. Recently, Interrelation among some of these variants of normality was studied in [4] and factorizations of normality are obtained in [4, 5, 12, 14]. In this paper, we tried to exhibit the interrelations that exist among these generalized notions of normality and introduced a simultaneous generalization of κ-normality and weak θ-normality called weak κ-normality. Intrestingly, the class of weakly κ-normal spaces contains the class of almost compact spaces whereas the class of κ-normal spaces does not contain the class of almost compact spaces. This newly introduced notion of weak normality

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is utilized to obtain a factorization of normality. Moreover, it is verified that some covering properties which need not imply κ-normality implies weak κ-normality.

Let X be a topological space and let $A \subset X$. Throughout the present paper, the closure and interior of a set A will be denoted by $\overline{A}$ (or clA) and $intA$ (or A^o) respectively. A set $U \subset X$ is said to be *regularly open* [15] if $U = int\overline{U}$. The complement of a regularly open set is called *regularly closed*. It is observed that an intersection of two regularly closed sets need not be regularly closed. A finite union of regular open sets is called π*-open* set and a finite intersection of regular closed sets is called π*-closed* set. It is obvious that the complement of a π-open set is π-closed and the complement of a π-closed set is π-open, the finite union (intersection) of π-closed sets is π closed, but the infinite union (intersection) of π-closed sets need not be π-closed (See [11]). A point $x \in X$ is called a θ*-limit point* (respectively δ*-limit point*) [21] of A if every closed (respectively regularly open) neighbourhood of x intersects A. Let $cl_\theta A$ (respectively $cl_\delta A$) denotes the set of all θ-limit point (respectively δ-limit point) of A. The set A is called θ*-closed* (respectively δ*-closed*) if $A = cl_\theta A$ (respectively $A = cl_\delta A$). The complement of a θ-closed (respectively δ-closed) set will be referred to as a θ*-open* (respectively δ*-open)* set. The family of θ-open sets as well as the family of δ open sets form topologies on X. The topology formed by the set of δ-open sets is the semiregularization topology whose basis is the family of regularly open sets.

Let Y be a subspace of X. A subset A of X is concentrated on Y [2] if A is contained in the closure of $A \cap Y$ in X. A subset A of Y is said to be strongly concentrated on Y [6] if $A \subset \overline{(A \cap Y)^o}$. It is obvious that every strongly concentrated set is concentrated. We say that X is normal on Y if every two disjoint closed subsets of X concentrated on Y can be separated by disjoint open neighbourhoods in X [2]. Similarly, X is said to be weakly normal on Y [6] if for every disjoint closed subsets A and B of X strongly concentrated on Y, there exist disjoint open sets in X separating A and B respectively.

A space X is called *densely normal* if there exists a dense subspace Y of X such that X is normal on Y [2]. A topological space X is said to be *weakly densely normal* [6] if there exist a proper dense subspace Y of X such that X is weakly normal on Y. It is easy to see that every densely normal space is weakly densely normal and every weakly densely normal space is κ-normal. On the other hand, the converses are not true, as were shown in [10] and [6].

Lemma 1.1. *A subset A of a topological space X is θ-open if and only if for each $x \in A$, there is an open set U such that $x \in U \subset \overline{U} \subset A$.*

Definition 1.2. A topological space X is said to be

(i) *quasi-normal* [23] if any two disjoint π-closed subsets A and B of X there exist two open disjoint subsets U and V of X such that $A \subset U$ and $B \subset V$.

(ii) *π-normal* [11] if for any two disjoint closed subsets A and B of X one of which is π-closed, there exist two open disjoint subsets U and V of X such that $A \subset U$ and $B \subset V$.
(iii) *Δ-normal* [9] if every pair of disjoint closed sets one of which is δ-closed are contained in disjoint open sets.
(iv) *weakly Δ-normal* [9] if every pair of disjoint δ-closed sets are contained in disjoint open sets.
(v) *weakly functionally Δ-normal* (*wf Δ-normal*) [9] if for every pair of disjoint δ-closed sets A and B there exists a continuous function $f : X \to$ [0,1] such that $f(A) = 0$ and $f(B)= 1$.
(vi) *θ-normal* [12] if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets;
(vii) *weakly θ-normal* [12] if every pair of disjoint θ-closed sets are contained in disjoint open sets;
(viii) *functionally θ-normal* [12] if for every pair of disjoint closed sets A and B one of which is θ-closed there exists a continuous function $f : X \to$[0,1] such that $f(A) = 0$ and $f(B)$=1;
(ix) *weakly functionally θ-normal* (*wf θ-normal*) [12] if for every pair of disjoint θ-closed sets A and B there exists a continuous function $f : X \to$ [0,1] such that $f(A) = 0$ and $f(B)= 1$.
(x) *β-normal* [1] if for any two disjoint closed subsets A and B of X, there exist open sets U and V of X such that $A \cap U$ is dense in A, $B \cap V$ is dense in B and $\overline{U} \cap \overline{V} = \phi$.
(xi) *almost β-normal* [3] if for every pair of disjoint closed sets A and B, one of which is regularly closed, there exist open sets U and V such that $\overline{A \cap U} = A$, $\overline{B \cap V} = B$ and $\overline{U} \cap \overline{V} = \phi$.
(xii) *θ-regular* [12] if for each closed set F and each open set U containing F, there exists a θ-open set V such that $F \subset V \subset U$.
(xiii) *semi-normal* [22] if for every closed set F and each open set U containing F, there exists a regular open set V such that $F \subset V \subset U$.
(xiv) *almost normal* [18] if every pair of disjoint closed sets one of which is regularly closed are contained in disjoint open sets.
(xv) *mildly normal* [19] (or *κ-normal* [20]) if every pair of disjoint regularly closed sets are contained in disjoint open sets.
(xvi) *Δ-regular* [9] if for every closed set F and each open set U containing F, there exists a δ-open set V such that $F \subset V \subset U$.

2. Weakly κ-normal spaces

Definition 2.1. A θ-closed set A is said to be a *regularly θ-closed* set if $\overline{intA} = A$. The complement of a regularly θ-closed set will be *regularly θ open*.

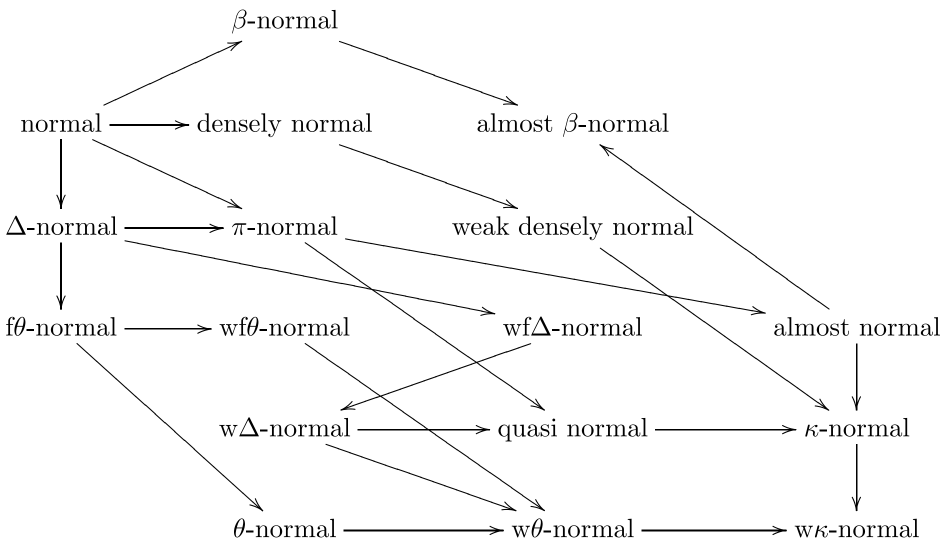
Clearly every regularly θ-closed set is regularly closed as well as θ-closed but the converse need not be true.

Example 2.2. Let X be the set of positive integers. Define a topology on X by taking every odd integer to be open and a set $U \subset X$ is open if for every even integer $p \in U$, the predecessor and the successor of p are also in U. Here the set $\{2k, 2k+1, 2k+2 : k \in Z^+\}$ is a regularly closed set which is not θ-closed.

Example 2.3. Let X denote the interior of the unit square S in the plane together with the points $(0,0)$ and $(1,0)$, i.e. $X = S^o \cup \{(0,0),(1,0)\}$. Every point in S^o has the usual Euclidean neighbourhoods. The points $(0,0)$ and $(1,0)$ have neighbourhoods of the form U_n and V_n respectively, where, $U_n = \{(0,0)\} \cup \{(x,y) : 0 < x < 1/2, 0 < y < 1/n\}$ and $V_n = \{(1,0)\} \cup \{(x,y) : 1/2 < x < 1, 0 < y < 1/n\}$. Clearly, the sets $\{(0,0)\}$ and $\{(1,0)\}$ are θ-closed but not regularly θ-closed.

Definition 2.4. A topological space X is said to be *weakly κ-normal* if for every pair of disjoint regularly θ-closed sets A and B there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

From the definitions it is obvious that every κ-normal space is weakly κ-normal and every weakly θ-normal space is weakly κ-normal. The following diagram illustrates the interrelations that exist between weakly κ-normal spaces and variants of normality that exist in literature. But none of the implications below is reversible (See [7], [9], [11], [12], [14], [18] and examples below).



Example 2.5. The space defined in Example 2.2 is weakly κ-normal but not κ-normal.

Example 2.6. The example of a Tychonoff κ-normal space which is not densely normal was given by Just and Tartir [10]. Since every regular space is θ-regular [12], this space is θ-regular but not normal. Thus the space is not weakly θ-normal as every θ-regular, weakly θ-normal space is normal [12].

Theorem 2.7. *A topological space X is weakly κ-normal if and only if for every regularly θ-closed set A and a regularly θ-open set U containing A there is an open set V such that $A \subset V \subset \overline{V} \subset U$.*

Proof. Let X be a weakly κ-normal space and U be a regularly θ-open set containing a regularly θ-closed set A. Then A and $X - U$ are disjoint regularly θ-closed sets in X. Since X is weakly κ-normal, there are disjoint open sets V and W containing A and $X - U$, respectively. Then $A \subset V \subset X - W \subset U$. Since $X - W$ is closed, $A \subset V \subset \overline{V} \subset U$. Conversely, let A and B be two disjoint regularly θ-closed sets in X. Then $U = X - B$ is a regularly θ-open set containing the regularly θ-closed set A. Thus by the hypothesis there exists an open set V such that $A \subset V \subset \overline{V} \subset U$. Then V and $X - \overline{V}$ are disjoint open sets containing A and B, respectively. Hence X is weakly κ-normal. □

Theorem 2.8. *Let X be a finite topological space. For a subset A of X, the following statements are equivalent.*
(a) A is clopen.
(b) A is θ-closed.
(c) A is θ-open.

Proof. The implication $(a) \implies (b)$ is obvious. To prove $(b) \implies (a)$, let A be a closed subset of X. Then $(X - A)$ is θ-open in X. By Lemma 1.3.4, for each $x \in X - A$ there exists an open set U_x containing x such that $x \in U_x \subset \overline{U_x} \subset X - A$. Since X is finite, $\bigcup_{x \in X - A} \overline{U_x} = X - A$, is the union of finitely many closed sets and hence closed. Thus A is open. By hypothesis A is θ-closed and hence closed. Consequently, A is clopen. The proofs of $(a) \implies (c)$ and $(c) \implies (a)$ are similar and hence omitted. □

From the above result the following observation is obvious.

Remark 2.9. Every finite topological space is weakly κ-normal whereas finite topological spaces need not be κ-normal.

Theorem 2.10 ([13])**.** *A space X is almost regular if and only if for every open set U in X, $int\overline{U}$ is θ-open.*

Theorem 2.11. *In an almost regular space, the following statements are equivalent*
(a) X is κ-normal.
(b) X is weakly κ-normal.

Proof. The proof of $(a) \implies (b)$ directly follows from definitions. To prove $(b) \implies (a)$, let X be an almost regular, weakly κ-normal space. Let A and B be two disjoint regularly closed sets in X. By Theorem 2.10, A and B are disjoint regularly θ-closed sets in X. Thus by weak κ-normality of X, there exist disjoint open sets separating A and B. Hence X is κ-normal. □

Theorem 2.12. *In an almost regular space, every π-closed set is θ-closed.*

Proof. Let X be an almost regular space and let $A \subset X$ be π-closed in X. Thus A is finite intersection of π-closed sets in X. Since in an almost regular space every regularly closed set is θ-closed [16] and finite intersection of θ-closed sets is θ-closed [21], A is θ-closed. □

Theorem 2.13. *Every almost regular, weakly θ-normal space is quasi normal.*

Proof. Let X be an almost regular, weakly θ-normal space. Let A and B be two disjoint π-closed sets in X. By Theorem 2.12, A and B are disjoint θ-closed sets which can be separated by disjoint open set as X is weakly θ-normal. □

Theorem 2.14. *Every almost regular, θ-normal space is π-normal.*

It is well known that every compact Hausdorff space is normal. However, in the absence of Hausdorffness or regularity a compact space may fail to be normal. Thus it is useful to know which topological property weaker than Hausdorffness with compactness implies normality. The property of being a T_1 space fails to do the job since the cofinite topology on an infinite set is a compact T_1 space which is not normal. However, it is well known that Every compact R_1-space is normal (See [17]). In [12], it is shown that every compact space in particular every paracompact space in absence of any separation axioms is θ-normal. It is also known that every Lindelöf spaces need not be κ-normal. However, by the following theorem of [12] it follows that every Lindelöf space is weakly κ-normal. Similarly, almost compactness need not implies κ-normality, but by Theorem of [12], every almost compact space is weakly κ-normal.

Theorem 2.15 ([12]). *Every Lindelöf space is weakly θ-normal.*

Corollary 2.16. *Every Lindelöf space is weakly κ-normal.*

Corollary 2.17. *Every almost regular, Lindelöf space is κ-normal.*

Proof. The prove immediately follows from Theorem 2.11, since in an almost regular space every weakly κ-normal space is κ-normal. □

Theorem 2.18 ([12]). *Every almost compact space is weakly θ-normal.*

Corollary 2.19. *Every almost compact space is weakly κ-normal.*

Corollary 2.20. *Every almost regular, almost compact space is κ-normal.*

Proof. The prove immidiately follows from Theorem 2.11, since in an almost regular space every weakly κ-normal space is κ-normal. □

Remark 2.21. Corollary 2.17 and Corollary 2.20 were independently prooved in [18]. In contrary to the above results the following example establishes that Lindelöf spaces need not be κ-normal and almost compactness need not imply κ-normality.

Example 2.22. Let X be the set of positive integers with the topology as defined in Example 2.2 and $Y = \{1, 2, 3, ..., 11\}$. Then the subspace topology on Y is compact but not κ-normal as disjoint regularly closed sets $\{2, 3, 4\}$ and $\{6, 7, 8\}$ can not be separated by disjoint open sets.

Definition 2.23 ([13]). A space X is said to be θ-compact if every open covering of X by θ-open sets has a finite subcollection that covers X.

The following result is useful to show that every almost regular, θ-compact space is κ-normal as well as weakly θ-normal.

Theorem 2.24 ([16]). *Let $A \subset X$ be θ-closed and let $x \notin A$. Then there exist regular open sets which separate x and A.*

Theorem 2.25. *In an almost regular space, every θ-compact space is weakly θ-normal.*

Proof. Let X be an almost regular θ-compact space. Let A and B be any two disjoint θ-closed subsets of X. By Theorem 2.24, for every $a \in A$, there exist disjoint regularly open sets U_a and V_a containing a and B respectively. Since X is almost regular, U_a and V_a are disjoint θ-open sets containing a and B. Now the collection $\{U_a : a \in A\}$ is a θ-open cover of A. Then $A \subset \bigcup\limits_{a \in A} U_a = O$. Since arbitrary union of θ-open sets is θ-open, $X - O = D$ is θ-closed. Since A is a θ-closed set disjoint from D, by Theorem 2.24, for every $d \in D$, there exist disjoint regularly open sets S_d and T_d containing A and d respectively. Again by almost regularity of X, T_d is a θ-open set which is disjoint from A. Now the collection $\mathcal{U} = \{U_a : a \in A\} \cup \{T_d : d \in D\}$ is a θ-open covering of X. By θ-compactness of X, $mathcalU$ has a finite subcollection $\mathcal{V}$ which covers X. Let the members of $\mathcal{V}$ which intersects A be $\mathcal{W}$. Each member of $\mathcal{W}$ is of the form U_a for some $a \in A$ as for each $d \in D$, $T_d \cap A = \phi$. Suppose $\mathcal{W} = \{U_{a_i} : i = 1, 2, 3, ..., n\}$. Then $\bigcup\limits_{i=1}^{n} U_{a_i} = U$ and $\bigcap\limits_{i=1}^{n} V_{a_i} = V$ are disjoint open sets containing A and B respectively. Hence X is weakly θ-normal. □

Corollary 2.26. *In an almost regular space, every θ-compact space is weakly κ-normal.*

Proof. The proof immidiately follows from the fact that every θ normal space is weakly κ-normal. □

Corollary 2.27. *In an almost regular space, every θ-compact space is κ-normal.*

Proof. The proof immidiately follows from Theorem 2.11. □

Corollary 2.28. *In an almost regular space, every almost compact space is weakly κ-normal.*

Proof. The proof is immediate as every almost compact space is θ-compact [13]. □

3. Decompositions of normality

Theorem 3.1. *An T_1-space is almost normal if and only if it is almost β-normal and weakly κ-normal.*

Proof. Necessary part is obvious. Conversely, let X be a T_1-amlost β normal, weakly κ-normal space. Since X is T_1-almost β-normal, by Theorem 2.9 of [3], X is almost regular. So by Theorem 2.11, X is κ- normal. Hence X is almost normal as every almost β-normal, κ-normal space is almost normal [3]. □

Corollary 3.2. *An T_1 space is normal if and only if it is almost β-normal, weakly κ-normal and semi-normal.*

Proof. The prove follows from the result that every almost normal, semi normal space is normal [18]. □

Definition 3.3. A space X is said to be strongly seminormal if for every closed set A contained in an open set U there exists a regularly θ-open set V such that $A \subset V \subset U$.

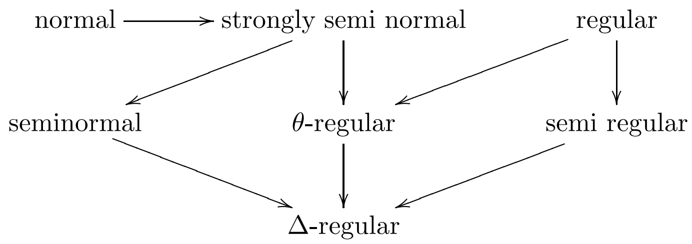
Theorem 3.4. *Every normal space is strongly seminormal.*

Proof. Let A be a closed set and U be an open set containing A. Let $B = X - U$. Then A and B are disjoint closed sets in X. By Urysohn's lemma there exists a continuous function $f : X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Let $V = f^{-1}[0,1/2)$ and $W = f^{-1}(1/2,1]$. Then $A \subset V \subset X - W \subset U$. Thus $A \subset V \subset \overline{V}^o \subset X - W \subset U$. We claim that $\overline{V}^o$ is a regularly θ-open set. $\overline{V}^o$ is regularly open, only we have to show that $\overline{V}^o$ is θ-open. let $x \in \overline{V}^o$. Then $f(x) \in [0,1/2)$. So there is a closed neighbourhood N of $f(x)$ contained in $[0,1/2)$. Let $U_x = (f^{-1}(N))^o$. Then $x \in U_x \subset f^{-1}(N) \subset \overline{V}^o$. By Lemma 1.1, $\overline{V}^o$ is θ-open. Hence X is strongly seminormal. □

Theorem 3.5. *Every strongly seminormal space is seminormal.*

Theorem 3.6. *Every strongly seminormal space is θ-regular.*

The following implications are obvious but none of these is reversible.



Example 3.7. Let X be the set of positive integers with the topology as defined in Example 2.2, then X is seminormal but not strongly seminormal.

Example 3.8. The space given in [10] by Just and Tartir is an example of a Tychonoff κ-normal space which is not densely normal. Since every seminormal κ-normal space is normal [18], thus becoming densely normal, this space is not seminormal but is θ-regular as every regular space is θ-regular.

Theorem 3.9. *A space X is normal if and only if it is strongly seminormal and weakly κ-normal.*

Proof. The necessary part i.e., a normal space is strongly seminormal as well as weakly κ-normal directly follows from the definition. Conversely, let X be a strongly seminormal and weakly κ-normal space. let A and B be two disjoint closed sets in X. Thus A is a closed set contained in an open set $U = X - B$. Since X is strongly seminormal, there exists an regularly θ-open set V such that $A \subset V \subset U$. Now $X - V$ is a regularly θ-closed set contained in an open set $X - A$. Again by strong seminormality of X, there exists a regularly θ-open set W such that $X - V \subset W \subset X - A$. Thus $X - V$ and $X - W$ are two disjoint regularly θ-closed sets in X containing B and A respectively. By weak κ-normality of X, there exist two disjoint open sets O and P separating $X - W$ and $X - V$. Hence X is normal. □

Corollary 3.10. *In the class of strongly seminormal spaces, the following statements are equivalent.*
(a) X is normal.
(b) X is Δ-normal.
(c) X is wfΔ-normal.
(d) X is weakly Δ-normal.
(e) X is functionally θ-normal.
(f) X is θ-normal.
(g) X is weakly functionally θ-normal.
(h) X is weakly θ-normal.
(i) X is π-normal.
(j) X is quasi normal.
(k) X is almost normal.
(l) X is κ-normal.
(m) X is weakly κ-normal.

Remark 3.11. In [9], it is shown that in the class of Δ-regular spaces statements (a)-(d) of Corollary 3.10 are equivalent and in the class of θ-regular spaces statements (a)-(h) are equivalent.

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