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A uniform approach to normality for topological spaces

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Abstract

(λ, μ)-regularity and (λ, μ)-normality are defined for generalized topological spaces. Several variants of normality existing in the literature turn out to be particular cases of (λ, μ)-normality. Uryshon's lemma and Tietze extension theorem are discussed in the light of (λ, μ)-normality.

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1. Introduction

A large amount of research in topology is devoted to the study of classes of subsets of topological spaces, which posses properties similar to those of open sets. In the literature, several such classes are available which include amongst others semi-open sets [10], α-open sets [12], β-open sets [1], pre-open sets [11], etc. Some other such classes are A-sets [15], B-sets [16], C-sets [7], etc. Since these classes have some features common in them, it is quite natural to enquire if these classes can be obtained by using one common definition? Á. Császàr has successfully provided an answer in this regard. The main tool he has used is, the class of mappings $\gamma : P(X) \to P(X)$ from the power set X into X itself possessing the property of monotonicity (that is, for $A \subseteq B$ implies $\gamma(A) \subseteq \gamma(B)$). In a topological space (X, τ), the operators such as *int*, *cl*, *int cl*, *cl int*, *int cl int*, *cl int cl* etc. are found to belong to this class of mappings. Accordingly, the weaker form of open sets including semi-open sets,

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pre-open sets, α-open sets, β-open sets are nothing but γ-open sets for different γ's. All these families form "generalized topologies" on X. In [4], Császàr has formulated separation axioms for such spaces. Accordingly, separation axioms using semi-open sets[5], β-open sets[13], etc. become particular cases in [4].

In the same spirit, we introduce and investigate a generalized form of normality called (λ, μ)-normality for generalized topologies in this paper. However, unlike in [4], we use two GT's simultaneously in our definition. This gives us a more general definition of normality, yet it covers almost all the relevant variants of normality existing in the literature. For example, if X has a topology, then by taking $\lambda = \mu = int$, we get normality for X; $\lambda = int$, $\mu = cl_\theta^*$ give θ-normality; $\lambda = int$, $\mu = cl_\delta^*$ give Δ-normality for X. If (X, τ_1, τ_2) is a bitopological space, then $\lambda = int_{\tau_1}$, $\mu = int_{\tau_2}$ gives rise to pairwise normality of (X, τ_1, τ_2). Thus our study provides a uniform approach towards various notions of normality existing in the literature. We have shown that the two most important results on normality- the Urysohn's lemma and Tietze extension theorem are valid for (λ, μ)-normality, although in a milder form. We have also defined and studied (λ, μ)-regularity in the process and provided its characterization.

2. Preliminaries

Á. Császàr has defined a generalized topological space [3] in the following way:

Definition 2.1. A collection $\mathcal{G}$ of subsets of X is called a *generalized topology* (in brief GT) [3] on X if

(i) $\varnothing \in \mathcal{G}$;

(ii) $G_i \in \mathcal{G}$ for $i \in I \neq \varnothing$, implies $G = \bigcup_{i \in I} G_i \in \mathcal{G}$.

The same has been defined and studied as *semi topological spaces* by Peleg[14]. For a topological space (X, τ), each family of semi-open sets, α-open sets, pre-open sets and β-open sets etc. form a generalized topology on X.
Á. Császàr [2] has used a map $\gamma : P(X) \longrightarrow P(X)$ where $P(X)$ is the power set of X, as his main tool for developing a generalized form of topological spaces. The map γ possesses the property of monotonicity, which says that, if $A \subseteq B$ then $\gamma(A) \subseteq \gamma(B)$.
The collection of all such mappings on X is denoted by $\Gamma(X)$, or simply by Γ.

Definition 2.2 ([2])**.** Consider a non empty set X and a map $\gamma \in \Gamma(X)$. We say that a subset A of X is γ-*open* if $A \subseteq \gamma(A)$.

For a topological space (X, τ), an open set (resp. semi-open, α-open, β-open, pre-open) is γ-open for $\gamma =$ int (resp. cl int, int cl int, cl int cl, int cl). Also for each $\gamma \in \Gamma(X)$, it may be verified that the γ-open sets form a generalized topology on X.

Definition 2.3 ([2])**.** Let A be a subset of X and γ be a monotonic mapping on X. Then the union of all γ-open sets contained in A is called the γ-*interior* of A, and is denoted by $i_\gamma(A)$.

Proposition 2.4 ([2])**.** *A subset A of X is γ-open if and only if $A = i_\gamma(A)$ if and only if A is i_γ-open.*

Definition 2.5 ([2])**.** A subset A of X is called γ-*closed* if $X \diagdown A$ is γ-open.

Definition 2.6 ([2])**.** The intersection of all γ-closed sets containing A is called γ-*closure* of A and is denoted by $c_\gamma(A)$.

It can be shown that $c_\gamma(A)$ is the smallest γ-closed set containing A.

Another operator called γ^* is defined, with the help of γ in the following way:

Definition 2.7 ([2])**.** For any $A \subseteq X$ and $\gamma \in \Gamma(X)$, we define

$$\gamma^*(A) = X \diagdown (\gamma(X \diagdown A))$$

Proposition 2.8 ([2])**.** *If $\gamma \in \Gamma$, then $\gamma^* \in \Gamma$.*

Proposition 2.9 ([2])**.** *A subset A of X is γ^*-closed if and only if $\gamma(A) \subseteq A$.*

Definition 2.10 ([19])**.** Let X be a topological space and let $A \subseteq X$. A point $x \in X$ is in θ-closure of A if every closed neighbourhood of x intersects A. The θ-closure of A is denoted by $cl_\theta(A)$. The set A is called θ-*closed* if $A = cl_\theta A$.

The complement of a θ-closed set is called θ-open set.

Definition 2.11 ([19])**.** Let X be a topological space and let $A \subseteq X$. A point $x \in X$ is in δ-closure of A if every regular open neighbourhood of x intersects A. The δ-closure of A is denoted by $cl_\delta(A)$. The set A is called δ-*closed* if $A = cl_\delta(A)$.

The complement of a δ-closed set is called δ-open set.

Definition 2.12. A topological space X is said to be

(1) [8] θ-*normal* if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets;
(2) [8] *Weakly* θ-*normal* if every pair of disjoint θ-closed sets are contained in disjoint open sets;
(3) [6] Δ-*normal* if every pair of disjoint closed sets one of which is δ-closed are contained in disjoint open sets;
(4) [6] *Weakly* Δ-*normal* if every pair of disjoint δ-closed sets are contained in disjoint open sets.

Definition 2.13 ([9])**.** A bitopological space (X, τ_1, τ_2) is said to be *pairwise normal* if given a τ_1-closed set A and a τ_2-closed set B with $A \cap B = \varnothing$, there exist τ_2-open set O_2 and τ_1-open set O_1 such that $A \subseteq O_2$, $B \subseteq O_1$, and $O_1 \cap O_2 = \varnothing$.

3. (λ, μ)-regularity and (λ, μ)-normality

For defining (λ, μ)-regularity and (λ, μ)-normality, no topology is required on X. It is because, for a non-empty set X, $\Gamma(X)$ is also non-empty. However, we may call X a space, once we define some topological property on X, such as (λ, μ)-regularity, (λ, μ)-normality etc.

Definition 3.1. Let X be a non-empty set and $\lambda, \mu \in \Gamma(X)$. Then X is said to be *λ-regular with respect to μ* if for each point $x \in X$ and each λ-closed set P such that $x \notin P$, there exist a λ-open set U and a μ-open set V such that $x \in U$, $P \subseteq V$ and $U \cap V = \varnothing$.

X is said to be *(λ, μ)-regular* if X is λ-regular with respect to μ and vice versa.

Definition 3.2. A non-empty set X is called *(λ, μ)-normal* if for a given λ-closed set A and a μ-closed set B with $A \cap B = \varnothing$, there exist a μ-open set U and a λ-open set V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \varnothing$.

Below we provide characterizations for (λ, μ)-regularity and (λ, μ)-normality.

Theorem 3.3. *Let X be a non-empty set. Then X is (λ, μ)-regular if and only if*

- (i) *for a given $x \in X$ and λ-open neighbourhood U of x, there exists a μ-closed neighbourhood V of x (that is, $x \in V_0 \subseteq V$, for some $V_0 \subseteq X$, where V_0 is λ-open and V is μ-closed) such that $x \in V \subseteq U$.* *and*
- (ii) *for a given $y \in X$ and μ-neighbourhood P of y, there exists a λ-closed neighbourhood Q of y such that $y \in Q \subseteq P$.*

Proof. Let $x \in X$ and U be a λ-open neighbourhood of x. Therefore $x \notin X \setminus U$, a λ-closed set. Thus, there exists a disjoint pair of λ-open set O and μ-open set W such that $x \in O$, $X \setminus U \subseteq W$ and $O \cap W = \varnothing$. That is, $X \setminus W \subseteq U$. Hence $x \in O \subseteq X \setminus W \subseteq U$, that is, $x \in V \subseteq U$, where $V = X \setminus W$, a μ-closed set.

Similarly, if X is μ-regular with respect to λ, then for a given point $x \in X$ and a μ-open neighbourhood U of x, there exists a λ-closed neighbourhood V of x such that $x \in V \subseteq U$.

Conversely, let $x \in X$ and F be a λ-closed set such that $x \notin F$. Then $x \in X \setminus F$ and $X \setminus F$ is λ-open. Hence by (i), there exists a λ-open set V_0 and a μ-closed set V such that $x \in V_0 \subseteq V \subseteq X \setminus F$. Then by (ii), there exists a λ-open set Q_0 and a μ-closed set Q such that $x \in Q_0 \subseteq Q \subseteq V_0$. Thus we have, $x \in Q_0$, $F \subseteq P_0$, where $P_0 = X \setminus V$, such that Q_0 is λ-open, P_0 is μ-open and $P_0 \cap Q_0 = \varnothing$. Hence X is (λ, μ)-regular. □

Theorem 3.4. *Let X be a non-empty set. Then X is (λ, μ)-normal if and only if for a given μ-closed set C and a λ-open set D such that $C \subseteq D$, there are a λ-open set G and a μ-closed set F such that $C \subseteq G \subseteq F \subseteq D$.*

Proof. Let C and D be the μ-closed and λ-open sets respectively such that $C \subseteq D$. Then $X \setminus D$ is a λ-closed set such that $C \cap (X \setminus D) = \varnothing$. Then, from the (λ, μ)-normality, there exist a λ-open set G and a μ-open set V such that $C \subseteq G$, $X \setminus D \subseteq V$ and $G \cap V = \varnothing$. Therefore $X \setminus V \subseteq D$ and hence $C \subseteq G \subseteq X \setminus V \subseteq D$, where $X \setminus V$ is a μ-closed set. Hence $C \subseteq G \subseteq F \subseteq D$, where $X \setminus V = F$(say).

Conversely, consider C as D are λ-closed and μ-closed sets respectively such that $C \cap D = \varnothing$. Then $X \setminus C$ is λ-open set containing D. Then by the given hypothesis, there exist a λ-open set G and a μ-closed set F such that $D \subseteq G \subseteq F \subseteq X \setminus C$. Thus, we have $D \subseteq G$, $C \subseteq V$ and $G \cap V = \varnothing$, where $V = X \setminus F$, μ-open set. Hence X is (λ, μ)-normal. □

In our next section, we provide generalized versions of Uryshon's lemma and Tietze extension theorem, which holds for (λ, μ)-normality.

4. Uryshon's lemma and Tietze extension Theorem for (λ, μ)-normality

Definition 4.1 ([18])**.** Let (X, λ) be a generalized topological space and $\mathbb{R}$ be the real line with the usual topology. A mapping $f : X \to \mathbb{R}$ is said to be *generalized upper semi-continuous* or g.u.s.c. in brief (resp. *generalized lower semi-continuous* or g.l.s.c. in brief) if for each $a \in \mathbb{R}$, the set $\{x \in X : f(x) < a\}$ (resp. $\{x \in X : f(x) > a\}$) is λ-open.

Unlike in topology, a mapping which is both generalized upper semi-continuous and generalized lower semi-continuous, may fail to be generalized continuous in generalized topology.

Example 4.2. Let $X = [0, 1]$, λ consist of the unions of the members of the type $[0, a)$, $(b, 1]$, $a, b \in [0, 1]$. Let $Y = X$, under the usual subspace topology of $\mathbb{R}$. Then the identity mapping $I : X \to Y$ is g.u.s.c. and g.l.s.c. but not generalized continuous as $f^{-1}(a, b) \notin \lambda$.

Theorem 4.3. *Let X be a (λ, μ)-normal space. Then for any disjoint pair of λ-closed set H and μ-closed set F, there exists a real valued function g on X such that*

(i) $g(x) = 0$ *for* $x \in F$, $g(x) = 1$ *for* $x \in H$, $0 \leq g(x) \leq 1$, *for all* $x \in X$;

(ii) g *is λ-upper semi-continuous and μ-lower semi-continuous.*

Proof. Let X be a (λ, μ)-normal space and G and H be two disjoint subsets of X such that G is μ-closed and H is λ-closed.
Let us consider, $G_0 = G$ and $K_1 = X \setminus H$. Then G_0 is μ-closed and K_1 is λ-open set such that $G_0 \subseteq K_1$. Since X is (λ, μ)-normal, therefore there exist a λ-open set $K_{1/2}$ and a μ-closed set $G_{1/2}$ such that $G_0 \subseteq K_{1/2} \subseteq G_{1/2} \subseteq K_1$. Again applying the hypothesis to each pair of sets (G_0 and $K_{1/2}$) and ($G_{1/2}$ and K_1), we obtain λ-open sets $K_{1/4}, K_{3/4}$ and μ-closed sets $G_{1/4}, G_{3/4}$ such that

$$G_0 \subseteq K_{1/4} \subseteq G_{1/4} \subseteq K_{1/2} \subseteq G_{1/2} \subseteq K_{3/4} \subseteq G_{3/4} \subseteq K_1.$$

Continuing this process, we obtain two families $\{G_i\}$ and $\{K_i\}$, where $i = p/2^q$, where $\{p = 1, 2, \ldots, 2^q - 1, q = 1, 2, \ldots\}$. If i is any other dyadic rational number other than $p/2^q$, then let $K_i = \varnothing$, whenever $i \leq 0$ and $K_i = X$, for $i > 1$. Similarly, $G_i = \varnothing$ for $i < 0$ and $G_i = X$ for $i \geq 1$. Thus, for every $r \leq s \leq t$, we have

$$K_r \subseteq K_s \subseteq G_s \subseteq G_t \text{ and for } s < t, \text{ we have } G_s \subseteq K_t.$$

Now, we define a function $g : X \to [0, 1]$ such that

$$g(x) = \inf\{t \mid x \in K_t\}$$

Clearly, $g(x) \in [0, 1]$. If $x \in G$, then $x \in K_i$ for all i, therefore $g(x) = 0$, when $x \in H = X \setminus K$, then $x \notin K_i$ for all $i \in [0, 1]$, hence $g(x) = 1$.
Now, we have to show that g is λ-upper semi-continuous and μ-lower semi-continuous.
First we show that

(i) if $x \in G_p$ then $g(x) \leq p$
(ii) if $x \notin K_p$, then $g(x) \geq p$.

Let $x \in G_p$, then $x \in K_s$ for every $s > p$. Therefore $g(x) \leq p$. Similarly, if $x \notin K_p$, then $x \notin K_s$ for any $s < p$, hence $g(x) \geq p$.
Thus we can say that whenever $g(x) > p$, we have $x \notin G_p$ and $g(x) < p$, we have $x \in K_p$.
Now, we consider, $x \in g^{-1}([0, a))$, then $g(x) \in [0, a)$, that is, there exists $t < a$ such that $g(x) < t$ and hence $x \in K_t$, therefore $g^{-1}([0, a)) \subseteq \bigcup\limits_{t<a} K_t$.
Conversely, let $x \in \bigcup\limits_{t<a} K_t$, then $x \in K_i$ for some $i < a$, thus $g(x) \leq i < a$. Therefore $g^{-1}([0, a)) = \bigcup\limits_{t<a} K_t$, a λ-open set. Hence g is λ-upper semi-continuous function.
Again, consider, $x \in g^{-1}((a, 1])$, then $g(x) \in (a, 1]$, that is, there exists t with $a < t$ such that $g(x) > t$ and hence $x \notin G_t$, that is, $x \in X \setminus G_t$, therefore $g^{-1}((a, 1]) \subseteq \bigcup\limits_{t>a} (X \setminus G_t)$.
Conversely, let $x \in \bigcup\limits_{t>a} (X \setminus G_t)$, then $x \in X \setminus G_i$ for some $i > a$, thus $g(x) \geq i > a$ and $x \in g^{-1}((a, 1])$. Therefore $g^{-1}((a, 1]) = \bigcup\limits_{t>a} (X \setminus G_t)$, a μ-open set. Hence g is μ-lower semi-continuous function. □

Our next theorem resembles with the classical Tietze extension theorem. But before that, we quote a result which will be used in our main theorem.

Theorem 4.4 ([3]). *Let (X, λ) be a generalized topological space and (Y, λ_y) be a generalized topological subspace of X. Then a subset A of Y is λ-closed in Y if and only if it is the intersection of Y with a λ-closed set in X.*

Now we come to our main proposed result.

Theorem 4.5. *Let X be a (λ, μ)-normal space. Let $A \subseteq X$ be a λ-closed as well as μ-closed set and f be a real valued function defined on A which is λ-upper semi-continuous as well as μ-lower semi-continuous function. Then there exists an extension F of f to the whole of X such that F is λ-upper semi-continuous and μ-lower semi-continuous in X.*

Proof. Let X be a (λ, μ)-normal space and A be a λ-closed and μ-closed subset of X. Suppose f is a real valued function on A which is λ-upper semi-continuous and μ-lower semi-continuous.
Let n be a positive integer, then for each integer k, let

$$U_k^n = \{x : f(x) \geq k/n\} \text{ and } L_k^n = \{x : f(x) \leq (k-1)/n\}$$

Then, for every integer k, U_k^n and L_k^n are λ-closed and μ-closed subsets of A respectively. Therefore U_k^n and L_k^n are λ-closed and μ-closed subsets of X also and $U_k^n \cap L_k^n = \varnothing$.
By Theorem 4.3, for each $k = 0, 1, 2, \ldots$, there is a function $\mathcal{U}_k$ defined on X, which is λ-upper semi-continuous and μ-lower semi-continuous, such that

$$\mathcal{U}_k(x) = 0, \text{ for } x \in L_k^n \quad \mathcal{U}_k(x) = 1/n, \text{ for } x \in U_k^n$$
$$\text{and } 0 \leq \mathcal{U}_k(x) \leq 1/n \text{ for all } x \in X$$

Also, for each $k = 0, -1, -2, \ldots$, there is a function $\mathcal{V}_k$ defined on X which is λ-upper semi-continuous and μ-lower semi-continuous function, such that

$$\mathcal{V}_k(x) = -1/n, \text{ for } x \in L_k^n \quad \mathcal{V}_k(x) = 0, \text{ for } x \in U_k^n$$
$$\text{and } -1/n \leq \mathcal{V}_k(x) \leq 0 \text{ for all } x \in X$$

Now, we know that $L_k^n \subseteq L_{k+1}^n$ and $U_{k+1}^n \subseteq U_k^n$. Thus for $k = 0, -1, -2, \ldots$, if $f(x) \geq 0$, then $x \in U_k^n$ and hence $\mathcal{V}_k(x) = 0$. Similarly, for $k = 1, 2, \ldots$, if $f(x) \leq 0$, then $x \in L_k^n$ and hence $\mathcal{U}_k(x) = 0$.
Now, we define a real valued function f_n on X as follows:

$$f_n(x) = \sum_{k=1}^{\infty} \mathcal{U}_k(x) + \sum_{k=-\infty}^{0} \mathcal{V}_k(x) \text{ for } x \in X$$

Since $\mathcal{U}_i$ and $\mathcal{V}_i$ are λ-upper semi continuous and μ-lower semi-continuous function therefore f_n is also a λ-upper semi continuous and μ-lower semi-continuous function.
Now, consider $C_k^n = U_{k-1}^n \cap L_{k+1}^n$ for $k \in \mathbb{Z}$. Thus $C_k^n \subseteq U_{k-1}^n$ and $C_k^n \subseteq L_{k+1}^n$, that is, $C_k^n = \{x : \frac{k-1}{n} \leq f(x) \leq \frac{k}{n}\}$. Therefore $A = \bigcup\{C_k^n, k \in \mathbb{Z}\}$.
Now, let $k \geq 1$, and $x \in C_k^n$, that is, $0 \leq f(x) \leq k/n$. Since $x \in L_{k+1}^n$, therefore $\mathcal{U}_j(x) = 0$ for $j \geq k+1$ and $x \in U_{k-1}^n$, therefore $\mathcal{U}_j(x) = 1/n$ for $1 \leq j \leq k-1$. Also $0 \leq \mathcal{U}_k(x) \leq 1/n$. Thus

$$f_n(x) = \mathcal{U}_1(x) + \mathcal{U}_2(x) + \ldots + \mathcal{U}_{k-1}(x) + \mathcal{U}_k(x) + \mathcal{U}_{k+1}(x) + \ldots$$
$$f_n(x) = \tfrac{k-1}{n} + \mathcal{U}_k(x)$$

Therefore

$$\begin{aligned} |f(x) - f_n(x)| &\leq |k/n - (k-1)/n - \mathcal{U}_k(x)| \\ &\leq 1/n + |\mathcal{U}_k(x)| \\ &\leq 2/n, \quad for\ x \in C_k^n. \end{aligned}$$

Now, let $k \leq 0$, and $x \in C_k^n$, that is, $-(k+1)/n \leq f(x) \leq 0$. Since $x \in L_{k+1}^n$, therefore $\mathcal{V}_j(x) = -1/n$ for $(k-1) \leq j \leq 0$ and $x \in U_{k-1}^n$, therefore $\mathcal{V}_j(x) = 0$ for $j \leq (k+1)$. Thus

$$f_n(x) = \mathcal{V}_0(x) + \mathcal{V}_{-1}(x) + \mathcal{V}_{-2}(x) + \ldots \mathcal{V}_{k-1}(x) + \mathcal{V}_k(x) + \mathcal{V}_{k+1}(x) + \ldots$$
$$f_n(x) = \tfrac{k}{n} + \mathcal{V}_k(x), \text{ for a non positive integer } k$$

Therefore
$|f(x) - f_n(x)| \leq 2/n$
Hence $|f(x) - f_n(x)| \leq 2/n$ for all $x \in A$. We recall that g-nets defined in [17] behave almost the same way in generalized topology as the nets do in topology and a sequence is just a particular case of g-nets in generalized topology. Due to this fact, $f_n|_A$ $for\ n = 1, 2, \ldots$ converge uniformly to f on A. Also $(f_n|A)$ forms a Cauchy sequence with respect to the uniform norm on A. As a result, as in[9], f has an extension F to X. It may be easily shown that F is λ-upper semi-continuous and μ-lower semi-continuous function. □

5. Conclusion

Under different set of conditions, we get different variants of normality. If we take

(i) $\lambda = \mu =$ interior operator of a topology on X, then the λ-closed sets and μ-closed sets are nothing but the closed sets of X. Therefore (λ, μ)-normality just becomes **normality**.

(ii) $\lambda = int_{\tau_1}$ and $\mu = int_{\tau_2}$, two different interior operators over two different topologies τ_1 and τ_2, then (λ, μ)-normality becomes ***pairwise normality*** [9] of (X, τ_1, τ_2)

(iii) λ =interior operator and $\mu = cl_\theta^*$ operator, then (λ, μ)-normality becomes **θ-normality**[8]. This is due to fact that $cl_\theta \in \Gamma$, that is, cl_θ operator is monotonic. First we verify that for $A, B \subseteq X$ such that $A \subseteq B$, we have $cl_\theta(A) \subseteq cl_\theta(B)$. Let $x \in cl_\theta(A)$, then every closed neighbourhood of x intersects A. Since $A \subseteq B$, therefore every closed neighbourhood of x intersects B also. Hence $x \in cl_\theta(B)$. Thus $cl_\theta(A) \subseteq cl_\theta(B)$. Therefore $cl_\theta \in \Gamma$. Hence by Proposition 2.8, $cl_\theta^* \in \Gamma$.
Now, let A be μ-closed, that is, cl_θ^*-closed set. Then by Proposition 2.9 $cl_\theta(A) = (cl_\theta^*)^*(A)$ and by Proposition 2.8 $cl_\theta(A) = (cl_\theta^*)^*(A) \subseteq A$, that is, A is θ-closed. Since every θ-open set is open therefore in the light of (λ, μ)-normality, we have disjoint pair of sets in which one is λ-closed, that is, closed set and the other is μ-closed, that is, θ-closed set separated

by disjoint μ-open, that is, θ-open set and hence open set and λ-open set, that is, open set respectively.

(iv) λ =interior operator and $\mu = cl^*_\delta$ operator, then (λ, μ)-normality becomes **Δ-normality** [6]. Because $cl_\delta \in \Gamma$, that is, cl_δ operator is monotonic. As, consider $A, B \subseteq X$ such that $A \subseteq B$. Then $cl_\delta(A) \subseteq cl_\delta(B)$. Since $x \in cl_\delta(A)$, then every regular open neighbourhood of x intersects A. Since $A \subseteq B$, therefore every regular open neighbourhood of x intersects B also. Hence $x \in cl_\delta(B)$. Thus $cl_\delta(A) \subseteq cl_\delta(B)$. Therefore $cl_\delta \in \Gamma$. From the Proposition 2.8 $cl^*_\delta \in \Gamma$ also.

Now, a set A is μ-closed set, that is, cl^*_δ-closed implies that $cl_\delta(A) = (cl^*_\delta)^*(A) \subseteq A$, that is, A is δ-closed. As every δ-open set is open therefore in the light of (λ, μ)-normality, we have disjoint pair of sets in which one is λ-closed, that is, closed set and the other is μ-closed, that is, δ-closed sets separated by disjoint μ-open, that is, δ-open set and hence open set and λ-open set, that is, open set respectively.

(v) $\lambda = \mu = cl^*_\theta$ operator, we have disjoint pair of θ-closed sets separated by disjoint pair of θ-open sets which is again open sets. Therefore (λ, μ)-normality becomes **weakly θ-normality** [8].

(vi) $\lambda = \mu = cl^*_\delta$ operator, we have disjoint pair of δ-closed sets separated by disjoint pair of δ-open sets which is again open sets. Therefore (λ, μ)-normality becomes **weakly Δ-normality** [6].

(vii) $\lambda = \mu =$ closure operator, then space is always (λ, μ)-normal. Because here every set is λ-open as well as μ-open.

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