Applied General Topology

Appl. Gen. Topol. 18, no. 1 (2017), 23-30
doi:10.4995/agt.2017.4092

A note on unibasic spaces and transitive quasi-proximities

Adalberto García-Máynez [a] and Adolfo Pimienta Acosta [b,*]

[a] Instituto de Matemáticas, Universidad Nacional Autónoma de México; Area de la Investigación Científica Circuito Exterior, Ciudad Universitaria Coyoacán, 04510. México, D. F. (agmaynez@matem.unam.mx)
[b] Departamento de Matemáticas,Universidad Autónoma Metropolitana, prolongación canal de miramontes #3855, Col. Ex-Hacienda San Juan de Dios, Delegación Tlalpan, C.P. 14387, México, D.F-México (pimienta@xanum.uam.mx)

Communicated by M. Sanchis

Abstract

In this paper we prove there is a bijection between the set of all annular bases of a topological spaces (X, τ) and the set of all transitive quasi-proximities on X inducing τ. We establish some properties of those topological spaces (X, τ) which imply that τ is the only annular basis

2010 MSC: *Primary 54E05; 54E15; 54D35; Secondary 05C50.*

Keywords: *annular basis; entourage; semi-block; quasi-proximity; transitive quasi-proximity-uniformity; unibasic spaces.*

1. Introduction

W. J. Pervin showed in [9] that every topological spaces (X, τ) has a quasi-proximity δ which induces the original topology. In this paper we give conditions for a topological space (X, τ) admits a unique compatible quasi-proximity in which the topology is the only annular basis.

*The second author was supported in part by Universidad de la Costa(CUC-www.cuc.edu.co) under grant of Department of Exact and Natural Sciences, address street 58 # 55-66, Barranquilla, Colombia.

Received 27 August 2015 – Accepted 06 November 2016

By a quasi-proximity (see [1]) on a set X we will mean a relation δ between the family of subsets of X satisfying the following axioms:

a) $(X, \varnothing) \notin \delta$ and $(\varnothing, X) \notin \delta$;
b) $(C, A \cup B) \in \delta$ if only if $(C, A) \in \delta$ or $(C, B) \in \delta$;
c) $(A \cup B, C) \in \delta$ if only if $(A, C) \in \delta$ or $(B, C) \in \delta$;
d) For every $x \in X$, $(\{x\}, \{x\}) \in \delta$;
e) If $(A, B) \notin \delta$, there exists a set $C \subseteq X$ such that $(A, C) \notin \delta$ and $(X \setminus C, B) \notin \delta$.

A quasi-proximity δ on X is a *proximity* on X if $\delta = \delta^{-1}$, i.e., $(A, B) \in \delta$ iff $(B, A) \in \delta$.

For brevity, we write $A\delta B$ instead of $(A, B) \in \delta$ and $A\overline{\delta}B$ instead of $(A, B) \notin \delta$.

Let δ be a quasi-proximity on a set X. For each $A \subseteq X$, define $\widetilde{A} = \{x \in X \colon \{x\}\delta A\}$. Then the assignment $A \to \widetilde{A}$ is a *Kuratowski-closure operator* on X and the corresponding topology on X is denoted as τ_δ (see [1]) , 1.27).

H.-P. Künzi and M. J. Pérez-Peñalver in [6] prove some interesting results about the number of quasi-proximities that a topological spaces admits. H.-P. Künzi in [3] studies the number of quasi-uniformities belonging to the Pervin quasi-proximity class.

J. Ferrer in [2] trying to solve the question of whether every T_1 topological space with a unique compatible quasi-proximity should be hereditarily compact, he shows that it is true for product spaces as well as for locally hereditarily Lindelöf spaces.

H.-P. Künzi and S. Watson in [7] construct a T_1-space X is not hereditarily compact, but each open susbset of X is the intersection of two compact open sets. The construction is carried out in ZFC, but the cardinality of the space is very large.

2. Unibasic spaces and transitive quasi-proximities

The main result of this section establishes a bijection between all annular bases of a topological space (X, τ) and all transitive quasi-proximities on X inducing τ.

A basis $\mathcal{B}$ for a topological space (X, τ) is *annular* if it satisfies the following conditions:

i) $\varnothing \in \mathcal{B}$ y $X \in \mathcal{B}$;
ii) $B_1, B_2 \in \mathcal{B}$ implies that $B_1 \cap B_2 \in \mathcal{B}$ and $B_1 \cup B_2 \in \mathcal{B}$.

Definition 2.1.

(1) An open set V in (X, τ) is *everywhere basic* (*e.b.*) if V belongs to every annular basis of X.
(2) A topological space (X, τ) is *unibasic* if τ is the only annular basis of X.
(3) (X, τ) is *minimally basic* if X has annular basis $\mathcal{B}_0$ which is contained in every other annular basis $\mathcal{B}$ of X.

Remark 2.2.

i) Every element of a minimum annular basis $\mathcal{B}_0$ of X is *e.b.* and every unibasic space is minimally basic.
ii) Every open and compact subset of a topological space X is *e.b.*. Hence, every hereditarily compact space is unibasic.

Lemma 2.3. *Let $\mathcal{B}$ be an annular basis of a topological space (X,τ). Define $A\delta B$ iff $A\cap H\neq\varnothing$ for every $H\in C(\mathcal{B})$ which contains B. Then δ is a transitive quasi-proximity on X which induces τ.*

Proof. Clearly $X\overline{\delta}\varnothing$ and $\varnothing\overline{\delta}X$. If $(A\cup B)\delta C$, we must have $A\delta C$ or $B\delta C$. Indeed, $A\overline{\delta}C$ and $B\overline{\delta}C$ imply the existence of $H_1,H_2\in C(\mathcal{B})$ such that $H_1\cap H_2\supseteq C$, $A\cap H_1=\varnothing=B\cap H_2$. Therefore $(A\cup B)\cap H_1\cap H_2=\varnothing$ and $H_1\cap H_2$ is an element of $C(\mathcal{B})$ containing C, that, $(A\cup B)\overline{\delta}C$, a contradiction. In a similar way one may prove that $C\delta(A\cup B)$ implies that $C\delta A$ or $C\delta B$. It is obvious that $\{x\}\delta\{x\}$ for each $x\in X$. Finally, suppose that $A\overline{\delta}B$. Therefore, there exists an element $H\in C(\mathcal{B})$ such that $H\supseteq B$ and $A\cap H=\varnothing$. Therefore, $A\overline{\delta}H$ and $(X\setminus H)\overline{\delta}B$.

Observe now that $(X\setminus H)\overline{\delta}H$ for every $H\in C(\mathcal{B})$ and $T(X\setminus H,H)=X\times X\setminus[(X\setminus H)\times H]=(H\times X)\cup[X\times(X\setminus H)]$. Hence, if $A\overline{\delta}B$ and $H\in C(\mathcal{B})$ satisfies $B\subseteq H\subseteq X\setminus A$, we have $T(X\setminus H,H)\subseteq[(X\setminus A)\times X]\cup[X\times(X\setminus B)]=T(A,B)$. This proves that the quasi-uniformity $\mathcal{U}_\delta$ is transitive.

Finally, we must prove that $\tau_\delta=\tau$. For this, take any set $C\subseteq X$ and consider the set $C_1=\{x\in X\colon\{x\}\delta C\}$. It is enough to prove that $C_1=\overline{C}$. If $x\in X\setminus\overline{C}$, there exists a set $B\in\mathcal{B}$ such that $x\in B\subseteq X\setminus\overline{C}$. Therefore, $X\setminus B\in C(\mathcal{B})$ and $X\setminus B\supseteq C$, that is, $\{x\}\overline{\delta}C$ and $X\setminus\overline{C}\subseteq X\setminus C_1$. On the other hand, if $x\in X\setminus C_1$, i.e., if $\{x\}\overline{\delta}C$, there exists a set $H\in C(\mathcal{B})$ such that $H\supseteq C$ y $x\notin H$. Therefore, $x\in X\setminus\overline{C}$ and the proof is complete. □

A quasi-proximity δ on a set X is:

(1) *Point-symmetric* if $A\delta\{x\}$ implies $\{x\}\delta A$. Equivalently, δ is point-symmetric if $\tau_\delta\subseteq\tau_{\delta^{-1}}$.
(2) *Locally-symmetric* if $A\delta G$ for every τ-neighborhood G of x implies that $\{x\}\delta A$.

Notation 2.4. If $\mathcal{G}$ is a family of subsets of X, we define: $C(\mathcal{G})=\{H\colon X\setminus H\in\mathcal{G}\}$.

Let $\mathcal{B}$ be an annular basis of a topological space (X,τ) is:

i) *Disjunctive* (or a *Wallman basis*) if whenever $x\in B\in\mathcal{B}$, there exists an element $H_x\in C(\mathcal{B})$ such that $x\in H_x\subseteq B$.
ii) *Regular* if whenever $x\in B\in\mathcal{B}$, there exists an element $D\in\mathcal{B}$ and an element $H\in C(\mathcal{B})$ such that $x\in D\subseteq H\subseteq B$.
iii) *Normal* is for every pair H,K of disjoint elements of $C(\mathcal{B})$, there exists a pair B,D of disjoint elements of $\mathcal{B}$ such that $H\subseteq B$ and $K\subseteq D$.

Theorem 2.5. *Let $\mathcal{B}$ be an annular basis of a topological space (X,τ) and let δ be the quasi-proximity on X associated to $\mathcal{B}$. Then:*

i) *$\mathcal{B}$ is disjunctive iff δ is point-symmetric.*
ii) *$\mathcal{B}$ is regular iff δ is locallly symmetric.*
iii) *$\mathcal{B}$ is normal iff δ is of Wallman type*[1].

Proof. We prove only *iii*). Suppose δ is of Wallman type and let $H,K \in C(\mathcal{B})$ be disjoint. Since H and K are δ- remote, there exists a neighborhood G of H such that $H\overline{\delta}(X\setminus G)$ and $K\overline{\delta}G$. This last condition implies the existence of an elements of $H_1 \in C(\mathcal{B})$ such that $K \subseteq X\setminus H_1 \subseteq X\setminus G$. The first condition implies the existence of an element $K_1 \in C(\mathcal{B})$ such that $X\setminus G \subseteq K_1 \subseteq X\setminus H$. Hence, $X\setminus K_1$ and $X\setminus H_1$ are disjoint elements of $\mathcal{B}$ and $\mathcal{B}$ is normal.

Assume now that $\mathcal{B}$ is normal. Let A,B be δ-remote. Let $H,K \in C(\mathcal{B})$ be disjoint sets such that $A \subseteq H$ and $B \subseteq K$. Since $\mathcal{B}$ is normal, there exist disjoint elements $C,D \in \mathcal{B}$ such that $H \subseteq C$ and $K \subseteq D$. Defining $G = C$, we have $H\overline{\delta}(X\setminus G)$ and $K\overline{\delta}G$, i.e., δ is of Wallman type. □

Corollary 2.6. *Every transitive point-symmetric quasi-proximity of Wallman type is locally symmetric and its induced topology is completely regular.*

Lemma 2.7. *Let δ be a transitive quasi-proximity on a topological space (X,τ) and suppose that $\tau_\delta = \tau$. Then $\mathcal{B} = \{V \in \tau \colon V\overline{\delta}(X\setminus V)\}$ is an annular basis of (X,τ).*

Proof. Clearly $\varnothing \in \mathcal{B}$ and $X \in \mathcal{B}$. Suppose now that B_1,B_2 both belong to $\mathcal{B}$. If $B_1\cup B_2 \notin \mathcal{B}$, we would have $(B_1\cup B_2)\delta(X\setminus B_1)\cap(X\setminus B_2)$. Therefore $B_1\delta(X\setminus B_1)\cap(X\setminus B_2)$ or $B_2\delta(X\setminus B_1)\cap(X\setminus B_2)$. This would imply that $B_1\delta(X\setminus B_1)$ or $B_2\delta(X\setminus B_2)$, a contradicition. Hence, $B_1\cup B_2 \in \mathcal{B}$. In a similar fashion we prove that $B_1\cap B_2 \in \mathcal{B}$. It remains to prove that $\mathcal{B}$ is a basis of (X,τ). Suppose then that $x \in V \in \tau$. Therefore $\{x\}\overline{\delta}(X\setminus V)$ (recall $\tau_\delta = \tau$). Let $R \in \mathcal{U}_\delta$ be a transiive entourage contained in $T(\{x\},X\setminus V)$. Let us prove that $R(x)\subseteq V$. If $y \in R(x)$, we have $(x,y) \in R \subseteq T(\{x\},X\setminus V) = [(X\setminus\{x\})\times X]\cup[\{x\}\times V]$. Therefore, $(x,y) \in \{x\}\times V$, that is, $y \in V$. Besides, $R(x)\overline{\delta}(X\setminus R(x))$ because $R(x)\delta(X\setminus R(x))$ would imply that $[R(x)\times(X\setminus R(x))]\cap S \neq \varnothing$ for every $S \in \mathcal{U}_\delta$, and, in particular, $[R(x)\times(X\setminus R(x))]\cap R \neq \varnothing$. But since R is transitive, this last statement is clearly false. Hence, we must have that $R(x)\overline{\delta}(X\setminus R(x))$. Since this implies that $R(x)\cap\overline{(X\setminus R(x))} = \varnothing$, we deduce that $R(x)$ is open. Therefore, $R(x) \in \mathcal{B}$ and $\mathcal{B}$ is an annular basis of (X,τ). □

Let (X,τ) be a topological space with topology τ. for $G \in \tau$ let $S_G = (G\times G)\cup((X\setminus G)\times X)$. The filter generated by $\{S_G \colon G \in \tau\}$ is a quasi-uniformity $\mathcal{P}$ for X called *Pervin quasi-uniformity* (see [8]).

[1] Two sets $A,B \subseteq X$ are said to be *δ-remote* if there exist disjoint sets $H,K \subseteq X$ such that $A \subseteq H$, $B \subseteq K$, $(X\setminus H)\overline{\delta}H$ and $(X\setminus K)\overline{\delta}K$. A quasi-proximity δ on a set X is of *Wallman type* if for every pair of δ-remote sets A,B, there exists a neighborhood G of A such that $A\overline{\delta}(X\setminus G)$ and $B\overline{\delta}G$.

Theorem 2.8. *Let (X, τ) be a topological space.Then there exists a bijective correspondence between the collection of annular bases of (X, τ) and the collection of totally bounded transitive quasi-proximities on X which induce τ. Hence, (X, τ) is minimally basic iff the family of totally bounded transitive quasi-uniformities on X inducing τ has a minimum element and (X, τ) is unibasic iff $\mathcal{P} = \mathcal{U}_{\delta_0}$ is the only totally bounded transitive quasi-uniformity inducing τ.*

Theorem 2.9. *Let B be an everywhere-basic set on a topological space (X, τ) and suppose that $B \neq X$. If $K \subseteq X$ is closed and $K \subseteq B$, then K is compact.*

Proof. Suppose that K is not compact. Then there exists a family $\mathcal{G} = \{B_i : i \in J\} \subseteq \tau$ such that $K \subseteq \cup\{B_i : i \in J\} \subseteq B$, but for each finite subset $J_0 \subseteq J$, we have $K \setminus \cup\{B_i : i \in J_0\} \neq \varnothing$. If $\mathcal{B}' = \{L \in \tau : L \subseteq \cup\{B_i : i \in J_0\}$ for some $J_0 \subseteq J$, finite$\}$ and let $\mathcal{B}'' = \{L \in \tau : L \cap K = \varnothing\} \cup \{X\}$, it is easy to check that $\mathcal{B} = \{L_1 \cup L_2 : L_1 \in \mathcal{B}'$ and $L_2 \in \mathcal{B}''\}$ is an annular basis of (X, τ).But $B \notin \mathcal{B}$, contradicting the fact that B is everywhere basic. Hence, K must be compact. □

Definition 2.10. A topological space (X, τ) is $\mathcal{R}_0$ if whenever $x \in V \in \tau$ there exists a closed set H_x such that $x \in H_x \subseteq V$ and (X, τ) is $\mathcal{R}_1$ if whenever $x, y \in X$ and $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets V, W such that $x \in V$ and $y \in W$.

A topological space (X, τ) is $\mathcal{R}_0$ if only if τ is a Wallman basis of (X, τ). Also (X, τ) is regular if only if τ admits a regular Wallman basis. It is also clear that every $\mathcal{R}_1$ space is $\mathcal{R}_0$ and every regular or Hausdorff space is $\mathcal{R}_1$.

Theorem 2.11. *Let $\mathcal{B}$ be an everywhere basic subset of an $\mathcal{R}_1$ topological space (X, τ) such that $B \neq X$. Then B is compact.*

Proof. According to Theorem (2.9), it is enough to prove that $Fr(B) = \varnothing$. Assume, on the contrary, there exists a point $p \in Fr(B)$. Define $\mathcal{B}_1 = \{V \in \tau : p \notin \overline{V}\}$ and $\mathcal{B}_2 = \{W \in \tau : p \in W\}$. If $\mathcal{B} = \{V \cup W : V \in \mathcal{B}_1$ and $W \in \mathcal{B}_2\}$ it is clear that $\mathcal{B}$ is an annular basis of (X, τ). Observe that for every $T = V \cup W \in \mathcal{B}$, we have $p \notin Fr(T)$ (because $Fr(T) \subseteq Fr(V) \cup Fr(W) \subseteq X \setminus \{p\}$). This implies that $B \notin \mathcal{B}$, contradicting the fact that B is everywhere basic. □

Definition 2.12. A topological space (X, τ) is *irreducible* if every non-empty open set $V \in \tau$ is dense in X. Equivalently, (X, τ) is irreducible if every pair of non-empty open subsets of X have a non-empty intersection.

Theorem 2.13. *Let $B \neq X$ be an everywhere basic subset of a topological space (X, τ). If $X \setminus B$ is irreducible, then B is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $\mathcal{B}$. Let $\mathcal{B}'$ be the family of open sets $L \in \tau$ which are contained in a finite union of members of $\mathcal{U}$ and let $\mathcal{B}'' = \{\varnothing\} \cup \{M \in \tau : M \setminus B \neq \varnothing\}$. Clearly $\mathcal{B} = \{L \cup M : L \in \mathcal{B}'$ y $M \in \mathcal{B}''\}$ is an annular basis of (X, τ). However, $B \notin \mathcal{B}$, a contradiction. □

Theorems (2.11) and (2.13) have the following consequences:

Corollary 2.14. *An $\mathcal{R}_1$ topological spaces (X,τ) is minimally basic iff (X,τ) is locally compact and 0-dimensional.*

Corollary 2.15. *Let (X,τ) be an unibasic space and let $x \in X$. Then $X \setminus \overline{\{x\}}$ is compact. Therefore, if X has a compact, closed and non-empty subspace, then X itself is compact.*

Corollary 2.16. *Every $\mathcal{R}_1$ unibasic space (X,τ) has a finite topology. In fact, for every $x \in X$, $\overline{\{x\}}$ is open and X is a finite union of point-closures.*

Definition 2.17. Let (X,τ) be an $\mathcal{R}_0$ topological space.

a) (X,τ) is $\mathcal{R}_1'$ if every compact open subset of X is closed.
b) (X,τ) is $\mathcal{R}_1''$ every intersection of compact open subspaces of X is compact.

Remark 2.18. $\mathcal{R}_1 \Rightarrow \mathcal{R}_1' \Rightarrow \mathcal{R}_1'' \Rightarrow \mathcal{R}_0$.

Proof. ($\mathcal{R}_1 \Rightarrow \mathcal{R}_1'$) It enough to observe that if (X,τ) is $\mathcal{R}_1$, $K \subseteq X$ is compact, $V \subseteq X$ is open and $K \subseteq V$, then $\overline{K} \subseteq V$. □

A subset S of X is a *semi-block* of a entourage E of X if $S \times S \subseteq E$.

Lemma 2.19. *Let R be a transitive entourage of a set X; let $x \in X$ and let $A \subseteq X$ be a semi-block of R intersecting $R(x)$. Then $A \subseteq R(x)$.*

Proof. Select a point $y \in A \cap R(x)$ and let $z \in A$. Therefore, $(x,y) \in R$ and $(y,z) \in A \times A \subseteq R$. Since R is transitive, we deduce that $(x,z) \in R$, i.e., $z \in R(x)$. □

Definition 2.20. Let α be a cover of a set X. For $x \in X$, define $Cost(x,\alpha) = \bigcap\{L \colon x \in L \in \alpha\}$. The indexed cover $\{Cost(x,\alpha) \colon x \in X\}$ is denoted as α^{∇} and is called the *cobaricentric cover* of α. Let α be any cover of a set X. Then the entourage $E(\alpha^{\nabla})$ of the cobaricentric cover α^{∇} is a transitive entourage of X.

A cover α of a topological space (X,τ) is *interior-preserving* if for each $x \in X$, $Cost(x,\alpha)$ is a τ-neighborhood of x.

Lemma 2.21. *Let R be a totally bounded transitive entourage on a set X. Then the family $\{L \colon L = R(x)$ for some $x \in X\}$ is finite.*

Proof. Let $\{A_1, A_2, \dots, A_n\}$ be a finite cover of X consisting of semi-blocks of R. By Lemma (2.19), each $R(x)$ is the union of the sets A_i which intersect $R(x)$. Hence the family $\{L \colon L = R(x)$ for some $x \in X\}$ has at most 2^n elements. □

Theorem 2.22. *Let (X,τ) be a topological space. Consider the following properties:*

(1) *τ is finite.*
(2) *$\mathcal{P}$ is the only quasi-uniformity on X which induces τ.*
(3) *Every interior-preserving cover of X is finite.*

(4) (X, τ) *is hereditarily compact.*
(5) $\delta_{\mathcal{P}}$ *is the only quasi-proximity on* X *which induces* τ.
(6) $\delta_{\mathcal{P}}$ *is the only transitive quasi-proximity on* X *which induces* τ.
(7) (X, τ) *is unibasic.*

Then 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) $\Rightarrow$ 6) $\Rightarrow$ 7)*; if* (X, τ) *is* $\mathcal{R}_1''$, 7) $\Rightarrow$ 4) *and if* (X, τ) *is* $\mathcal{R}_1'$, 7) $\Rightarrow$ 1).

Proof. The proofs of the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) appear in ([1]). However, using Lemma (2.21) we obtain a quick proof of the implication 2) $\Rightarrow$ 3). Assuming 2), we deduce that $\mathcal{P} = \mathcal{FT}$. Hence, if α is an interior-preserving cover of X, the entourage $R = E(\alpha^{\nabla})$ is totally bounded and transitive. Therefore, by Lemma (2.21), the family $\{L \colon L = R(x) \text{ for some } x \in X\}$ is finite. This, in turn, implies that α is finite. Indeed, consider the topology of X whose closed sets are arbitrary unions of arbitrary intersections of elements of α. The point-closures in this topology are precisely the sets $Cost(x, \alpha)$, where $x \in X$. Since every closed set in this topology is finite we conclude that this topology is finite and hence, α is finite. The implication 5) $\Rightarrow$ 6) is evident and 6) $\Rightarrow$ 7) is a consequence of Theorem (2.8). If (X, τ) is $\mathcal{R}_1''$ and $V \in \tau$, $V \neq X$, clearly V is the intersection of all the compact open sets $X \setminus \overline{\{x\}}$, where $x \in X \setminus V$. By hypothesis, V must be compact. We have proved then that 7) $\Rightarrow$ 4) when (X, τ) is $\mathcal{R}_1''$-space. Finally, if (X, τ) is $\mathcal{R}_1'$, each set $X \setminus \overline{\{x\}}$ is compact and open and, hence, it is also closed. Therefore, each point-closure is open. Since X is compact, X is the closure of a finite subset of X. Since (X, τ) is $\mathcal{R}_0$, the topology τ must be finite. ☐

H.-P. Künzi has proved that properties 3), 4), 5), 6), 7) and

2') $\mathcal{P}$ is the only totally bounded quasi-uniformity on X which induces τ are equivalent (see [4]).

The validity of the implication 7) $\Rightarrow$ 2) is still open.

Typical examples of topological spaces admitting a unique totally bounded quasi-uniformity are the hereditarily compact spaces and set ω_0 equipped with the lower topology $\{[0, n] \colon n \in \omega_0\} \cup \{\varnothing, \omega_0\}$.

The space with carrier set $\omega_0 + 2$ and topology $\{[0, n] \colon n \in \omega_0\} \cup \{(\omega_0 + 2) \setminus \{\omega_0 + 1\}, \omega_0 + 2, (\omega_0 + 2) \setminus \{\omega_0\}, \varnothing\}$ admits a unique totally bounded quasi-uniformity, while this is not true for its subspace $(\omega_0 + 2) \setminus \{\omega_0\}$ (see example page 148 [4]).

Example 2.23 (see example 1 in [5])**.** Let N be the set of the positive integers equipped with the topology $\tau = \{\{1, \ldots, n\} \colon n \in N\} \cup \{\varnothing, N\}$. Obviously, every proper open subset of N is compact, but N is not compact. This example shows that a topological space that admits a unique compatible quasi-proximity need not be compact.

Question: If (X, τ) is an unibasic space is equivalently to say the $\mathcal{P}$ is the only compatible quasi-uniformity?

References

[1] P. Fletcher and W. Lindgren, Quasi-uniformity spaces, vol 77, Marcel Dekker, Inc., New York, First edition, 1982.

[2] J. Ferrer, On topological spaces with a unique quasi-proximity, Quaest. Math 17 (1994), 479–486.

[3] H.-P. Künzi, Nontransitive quasi-uniformities in the Pervin quasi-proximity class, Proc. Amer. Math. Soc. 130 (2002), 3725–3730.

[4] H.-P. Künzi, Quasi-uniform spaces-eleven years later, Topology Proceedings 18 (1993), 143–171.

[5] H.-P. Künzi, Topological spaces with a unique compatible quasi-proximity, Arch. Math. 43 (1984), 559–561.

[6] H.-P. Künzi and M. J. Pérez-Peñalver, The number of compatible totally bounded quasi-uniformities, Act. Math. Hung. 88 (2000), 15–23.

[7] H.-P. Künzi and S. Watson, A nontrivial T_1-spaces admitting a unique quasi-proximity, Glasg. Math. J. 38 (1996), 207–213.

[8] W. J. Pervin. Quasi-uniformization of topological spaces, Math. Annalen 147 (1962), 116–117.

[9] W. J. Pervin, Quasi-proximities for topological spaces, Math. Annalen 150 (1963), 325–326.