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On the locally functionally countable sub algebra of $C(X)$

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Abstract

Let $C_c(X) = \{f \in C(X) : |f(X)| \leq \aleph_0\}$, $C^F(X) = \{f \in C(X) : |f(X)| < \infty\}$, and $L_c(X) = \{f \in C(X) : \overline{C_f} = X\}$, where C_f is the union of all open subsets $U \subseteq X$ such that $|f(U)| \leq \aleph_0$, and $C_F(X)$ be the socle of $C(X)$ (i.e., the sum of minimal ideals of $C(X)$). It is shown that if X is a locally compact space, then $L_c(X) = C(X)$ if and only if X is locally scattered. We observe that $L_c(X)$ enjoys most of the important properties which are shared by $C(X)$ and $C_c(X)$. Spaces X such that $L_c(X)$ is regular (von Neumann) are characterized. Similarly to $C(X)$ and $C_c(X)$, it is shown that $L_c(X)$ is a regular ring if and only if it is $\aleph_0$-selfinjective. We also determine spaces X such that $\mathrm{Soc}(L_c(X)) = C_F(X)$ (resp., $\mathrm{Soc}(L_c(X)) = \mathrm{Soc}(C_c(X))$). It is proved that if $C_F(X)$ is a maximal ideal in $L_c(X)$, then $C_c(X) = C^F(X) = L_c(X) \cong \prod_{i=1}^{n} R_i$, where $R_i = \mathbb{R}$ for each i, and X has a unique infinite clopen connected subset. The converse of the latter result is also given. The spaces X for which $C_F(X)$ is a prime ideal in $L_c(X)$ are characterized and consequently for these spaces, we infer that $L_c(X)$ can not be isomorphic to any $C(Y)$.

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1. Introduction

$C(X)$ denotes the ring of all real valued continuous functions on a topological space X. In [10] and [11], $C_c(X)$, the subalgebra of $C(X)$, consisting of functions with countable image are introduced and studied. It turns out that $C_c(X)$, although not isomorphic to any $C(Y)$ in general, enjoys most of the important properties of $C(X)$. This subalgebra has recently received some attention, see [10], [23], [24], [4], and [11]. Since $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, this motivates us to consider a natural subring of $C(X)$, namely $L_c(X)$, which lies between $C_c(X)$ and $C(X)$. Our aim in this article, similarly to the main objective of working in the context of $C(X)$, is to investigate the relations between topological properties of X and the algebraic properties of $L_c(X)$. In particular, we are interested in finding topological spaces X for which $L_c(X) = C(X)$. An outline of this paper is as follows: In Section 2, we show that if X is a locally compact space, then $L_c(X) = C(X)$ if and only if X is locally scattered, which is somewhat similar to a classical result due to Rudin in [27], and Pelczynski and Semadeni in [25] (of course, by no means as significant). This classical result says that a compact space X is scattered if and only if $C(X) = C_c(X)$. Let us for the sake of the brevity, call the latter classical result, RPS-Theorem. If X is an almost discrete space or a P-space, then $L_1(X) = L_F(X) = L_c(X) = C(X)$, where $L_F(X)$ and $L_1(X)$ are the locally functionally finite (resp., constant) subalgebra of $C(X)$, see Definition 2.7.

In Section 3, we introduce z_l-ideals in $L_c(X)$ and trivially observe that most of the facts related to z-ideals are extendable to z_l-ideals. In Section 4, topological spaces in which points and closed sets are separated by elements of $L_c(X)$, are called locally countable completely regular space (briefly, lc-completely regular). Clearly, every zero-dimensional space is lc-completely regular (note, in the zero-dimensional case, points and closed sets are separated even by the elements of $C_c(X)$, which is a subring of $L_c(X)$), see [10, Proposition 4.4]. Spaces X, for which $L_c(X)$ is regular, are called locally countably P-space (briefly, LCP-space) and are characterized both algebraically and topologically in this section. It is shown that P-spaces and LCP-spaces coincide when X is lc-completely regular. Finally, in this section similar to $C(X)$ and $C_c(X)$, we prove that $L_c(X)$ is a regular ring if and only if it is $\aleph_0$-selfinjective. The socle of $C(X)$ (i.e., $C_F(X)$) which is in fact a direct sum of minimal ideals of $C(X)$ is characterized topologically in [20, Proposition 3.3], and it turns out that $C_F(X)$ is a useful object in the context of $C(X)$, see [20], [1], [2], [8], [3], and [6]. The socle of $C_c(X)$, denoted by $\mathrm{Soc}\big(C_c(X)\big)$, is studied in [11, Proposition 5.3], and spaces X for which $\mathrm{Soc}\big(C_c(X)\big) = C_F(X)$ are determined in [11, Theorem 5.6]. Motivated by the latter facts, we characterize the socle of $L_c(X)$ both topologically and algebraically, in Section 5. Spaces X for which $\mathrm{Soc}\big(L_c(X)\big) = \mathrm{Soc}\big(C_c(X)\big)$ and $\mathrm{Soc}\big(L_c(X)\big) = C_F(X)$ are also characterized. In [8, Proposition 1.2], [3, Remark 2.4], it is shown that $C_F(X)$ can not be a prime ideal in $C(X)$, where X is any space. But, in [11, Proposition 6.2], spaces

X such that $C_F(X)$ is prime in $C_c(X)$ are characterized. The latter characterization is similarly extended to $L_c(X)$. Consequently, this implies that $L_c(X)$ is not isomorphic to any $C(Y)$ in general. All topological spaces that appear in this article are assumed to be infinite completely regular Hausdorff, unless otherwise mentioned. For undefined terms and notations the reader is referred to [13], [7].

2. The subalgebra $L_c(X)$ of $C(X)$

Definition 2.1. Let $f \in C(X)$ and C_f be the union of all open sets $U \subseteq X$, such that $f(U)$ is countable. We define $L_c(X)$ to be the set of all $f \in C(X)$ such that C_f is dense in X, i.e.,

$$C_f = \bigcup \{U |\ U \text{ is open in } X \text{ and } |f(U)| \leq \aleph_0\}$$

$$L_c(X) = \{f \in C(X) : \overline{C_f} = X\}$$

We shall briefly and easily notice that, $L_c(X)$ is a subalgebra as well as a sublattice of $C(X)$ containing $C_c(X)$, and we call it the *locally functionally countable subalgebra of* $C(X)$.

It is manifest that $C_F(X) \subseteq C^F(X) \subseteq C_c(X) \subseteq L_c(X) \subseteq C(X)$, where $C^F(X) = \{f \in C(X) : |f(X)| < \infty\}$, see [10]. The following example shows that the equality between any two of these objects may not necessarily hold.

Example 2.2. Let the basic neighborhood of x be the set $\{x\}$, for each point $x \geq \sqrt{2}$ and for the rest of the real numbers (i.e., $x < \sqrt{2}$) the basic neighborhoods be the usual open intervals containing x. This is a topology $\mathcal{T}$ on $\mathbb{R}$ and in this case we put $X = \mathbb{R}$. Clearly, X is a completely regular Hausdorff space which is finer than the usual topology of $\mathbb{R}$. The function $f : X \to \mathbb{R}$, where $f(x) = 1$ for $x \geq \sqrt{2}$, and $f(x) = 0$ otherwise, is continuous and $X \backslash Z(f)$ is infinite, hence $f \in C^F(X) \backslash C_F(X)$, see [20, Proposition 3.3]. We define $g : X \to \mathbb{R}$, such that $g(x) = x$ for $x \in [\sqrt{2}, \infty) \cap Q$ and $g(x) = 0$ for $x \in ([\sqrt{2}, \infty) \cap Q^c) \cup (-\infty, \sqrt{2})$, hence $g \in C_c(X) \backslash C^F(X)$. Also we observe that for the function $h : X \to \mathbb{R}$, where $h(x) = x$ for $x \geq \sqrt{2}$, and $h(x) = \sqrt{2}$ otherwise, we have $h \in L_c(X) \backslash C_c(X)$. The identity function $i : X \to \mathbb{R}$ is continuous and $C_i = [\sqrt{2}, \infty)$, see Definition 2.1. Hence $i \in C(X) \backslash L_c(X)$.

We note that $\overline{C_f} = X$ if and only if for every open subset $G \subseteq X$, there exists an open subset $U \subseteq X$ such that $|f(U)| \leq \aleph_0$ and $U \cap G \neq \varnothing$ or equivalently if and only if for each open subset $G \subseteq X$, there exists a nonempty open subset $V \subseteq G$ with $|f(V)| \leq \aleph_0$.

Lemma 2.3. *For the space X the following statements hold.*

1. *If $f, g \in C(X)$, then $C_{f+g} \supseteq C_f \cap C_g$.*
2. *If $f, g \in C(X)$, then $C_{fg} \supseteq C_f \cap C_g$.*
3. *If $f \in C(X)$, then $C_{|f|} = C_f$.*
4. *If $f \in C(X)$, then $C_{\frac{1}{f}} = C_f$.*
5. *If $f, g \in L_c(X)$, then $\overline{C_f \cap C_g} = X$.*

Proof. Let $C_f = \bigcup_{\substack{U\subseteq X\\ |f(U)|\leq\aleph_0}} U$ and $C_g = \bigcup_{\substack{V\subseteq X\\ |g(V)|\leq\aleph_0}} V$, where U and V are open subsets of X, then

$$C_f \cap C_g = \bigcup_{\substack{U,V\subseteq X\\ |f(U)|,|g(V)|\leq\aleph_0}} (U\cap V)$$

Hence (1), (2), (3), (4) are evident. For part (5) we recall that if Y is a dense subset of X and G is an open subset of X, then $\overline{G\cap Y} = \overline{G}$. Since C_f, C_g are open and dense in X we infer that $\overline{C_f\cap C_g} = \overline{C_f} = \overline{C_g} = X$. □

The following examples show that the equalities in (1), (2) of the previous lemma do not necessarily hold, in general.

Example 2.4. (1) Let $i:\mathbb{R}\to\mathbb{R}$ be the identity function and $f:\mathbb{R}\to\mathbb{R}$ with $f(x) = -x$, then $C_i = C_f = \varnothing$, but $C_{i+f} = \mathbb{R}$. Hence $C_{i+f} \supsetneqq C_i\cap C_f$.

(2) Let $i:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be the identity function and $f:\mathbb{R}\backslash\{0\}\to\mathbb{R}$ with $f(x) = 1/x$, then $C_i = C_f = \varnothing$, but $C_{if} = \mathbb{R}\backslash\{0\}$. Hence $C_{if} \supsetneqq C_i\cap C_f$.

The following fact shows that $L_c(X)$ is indeed a subalgebra of $C(X)$ such that whenever $Z(f) = \varnothing$ where $f\in L_c(X)$, then f is a unit in $L_c(X)$. We remind the reader that the latter fact is not true for $C^*(X)$.

Corollary 2.5. *For the space X the following statements hold.*

1. *If $f,g\in L_c(X)$, then $f+g\in L_c(X)$ and $fg\in L_c(X)$.*
2. *$f\in L_c(X)$ if and only if $|f|\in L_c(X)$.*
3. *Let f be a unit element in $C(X)$, then $f\in L_c(X)$ if and only if $\frac{1}{f}\in L_c(X)$.*

Corollary 2.6. *$L_c(X)$ is a sublattice of $C(X)$.*

Definition 2.7. Let $f\in C(X)$ and C_f^F be the union of all open sets $U\subseteq X$ such that $f(U)$ is finite. We define $L_F(X)$ to be the set of all $f\in C(X)$ such that C_f^F is dense in X, and call it *locally functionally finite subalgebra of* $C(X)$, i.e.,

$$C_f^F = \bigcup\{U|\ U \text{ is open in } X \text{ and } |f(U)|<\infty\}$$
$$L_F(X) = \{f\in C(X) : \overline{C_f^F} = X\}$$

In particular, let $f\in C(X)$ and C_f^c be the union of all open sets $U\subseteq X$ such that $f(U)$ is constant. We define $L_1(X)$ to be the set of all $f\in C(X)$ such that C_f^c is dense in X, and we call it *locally functionally constant subalgebra of* $C(X)$, i.e.,

$$C_f^c = \bigcup\{U|\ U \text{ is open in } X \text{ and } |f(U)|=1\}$$
$$L_1(X) = \{f\in C(X) : \overline{C_f^c} = X\}$$

Clearly, $L_F(X)$ and $L_1(X)$ are subalgebras of $L_c(X)$. In [26] and [15], $E_0(X)$ is defined, and by the above notation we have $E_0(X) = L_1(X)$. It is evident that $C^F(X)\subseteq L_F(X)$.

Remark 2.8. We note that Lemma 2.3, Corollary 2.5, and Corollary 2.6 are also valid for $L_F(X)$ and $L_1(X)$.

Remark 2.9. It is manifest that $C_c(X) = \mathbb{R}$, where $X = [0, 1]$. But the Cantor function f is a monotonic nonconstant continuous function, and $\overline{C_f^c} = \overline{[0, 1]\setminus C} = [0, 1]$, where C is the Cantor set, see [9], [5]. Therefore the Cantor function f belongs to $L_1([0, 1])$, and $\mathbb{R} \subsetneq L_1([0, 1])$, hence $\mathbb{R} \subsetneq L_c([0, 1])$. We emphasize that $C_c(X) = \mathbb{R}$, but $\mathbb{R} \subsetneq L_c(X)$, and this can be considered as an advantage of $L_c(X)$ over $C_c(X)$, in this case.

Remark 2.10. In [15], a first countable compact space X (resp., in [26], a non-first countable compact space X) is constructed such that $L_1(X) = \mathbb{R}$.

We are interested in characterizing topological spaces X for which $L_c(X) = C(X)$. In the following proposition we have a simple result, which is similar to RPS-Theorem. Let us recall that in a commutative ring R by an annihilator ideal I, we mean $I = \mathrm{Ann}(S) = \{r \in R : rS = 0\}$, where $S \neq \{0\}$ is a nonempty subset of R.

Proposition 2.11. *If X is an almost discrete space (i.e., $I(X)$, the set of isolated points of X, is dense in X), then $L_1(X) = L_F(X) = L_c(X) = C(X)$. In particular, if every nonzero annihilator ideal of $C(X)$, where X is any space, contains a nonzero minimal ideal, then the latter equalities hold.*

Proof. If $f \in C(X)$, then $C_f^c \supseteq \bigcup_{x\in I(X)}\{x\} = I(X)$. Hence $\overline{C_f^c} = X$, i.e., $f \in L_1(X)$. Finally, we first recall that $C(X)$ contains many nonzero zero-divisors (note, for each $0 \neq f \in C(X), (f - |f|)(f + |f|) = 0$. Hence nontrivial annihilator ideals in $C(X)$ always exist. Consequently, by our assumption the socle of $C(X)$ is not zero, i.e., $C_F(X) \neq 0$. We now claim that $\mathrm{Ann}\big(C_F(X)\big) = 0$. To see this, if $I = \mathrm{Ann}\big(C_F(X)\big) \neq 0$, then I must contain a nonzero minimal ideal, hence $I \cap C_F(X) \neq 0$. But, $(I \cap C_F(X))^2 = 0$ and since $C(X)$ is reduced, we infer that $I \cap C_F(X) = 0$, which is absurd. This means that we have already shown that $\mathrm{Ann}\big(C_F(X)\big) = 0$, which by [20, Proposition 2.1] is equivalent to the density of $I(X)$ in X, hence we are done by the first part. □

Before, presenting the next fact, we evidently note that every scattered space is an almost discrete space, for if $x \in X$ and U_x is a neighborhood of x, then U_x has an isolated point x_0. Since U_x is open, x_0 is an isolated point of X, too. Hence $x_0 \in U_x \cap I(X) \neq \varnothing$, therefore $\overline{I(X)} = X$.

Proposition 2.12. *If X is a scattered space, then $L_1(X) = L_F(X) = L_c(X) = C(X)$. In particular, if X is a compact scattered space, then the latter rings coincide with $C_c(X)$.*

Proof. By the above comment and RPS-Theorem we are done. □

The following example shows that the converse of the above corollary is not valid.

Example 2.13. Let for each point $x \in \mathbb{Q}$, the basic neighborhood of x be the singleton $\{x\}$, and for each $x \in \mathbb{Q}^c$, the basic neighborhood of x be the usual open interval containing x. This constitutes a topology on $X = \mathbb{R}$, and it is clearly, a Hausdorff normal space which is almost discrete, since $I(X) = \mathbb{Q}$. Hence $L_c(X) = C(X)$, but X is not scattered.

In view of RPS-Theorem we may naturally define a compact space X to be scattered if given any $f \in C(X)$ and any $x \in X$, there exists a compact neighborhood V_f of x such that $|f(V_f^\circ)| \leq \aleph_0$. Motivated by this we give the following definition.

Definition 2.14. A space X is called *locally scattered* if given any $f \in C(X)$ and a nonempty open set G, there exists a compact subset V_f of X in G, with $\varnothing \neq V_f^\circ \subseteq G$ and $|f(V_f^\circ)| \leq \aleph_0$.

The space βX where X is discrete is locally scattered. Clearly, every scattered space is a locally scattered space, but the converse is not true. For example, $\beta\mathbb{N}$ is a locally scattered space which is not scattered, for $\beta\mathbb{N}\backslash\mathbb{N}$ has no isolated point (note, each clopen subset of $\beta\mathbb{N}\backslash\mathbb{N}$ has the same cardinality as $\beta\mathbb{N}\backslash\mathbb{N}$, see [13, 6S(4)]).

Lemma 2.15. *Let X be a locally scattered space. Then every open C-embedded subset of X (e.g., any clopen subset) is also locally scattered.*

Proof. Let Y be an open C-embedded subset of X, and G be an open subset in Y, and $f \in C(Y)$. Since Y is C-embedded in X, we infer that there exists $g \in C(X)$ such that $g|_Y = f$. Clearly, G is open in X and by our assumption, there exists a compact subset V_g in G such that $\varnothing \neq V_g^\circ \subseteq G \subseteq Y$, $|g(V_g^\circ)| \leq \aleph_0$. Thus V_g is compact in Y in G with $|f(V_g^\circ)| = |g(V_g^\circ)| \leq \aleph_0$, i.e., Y is locally scattered. $\square$

Let us recall that a Hausdorff space X is locally compact if and only if each point in X has a compact neighborhood. Clearly, every compact Hausdorff space is locally compact. The following result is somewhat similar to RPS-Theorem.

Theorem 2.16. *Let X be a compact space. Then $L_c(X) = C(X)$ if and only if X is locally scattered. In particular, if X is a discrete space and Y is a non-scattered clopen subset of βX (e.g., $X = \mathbb{N}$ and $Y = \beta\mathbb{N}$), then $L_c(Y) = C(Y) = C^*(Y) \neq C_c(Y)$.*

Proof. First, we assume that X is compact and $L_c(X) = C(X)$. Now, for each $f \in C(X)$ we have $\overline{C_f} = X$. Hence for any nonempty open subset G in X there exists an open subset U_f in X such that $|f(U_f)| \leq \aleph_0$, $U_f \cap G \neq \varnothing$. Since the open subsets of a locally compact space are locally compact, we infer that $U_f \cap G$ is locally compact. Consequently, any neighborhood of a point $x \in U_f \cap G$ contains a compact neighborhood, V_f say, of x. Hence $x \in V_f^\circ \subseteq V_f \subseteq U_f \cap G \subseteq X$ and $|f(V_f^\circ)| \leq |f(U_f)| \leq \aleph_0$, which means that X is locally scattered and we are done. The converse is evident by Definition 2.14, and the definition of $L_c(X)$. For the last part, we notice that Y as a closed subset of βX is compact and by Lemma 2.15, it is locally scattered. Now by the first part and the compactness of Y, we have $L_c(Y) = C(Y) = C^*(Y)$. But, in view of RPS-Theorem and the fact that Y is not scattered, we infer that $C_c(Y) \neq C(Y)$, and we are done. $\square$

The previous proof immediately yields the following fact, too.

Corollary 2.17. *Let X be a locally compact space. Then $L_c(X) = C(X)$ if and only if X is locally scattered.*

We recall that if the set of open neighborhoods of a point P in X is closed under countable intersection, then P is called a P-point. The set of all P-points of X is denoted by $\mathcal{P}_X$ and X is called a P-space if $\mathcal{P}_X = X$. An interesting result due to A. W. Hager asserts that a P-space X is functionally countable (i.e., $C(X) = C_c(X)$) if and only if it is pseudo-$\aleph_1$-compact (i.e., each locally finite family of open sets is countable), see [21, Proposition 3.2]. This result is extended to $C_c(X) = C^F(X)$ in [11, Proposition 4.2]. The following is also a counterpart of the latter result.

Proposition 2.18. *If $\overline{\mathcal{P}_X} = X$ (in particular, if X is a P-space), then $L_1(X) = L_F(X) = L_c(X) = C(X)$.*

Proof. For each $f \in C(X)$ and $x \in \mathcal{P}_X$ there exists an open neighborhood U_x of x such that f is constant on U_x, see [13, 4L(3)]. Therefore $C_f^c \supseteq \bigcup_{|f(U_x)|=1} U_x \supseteq \mathcal{P}_X$, hence $f \in L_1(X)$. $\square$

We note that $\beta\mathbb{N}$ is not a P-space while $L_1(\beta\mathbb{N}) = L_F(\beta\mathbb{N}) = L_c(\beta\mathbb{N}) = C(\beta\mathbb{N})$. By [13, 6V(6)], $\beta\mathbb{N}\backslash\mathbb{N}$ has a dense set of P-points, hence $L_1(\beta\mathbb{N}\backslash\mathbb{N}) = L_F(\beta\mathbb{N}\backslash\mathbb{N}) = L_c(\beta\mathbb{N}\backslash\mathbb{N}) = C(\beta\mathbb{N}\backslash\mathbb{N})$.

Remark 2.19. Let X be a P-space without isolated points, see [13, 13 P], then X is not almost discrete. But by Proposition 2.18, $L_1(X) = L_F(X) = L_c(X) = C(X)$, see also Proposition 2.11.

Let us borrow the following definition from [16].

Definition 2.20. A topological space X is called *locally functionally countable* if every point $x \in X$ is *countably P-point*, in the sense that there exists an open neighborhood U_x of x such that $C(U_x) = C_c(U_x)$.

The following result implies that if a space X is second countable or a compact space, then X is locally functionally countable if and only if it is functionally countable (i.e., $C(X) = C_c(X)$).

Proposition 2.21. *Let X be a Lindelöf space. Then X is locally functionally countable if and only if it is functionally countable.*

Proof. It is evident that every functionally countable space is locally functionally countable (note, for each $x \in X$ take $U_x = X$). Conversely, let X be locally functionally countable, then for each $f \in C(X)$, $f(X) = f(\bigcup_{x\in X} U_x)$, where

U_x is an open neighborhood of x with $C(U_x) = C_c(U_x)$. Since X is Lindelöf and $C(X) \subseteq C(U_x)$, for each $x \in X$, we infer that $f(X) = f(\bigcup_{i=1}^{\infty} U_{x_i}) = \bigcup_{i=1}^{\infty} f(U_{x_i})$ is countable, and we are done. □

The next result shows that for every locally functionally countable space X, $C(X)$ coincides with $L_c(X)$. But the converse is not true in general, see Example 2.13 (note, $\mathbb{R}$ with the topology in this example is not locally functionally countable, for no irrational number is a countably P-point).

Proposition 2.22. *If X is a locally functionally countable space, then $L_c(X) = C(X)$.*

Proof. We must show that for each $f \in C(X)$, $\overline{C_f} = X$. Let $G \subseteq X$ be an open set in X and $x \in G$. Since X is locally functionally countable, there exists an open neighborhood U_x of x such that $C(U_x) = C_c(U_x)$. Clearly $|f(U_x)| = |(f|_{U_x})(U_x)| \leq \aleph_0$. Now, $x \in U_x \bigcap G \neq \varnothing$ and $U_x \subseteq C_f$ imply that $C_f \bigcap G \neq \varnothing$, hence $\overline{C_f} = X$. □

It is clear that if Y is a subset of X such that for each $f \in C(X)$, $f|_Y$ is constant, then Y must be a singleton. For otherwise, if $y_1, y_2 \in Y$ and $y_1 \neq y_2$, then by complete regularity of X there exists $f \in C(X)$ such that $f(y_1) \neq f(y_2)$, which is absurd. Hence the following definition, which is also needed, is now in order.

Definition 2.23. If Y is a subset of a space X, then the set of all $f \in C(X)$ such that $f|_Y$ is constant is a subalgebra of $C(X)$, denoted by $C_1(Y)$. Naturally, we say that Y is *constant with respect to a subring A of $C(X)$* if $A \subseteq C_1(Y)$.

We note that for every topological space X, $C_1(X) = C(X)$ if and only if X is singleton. If Y is a proper closed subset of X, then $\mathbb{R} \subsetneq C_1(Y)$.

The following proposition is evident.

Proposition 2.24. *Let X be a topological space and Y be a connected subset of X, then $C_c(X) \subseteq C_1(Y)$. In particular, if $X \backslash Y$ is countable, then $A \subseteq C_1(Y)$ if and only if $A \subseteq C_c(X)$.*

We conclude this section with the following fact whose proof is evident by the complete regularity of X.

Corollary 2.25. *For any subspace Y of X, $\mathbb{R} \subseteq C_1(Y) \subseteq C(X)$. Moreover, $C_1(Y) = \mathbb{R}$ if and only if Y is dense in X.*

Proof. For the last part we note that if $x \notin \overline{Y}$, then there exists $f \in C(X)$ with $f(x) = 0$ and $f(\overline{Y}) = 1$, i.e., $C_1(Y) \neq \mathbb{R}$. This implies that $\overline{Y} = X$ in case $C_1(Y) = \mathbb{R}$. Conversely, let $\overline{Y} = X$ and take $f \in C(X)$ such that $f \in C_1(Y)$, then $f(Y) = c$, where $c \in \mathbb{R}$. Consequently, $f = c$ in $C(X)$, for Y is dense in X, hence we are done. □

3. z_l-IDEALS

We remind the reader that many facts in the context of $C(X)$ can be extended naturally to $L_c(X)$, similarly to $C_c(X)$, see [10]. The proofs of most of the results in this section follow mutatis mutandis from the proofs of their corresponding results in [10]. Therefore, we state them without proofs, for the record, but give pertinent references for their corresponding proofs (note, the reason that we emphasize on the recording of these facts here is because we do believe that $L_c(X)$ and $C_c(X)$, are eligible to play appropriate roles as companions of $C(X)$, in the future studies in the context of $C(X)$, see for example, the comment in the first two lines of the introduction in [4].

Definition 3.1. A space X is said to be *locally countably pseudocompact* (briefly, lc-pseudocompact) if $L_c^*(X) = L_c(X)$, where $L_c^*(X) = L_c(X) \cap C^*(X)$.

The next three results are the counterparts of [13, Theorem 1.7, Corollary 1.8, and Theorem 1.9].

Proposition 3.2. *Every homomorphism $\varphi : L_c(X) \to L_c(Y)$ takes $L_c^*(X)$ into $L_c^*(Y)$.*

Corollary 3.3. *If Y is not a lc-pseudocompact space, then $L_c(Y)$ can not be a homomorphic image of any $L_c^*(X)$.*

Corollary 3.4. *Let φ be a homomorphism from $L_c(X)$ into $L_c(Y)$ whose image contains $L_c^*(Y)$, then $\varphi(L_c^*(X)) = L_c^*(Y)$.*

If $f \in L_c(X)$ and $f > 0$, then there exists $g \in L_c(X)$ with $f = g^2$. We also note that whenever $f \in L_c(X)$ and $f^r \in C(X)$ where $r \in \mathbb{R}$, then $f^r \in L_c(X)$. We recall that all positive units in $L_c(X)$ have the same number of square roots, see [13, 1B(1)]. The following proposition and its corollary are the counterparts of [13, 1D(1)] and [10, Lemma 2.4] for $L_c(X)$. Since the latter facts play a basic role in the context of $C(X)$, we present sketch of proofs for these counterparts.

Proposition 3.5. *If $f, g \in L_c(X)$ and $Z(f)$ is a neighborhood of $Z(g)$, then $f = gh$ for some $h \in L_c(X)$.*

Proof. We have $Z_l(g) \subseteq intZ_l(f)$. Put

$$h(x) = \begin{cases} 0 & , \ x \in Z_l(f) \\ \frac{f(x)}{g(x)} & , \ x \notin intZ_l(f) \end{cases}$$

therefore $h \in C(X)$, and $\overline{C_h} \supseteq \overline{C_f \cap C_{1/g}} = \overline{C_f \cap C_g} = X$. Hence $h \in L_c(X)$ and $f = gh$. $\square$

Corollary 3.6. *If $f, g \in L_c(X)$, and $|f| \leq |g|^r, r > 1$, then $f = gh$ for some $h \in L_c(X)$. In particular, if $|f| \leq |g|$, then whenever f^r is defined for $r > 1$, f^r is a multiple of g.*

Proof. Let

$$h(x) = \begin{cases} 0 & , \ x \in Z_l(g) \\ \frac{f(x)}{g(x)} & , \ x \notin Z_l(g) \end{cases}$$

then $h \in C(X)$, $\overline{C_h} \supseteq \overline{C_f \cap C_g} = X$. Hence $h \in L_c(X)$ and $f = gh$. $\square$

Proposition 3.7. *If $f \in L_c(X)$, then there exists a positive unit $u \in L_c(X)$ with $(-1 \vee f) \wedge 1 = uf$.*

Proof. Put

$$u(x) = \begin{cases} 1 & , \quad -1 \leq f(x) \leq 1 \\ \frac{1}{|f(x)|} & , \quad 1 \leq |f(x)| \end{cases}$$

Clearly $C_u = C_f$, hence $u \in L_c(X)$ and $(-1 \vee f) \wedge 1 = uf$. So if $f \in L_c(X)$, then f and $(-1 \vee f) \wedge 1$ belongs to an ideal of $L_c(X)$ $\square$

Remark 3.8. The previous results are also true if we replace $L_c(X)$ by either $L_F(X)$ or $L_1(X)$.

Convention. Let us put $Z_l(X) = \{Z(f) : f \in L_c(X)\}$, $Z_F(X) = \{Z(f) : f \in L_F(X)\}$, and $Z_1(X) = \{Z(f) : f \in L_1(X)\}$, where X is a topological space.

Definition 3.9. Two subsets A and B of a topological space X are said to be *locally countably separated* (briefly, lc-separated) in X if there is an element $f \in L_c(X)$ such that $f(A) = 1$, $f(B) = 0$.

The following result is the counterpart of [13, Theorem 1.15], [10, Theorem 2.8] .

Theorem 3.10. *Two subsets A, B of a space X are lc-separated if and only if they are contained in disjoint members of $Z_l(X)$. Moreover, lc-separated sets have disjoint zero-set neighborhoods in $Z_l(X)$.*

Clearly, if $a < b$ and $f \in L_c(X)$ such that $f(x) \leq a$, $\forall x \in A$, and $f(x) \geq b$, $\forall x \in B$, where A, B are subsets of X, then A, B are lc-separated in X.

Corollary 3.11. *If A, B are lc-separated in X, then there are zero-sets Z_1, Z_2 in $Z_l(X)$ with $A \subseteq X \setminus Z_1 \subseteq Z_2 \subseteq X \setminus B$.*

Definition 3.12. $\varnothing \neq F \subseteq Z_l(X)$ is called a z_l*-filter* on X if F satisfies the following conditions.

(1) $\varnothing \notin F$.
(2) $Z_1, Z_2 \in F$, then $Z_1 \cap Z_2 \in F$.
(3) $Z \in F$, $Z' \in Z_l(X)$ with $Z' \supseteq Z$, then $Z' \in F$.

Prime z_l-filter and z_l-ultrafilter are defined similarly to their counterparts in [13]. If I is an ideal of $L_c(X)$, then $Z_l[I] = \{Z(f) : f \in I\}$ is a z_l-filter on X. Conversely, if F is a z_l-filter on X, then $Z^{-1}[F] = \{f \in L_c(X) : Z(f) \in F\}$ is an ideal in $L_c(X)$. Moreover, every z_l-filter F is of the form $F = Z_l[I]$ for some ideal I in $L_c(X)$ and for any ideal J in $L_c(X)$, $Z^{-1}[Z_l[J]]$ is an ideal in $L_c(X)$ containing J. In Example 2.13, we consider the identity function $i : (\mathbb{R}, \mathcal{T}) \to \mathbb{R}$, clearly $i \in L_c(\mathbb{R}) = C(\mathbb{R})$. Now, put $I = (i)$, then $Z_l(I) = \{0\}$. Clearly, $f(x) = x^{1/3} \in L_c(\mathbb{R})$, $f \in Z^{-1}[Z_l[I]] \setminus I$. Hence the following definition is in order.

Definition 3.13. An ideal I in $L_c(X)$ is called a z_l*-ideal* if whenever $Z(f) \in Z_l[I]$ and $f \in L_c(X)$, then $f \in I$. Similarly, z_F-ideal and z_1-ideal are defined, see the previous convention.

Clearly, every z_l-ideal is an intersection of prime ideals in $L_c(X)$. Similarly, every z_F-ideal and z_1-ideal is an intersection of prime ideals in $L_F(X)$ and $L_1(X)$.

We emphasize again that the proofs of the following results are the same as the proofs of their counterparts in $C(X)$ and $C_c(X)$, see [10], and [13]. One can easily observe that these results up to Proposition 3.24 and including it are also valid for $L_c(X)$ and $L_F(X)$. The next theorem is the counterpart of [13, Theorem 2.9], [10, Theorem 2.13].

Theorem 3.14. *Let P be any z_l-ideal in $L_c(X)$. Then the following statements are equivalent.*

1. *P is a prime ideal in $L_c(X)$.*
2. *P contains a prime ideal in $L_c(X)$.*
3. *For all $f, g \in L_c(X)$, if $fg = 0$, then $f \in P$ or $g \in P$.*
4. *For each $f \in L_c(X)$, there exists a zero-set in $Z_l[P]$ on which f does not change sign.*

Corollary 3.15. *Every prime ideal in $L_c(X)$ is contained in a unique maximal ideal in $L_c(X)$.*

Clearly if P is a prime ideal in $L_c(X)$, then $Z_l[P]$ is a prime z_l-filter, and if F is a prime z_l-filter, then $Z_l^{-1}[F]$ is a prime z_l-ideal. It is evident that every prime z_l-filter is contained in a unique z_l-ultrafilter. The following lemma is the counterpart of [10, Lemma 3.1], also see [28].

Lemma 3.16. *Let $f, g, l \in L_c(X)$, $Z(f) \supseteq Z(g) \cap Z(l)$ and define*

$$h(x) = \begin{cases} 0 & , \ x \in Z(g) \cap Z(l) \\ \frac{fg^2}{g^2+l^2} & , \ x \notin Z(g) \cap Z(l) \end{cases}, \quad k(x) = \begin{cases} 0 & , \ x \in Z(g) \cap Z(l) \\ \frac{fl^2}{g^2+l^2} & , \ x \notin Z(g) \cap Z(l) \end{cases}$$

Then we have the following conditions.

1. $|k| \vee |h| \leq |f|$.
2. $f = h + k$.
3. $fl^2 = k(g^2 + l^2)$, $fg^2 = h(g^2 + l^2)$.
4. $h, k \in L_c(X)$.
5. $C_h \supseteq C_f \cap C_g \cap C_l$ *and* $C_k \supseteq C_f \cap C_g \cap C_l$.

The following results are the counterparts of [10, Corollary 3.2 to Corollary 3.8].

Lemma 3.17. *Let A, B be two z_l-ideals in $L_c(X)$. Then either $A + B = L_c(X)$ or $A + B$ is a z_l-ideal.*

Corollary 3.18. *Let $F = \{A_i\}_{i \in I}$ be a collection of z_l-ideals in $L_c(X)$. Then either $\sum_{i \in I} A_i = L_c(X)$ or $\sum_{i \in I} A_i$ is a z_l-ideal.*

Proposition 3.19. *Every minimal prime ideal in $L_c(X)$ is a z_l-ideal.*

Corollary 3.20. *Let $F = \{P_i\}_{i\in I}$ be a collection of minimal prime ideals in $L_c(X)$. Then either $\sum\limits_{i\in I} P_i = L_c(X)$ or $P = \sum\limits_{i\in I} P_i$ is a prime ideal in $L_c(X)$.*

Proposition 3.21. *A prime ideal P in $L_c(X)$ is absolutely convex.*

Proposition 3.22. *The sum of a collection of semiprime ideals in $L_c(X)$ is a semiprime ideal or is the entire ring $L_c(X)$.*

Proposition 3.23. *Let P be a prime ideal in $L_c(X)$. Then the ring $L_c(X)/P$ is totally ordered and its prime ideals are comparable.*

The next corollary is much stronger than Corollary 3.20 whose proof is similar to [10, Corollary 3.9].

Proposition 3.24. *Let $\{P_i\}_{i\in I}$ be a collection of semiprime ideals in $L_c(X)$ such that at least one of P_i's is prime, then $\sum_{i\in I} P_i$ is a prime ideal or all of $L_c(X)$.*

All the previous results beginning with Theorem 3.14, are also valid for $L_1(X)$. The following theorem is the counter part of [10, Theorem 3.10], see also the comment preceding [10, Theorem 3.10].

Theorem 3.25. *Let I be an ideal in $L_c(X)$. Then I and $\sqrt{I}$ have the same largest z_l-ideal.*

4. LOCALLY COUNTABLE COMPLETELY REGULAR SPACES

Definition 4.1. A Hausdorff space X is called *locally countable completely regular* (briefly, lc-completely regular) if whenever $F \subseteq X$ is a closed set and $x \in X\backslash F$, then there exists $f \in L_c(X)$ with $f(F) = 0$ and $f(x) = 1$.

We should remind the reader that, in this section, whenever the proof of a result is very similar to the proof of its counterpart in the literature, the proof is avoided.

The proof of the following result is evident.

Proposition 4.2. *A Hausdorff space X is lc-completely regular if and only if whenever $F \subseteq X$ is closed and $x \in X \setminus F$, then x and F have two disjoint zero-set neighborhoods in $Z_l(X)$. Consequently, there exist $g, h \in L_c(X)$ with $x \in X \setminus Z(h) \subseteq Z(g) \subseteq X \setminus F$.*

Clearly X is a lc-completely regular space if and only if $F = \{Z(f) : f \in L_c(X)\}$ is a base for the closed sets in X or equivalently if and only if $B = \{\mathrm{int(Z(f))} : \mathrm{f} \in \mathrm{L_c(X)}\}$ is a base for the open sets in X. The next proposition is the counterpart of [13, 3.11(a)], [10, Proposition 4.3].

Proposition 4.3. *Let X be a lc-completely regular space and A, B be two disjoint closed sets in X such that A is compact, then there is $f \in L_c(X)$ with $f(A) = 0$ and $f(B) = 1$.*

Proposition 4.4. *Let X be a compact space. Then X is lc-completely regular if and only if $L_c(X)$ separates points in X.*

Since $C_c(X)$ is a subring of $L_c(X)$, the next result is evident, see [10, Proposition 4.4]

Proposition 4.5. *If X is a zero-dimensional space, then X is a lc-completely regular space.*

The following fact is also similar to [13, Theorem 3.6], and [10, Corollary 4.5].

Proposition 4.6. *Let X be a Hausdorff space. Then X is a lc-completely regular space if and only if its topology coincides with the weak topology induced by $L_c(X)$.*

We recall that X is a P-space (resp., CP-space) if and only if $C(X)$ (resp., $C_c(X)$) is a regular ring, see [13, 4J] and [10]. In [10], it is shown that if $C(X)$ is regular, then so too is $C_c(X)$. If X is zero-dimensional, then the regularity of $C(X)$ and $C_c(X)$ coincide. We have already observed, see Proposition 2.18, that if X is a P-space, then $L_1(X) = L_F(X) = L_c(X) = C(X)$. The next definition is now in order.

Definition 4.7. A space X is called a *locally countably P-space* (briefly, LCP-space) if $L_c(X)$ is regular.

By the above comment we have the following result.

Proposition 4.8. *Every P-space is LCP-space.*

Proposition 4.9. *If A is any regular subring of $C(X)$ such that $C_c(X) \subseteq A \subseteq C(X)$, then $C_c(X)$ is regular. In particular, if $L_c(X)$ is regular, then $C_c(X)$ is regular, too.*

Proof. Let A be a regular ring, we must show that for each $f \in C_c(X)$, there exists $g \in C_c(X)$ such that $f = f^2g$. Since A is regular, there is $h \in A$ with $f = f^2h$. Consequently, $f = f^2g$, where $g = h^2f$. It is also evident that $Z(f) \subseteq Z(g)$ and $g(x) = \frac{1}{f(x)}$, whenever $x \notin Z(f)$. Hence $|g(X)| = |f(X)|$, i.e., $g \in C_c(X)$, and we are done. $\square$

Corollary 4.10. *Let X be a zero-dimensional space. Then X is P-space if and only if any of the rings $C_c(X)$, $L_c(X)$ is regular.*

Remark 4.11. It is wroth mentioning that if X is a zero-dimensional space, then the regularity of $C(X)$, $L_c(X)$, $C_c(X)$, and $C(X, K)$ (where $C(X, K)$, is a subring of $C(X)$ whose elements take values in K, a subfield of $\mathbb{R}$) coincide, see the above proposition and [10, Remark 7.5].

The following theorem is the counterpart of [10, Theorem 5.5] and its proof is also the same as the proof of its counterpart. We present a proof for the sake of completeness.

Theorem 4.12. *A space X is a LCP-space if and only if every zero-set in $Z_l(X)$ is open. Moreover, in this case whenever $\{f_i\}_{i=1}^{\infty}$ is a countable set in $L_c(X)$, then $\bigcap\limits_{i=1}^{\infty} Z_l(f_i)$ is an open zero-set in $Z_l(X)$.*

Proof. Let X be a LCP-space and $f \in L_c(X)$, hence $f = f^2 g$ for some $g \in L_c(X)$. It is evident that $e = fg$ is an idempotent in $L_c(X)$ and $Z(f) = Z(e) = X \backslash Z(1-e)$ is clopen. Conversely, let $Z(f)$ be open for each $f \in L_c(X)$, we are to show that $L_c(X)$ is regular. Since $Z(f) = Z(f^2)$ for all $f \in L_c(X)$, we infer that $f = f^2 g$ for some $g \in L_c(X)$, by Proposition 3.5. Hence $L_c(X)$ is regular. Finally, let I be an ideal in $L_c(X)$, generated by $\{f_i\}_{i=1}^{\infty}$, i.e., $I = \sum_{i=1}^{\infty} f_i L_c(X)$. If $I = C(X)$, then we are done in this case, for $\bigcap\limits_{i=1}^{\infty} Z_l(f_i) = \varnothing$. Hence we assume that $I \neq C(X)$. Since $L_c(X)$ is regular, we infer that $I = \sum_{i=1}^{\infty} \oplus e_i L_c(X)$, where each e_i, $i \in I$ is an idempotent in $L_c(X)$, and for each $i \neq j$, $e_i e_j = 0$, see [10, Theorem 5.5], or [17, Lemma 2], [8, Proposition 1.4]. If $x \in X$, $e_j(x) \neq 0$, then for each $i \neq j$, $e_i(x) = 0$, and

$$\bigcap Z[I] = \bigcap_{i=1}^{\infty} Z(f_i) = \bigcap_{i=1}^{\infty} Z(e_i)$$

Now, we may define $g = \sum_{i=1}^{\infty} \frac{e_i}{p^i(1+e_i)}$, where $p \geq 2$ is a real number. Clearly $g \in C(X)$, and $Z(g) = \bigcap_{i=1}^{\infty} Z(e_i)$. On the other hand for each $x \in X$, there exists at most a unique $i \geq 1$ such that $e_i(x) \neq 0$. Therefore $g(x) = \frac{e_i(x)}{p^i(1+e_i(x))} = \frac{1}{2p^i}$. Hence $g(X) \subseteq \{0, \frac{1}{2p}, \frac{1}{2p^2}, \ldots\}$ i.e., $g \in C_c(X)$, therefore $g \in L_c(X)$. □

Remark 4.13. In view of the previous proof we may record an interesting fact, which follows. Let X be a LCP-space and $\{f_i\}_{i\in I}$ be an infinite countable set of elements in $C(X)$, then $\bigcap_{i\in I} Z(f_i) = Z(g)$, where $g \in C_c(X) \subseteq L_c(X)$ can be chosen with the property that $g(X)$ is an infinite subset of an arbitrary subfield of $\mathbb{R}$.

It is well known that X is a P-space if and only if every G_δ-set is open, see [13, 4J(3)]. The following theorem is the counterpart of this result, see also [10, Corollary 5.7].

Corollary 4.14. *Let X be a lc-completely regular LCP-space. Then every G_δ-set A containing a compact set S contains a zero-set in $Z_l(X)$ containing S. In particular, every lc-completely regular LCP-space is a P-space.*

If $M_p^l = M_p \cap L_c(X)$ and $O_p^l = O_p \cap L_c(X)$, where $p \in X$ and O_p is the ideal of $C(X)$ consisting of all f in $C(X)$ for which $Z(f)$ is a neighborhood of p. It goes without saying that M_p^l is a maximal ideal in $L_c(X)$ and O_p^l is a z_l-ideal in $L_c(X)$. The following theorem is the counterpart of [13, 4J], [10, Theorem 5.8].

Theorem 4.15. *Let X be a topological space. Then the following statements are equivalent.*

(1) *X is a LCP-space.*
(2) *$L_c(X)$ is a regular ring.*
(3) *Each ideal in $L_c(X)$ is a z_l-ideal.*
(4) *Each prime ideal in $L_c(X)$ is a maximal ideal.*
(5) *For each $p \in X$, $M_p^l = O_p^l$.*
(6) *Every zero-set in $Z_l(X)$ is open.*
(7) *Each ideal in $L_c(X)$ is an intersection of maximal ideals.*
(8) *For all $f, g \in L_c(X)$, $(f,g) = (f^2 + g^2)$.*
(9) *For every $f \in L_c(X)$, $Z_l(f)$ $(X \setminus Z_l(f))$ is C-embedded.*
(10) *If $\{f_i : i \in \mathbb{N}\} \subseteq L_c(X)$, then $\bigcap_{i=1}^{\infty} Z_l(f_i)$ is an open zero-set in $Z_l(X)$.*

The following results are the counterparts of [11, Proposition 2.5] and [11, Corollary 2.6].

Proposition 4.16. *$L_c(X)$ is regular if and only if every pseudoprime ideal in $L_c(X)$ is prime.*

Corollary 4.17. *Let X be a lc-completely regular. Then every pseudoprime ideal in $C(X)$ is prime if and only every pseudoprime ideal in $L_c(X)$ is prime.*

The following theorem is similar to [13, Theorem 4.11], [11, Theorem 3.8].

Theorem 4.18. *Let X be a lc-completely regular space, then the following statements are equivalent.*

(1) *X is compact.*
(2) *Every ideal of $L_c(X)$ is fixed.*
(3) *Every maximal ideal of $L_c(X)$ is fixed.*
(4) *Every prime ideal of $L_c(X)$ is fixed.*

If X is any topological space and $x \in X$, $M_x^l = M_x \cap L_c(X)$, then as we pointed out earlier M_x^l is a maximal ideal of $L_c(X)$ and in fact $\frac{L_c(X)}{M_x^l} \cong \mathbb{R}$. Consequently, the Jacobson radical of $L_c(X)$ is zero.

Definition 4.19. A maximal ideal M in $L_c(X)$ is called a *real maximal ideal of $L_c(X)$* if $\frac{L_c(X)}{M} \cong \mathbb{R}$. A topological space X is called *locally countably realcompact space* (briefly, lc-realcompact) if every real maximal ideal M of $L_c(X)$ is of the form $M = M_x^l$ for some $x \in X$.

The following results are the counterparts of [13, 10.5(c)] and [11, Theorem 3.11].

Theorem 4.20. *X is a lc-realcompact space if and only if each nonzero homomorphism from $L_c(X)$ into $\mathbb{R}$ is a valuation map.*

If X is a compact zero-dimensional space, the corresponding $x \to M_x^l$ is one-one from X onto the set of maximal ideals of $L_c(X)$, say $\mathrm{Max}(L_c(X))$, and hence the space X is homeomorphic to $\mathrm{Max}(L_c(X))$ with the Stone topology (note, the proof is similar to [13, 4.9(a)], see also the comment above [10, Theorem 3.9] and [19]). The proof of the following result which is similar to its counterpart in [13, Theorem 8.3], is omitted.

Proposition 4.21. *Two zero-dimensional lc-realcompact spaces X and Y are homeomorphic if and only if $L_c(X) \cong L_c(Y)$.*

We recall that if X, Y are compact zero-dimensional spaces, then $C(X) \cong C(Y)$ if and only if $C_c(X) \cong C_c(Y)$. In what follows we show that this result also holds if we replace $C_c(X)$ by $L_c(X)$, but the proof is not as evident.

Theorem 4.22. *Let X and Y be two lc-completely regular compact spaces (e.g., zero-dimensional compact spaces). Then X and Y are homeomorphic if and only if $L_c(X) \cong L_c(Y)$. In particular, if X, Y are compact zero-dimensional spaces, then $L_c(X) \cong L_c(Y)$ if and only if $C_c(X) \cong C_c(Y)$ if and only if $C^F(X) \cong C^F(Y)$ if and only if $C(X) \cong C(Y)$.*

Proof. Clearly if $L_c(X) \cong L_c(Y)$, then $\mathrm{Max}(L_c(X))$ and $\mathrm{Max}(L_c(Y))$ are homeomorphic (with the Stone topology), i.e., X, Y are homeomorphic, see the comment preceding Proposition 4.21. Conversely, let $\varphi : X \to Y$ be a homeomorphism from X onto Y. If $f \in L_c(Y)$, then we claim that $fo\varphi \in L_c(X)$. To see this, since $f \in L_c(Y)$, we infer that $Y = \overline{C_f} = \overline{\bigcup_{i\in I} V_i}$, where for each $i \in I$, V_i is open in Y and $|f(V_i)| \leq \aleph_0$. Let us put $U_i = \varphi^{-1}(V_i)$, where $i \in I$. Clearly U_i is open in X and $|fo\varphi(U_i)| = |fo\varphi(\varphi^{-1}(V_i))| = |f(V_i)| \leq \aleph_0$, hence $C_{fo\varphi} \supseteq \bigcup_{i\in I} U_i$. Since φ is open (note, φ^{-1} is continuous), we infer that

$$X = \varphi^{-1}(Y) = \varphi^{-1}(\overline{\bigcup_{i\in I} V_i}) \subseteq \overline{\varphi^{-1}(\bigcup_{i\in I} V_i)} = \overline{\bigcup_{i\in I} \varphi^{-1}(V_i)} = \overline{\bigcup_{i\in I} U_i}$$

Therefore $\overline{C_{fo\varphi}} = X$, i.e., $fo\varphi \in L_c(X)$. Now we define $\sigma : L_c(Y) \to L_c(X)$ with $\sigma(f) = fo\varphi$. It is evident that σ is an isomorphism from $L_c(Y)$ onto $L_c(X)$. The last part is evident. □

Remark 4.23. The above result shows that if X, Y are compact zero-dimensional spaces, such that $C(X) \cong C(Y)$, then $L_c(X) \cong L_c(Y)$. In the comment following [11, Corollary 9.5], it is observed that whenever X, Y are two arbitrary spaces (not necessary compact zero-dimensional) and $C(X) \cong C(Y)$, then $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$ (i.e., $C_c(X)$ and $C^F(X)$ are algebraic objects). This naturally raises the question that whether $L_c(X)$ is also an algebraic object, too (i.e., if $C(X) \cong C(Y)$, then is $L_c(X) \cong L_c(Y)$)? Clearly, if X, Y are strongly zero-dimensional spaces with $C(X) \cong C(Y)$, then $L_c(\beta X) \cong L_c(\beta Y)$.

Let us recall that a commutative ring R is selfinjective (resp., $\aleph_0$-selfinjective), if every homomorphism $f : I \to R$, where I is an ideal (resp., countably generated ideal) in R, can be extended to $\widehat{f} : R \to R$. We recall that a subset S of a commutative ring R is said to be orthogonal, provided $xy = 0$ for all $x, y \in S$ with $x \neq y$. In the following result we show that [10, Theorem 6.10] is also true for $L_c(X)$. In contrast to the proofs of some of the previous results, we should emphasize that the next proof can not be easily obtained from the proof of its counterpart (i.e., [10, Theorem 6.10]). It is well known that the $\aleph_0$-selfinjectivity of a ring is not a consequence of its regularity, in general, see [14, Examples 14.7, 14.9]. But, the following worthwhile fact shows that $L_c(X)$ as well as $C(X)$ and $C_c(X)$ have this rare property, see [8], [10]. We should remind the reader that $C^F(X)$ does not satisfy this property in general, see [10, Remark 6.11, Example 7.1] (note, $C^F(X)$ is always regular, see [11, the comment preceding Proposition 4.2].

Theorem 4.24. *Let X be a topological space. Then $L_c(X)$ is regular if and only if $L_c(X)$ is $\aleph_0$-selfinjective.*

Proof. If $L_c(X)$ is $\aleph_0$-selfinjective, then $L_c(X)$ is regular by [18, Proposition 1.2], or [10, Lemmas 6.7, 6.8, Remark 6.9]. Conversely, by [18, Lemma 1.9] and [10, Lemma 6.8, Remark 6.9], it suffices to show that if S is an orthogonal subset in $L_c(X)$, then there exists $f \in L_c(X)$ such that for each $g \in S$, $fg = g^2$. Let $S = \{f_i\}_{i=1}^{\infty}$, where $f_i \neq 0$, for each $i \in I$. Since $L_c(X)$ is regular, $\bigcap_{i=1}^{\infty} Z(f_i) = Z(h)$ is an open zero-set in $L_c(X)$, by Theorem 4.12. Put $G_i = X \backslash Z(f_i)$, for each $i \geq 1$. Since $f_i f_j = 0$, hence $G_i \cap G_j = \varnothing$, for each $i \neq j$, and G_i's are clopen for each $i \geq 1$. Let us put $G = \bigcup_{i=1}^{\infty} G_i$, hence $X = \bigcup_{i=1}^{\infty} G_i \cup (X \backslash G)$. We may define $f : X \to \mathbb{R}$ by $f(x) = \begin{cases} f_i(x) & , \ x \in G_i \\ 0 & , \ x \notin G \end{cases}$ i.e., $f|_{G_i} = f_i$ for all $i \geq 1$ and $f(x) = 0$ for all $x \in X \backslash G$. Hence f is continuous by [13, 1A(2)] and we must show that $f \in L_c(X)$. Let $V \subseteq X$ be an arbitrary open set, then we are to show that there exists an open set U in X such that $|f(U)| \leq \aleph_0$ and $U \cap V \neq \varnothing$. Now we consider two cases. First let $V \subseteq X \backslash G$, then $f(V) = 0$, hence $V \subseteq C_f$. Otherwise $V \cap G \neq \varnothing$, hence there exists a nonempty open subset G_i such that $V \cap G_i \neq \varnothing$. Since $f_i \in L_c(X)$ i.e., $\overline{C_{f_i}} = X$, hence there exists an open set $H \subseteq C_{f_i}$ such that $|f_i(H)| \leq \aleph_0$ and $\varnothing \neq H \cap (V \cap G_i) = U$. Now clearly, $|f(U)| = |f_i(U)| \leq |f_i(H)| \leq \aleph_0$ i.e., we are done. Finally, we claim that $ff_i = f_i^2$, for each $f_i \in S$ and this complete the proof, by [18, Lemma 1.9]. To this end, we note that if $f(x) = 0$, then $x \notin G$, hence $x \notin G_i$ for all $i \geq 1$, i.e., $x \in Z(f_i)$, for all $i \geq 1$. Thus $ff_i = f_i^2$, on $Z(f)$ for each $f_i \in S$. Since $f(x) = f_i(x)$, for each $x \in G_i = X \backslash Z(f_i)$ and $Z(f) \subseteq Z(f_i)$ for each $i \geq 1$, we infer that $ff_i = f_i^2$, for each $f_i \in S$, hence we are done. □

Remark 4.25. Let X be an uncountable discrete space, then $C(X) = L_c(X)$ is selfinjective but $C_c(X)$ is not selfinjective, see [10, Example 7.1, Remark 7.5]. More generally, if $C(X)$ is $\aleph_0$-selfinjective, then by [8, Theorem 1], X is a P-space. Hence in view of Proposition 2.18, we have $L_1(X) = L_F(X) = L_c(X) = C(X)$. Moreover in view of Theorem 4.22 and Remark 4.11, we note that the $\aleph_0$-selfinjectivity of $C(X)$, $L_c(X)$, $C_c(X)$, and $C(X, K)$ coincide if X is a zero-dimensional space.

5. The socle of $L_c(X)$

We recall that the socle of any commutative ring R, $\mathrm{Soc}(R)$, is the sum of its nonzero minimal ideals (in fact, it can be written as the direct sum of some of its nonzero minimal ideals). We recall that $C_F(X) = \{f \in C(X) : |X \setminus Z(f)| < \infty\}$, the socle of $C(X)$, is a z-ideal. Clearly, $C_F(X) \subseteq L_F(X) \subseteq L_c(X)$. We show that $C_F(X) \subseteq \mathrm{Soc}\big(L_c(X)\big)$, i.e., $C_F(X)$ is a sum of minimal ideals of $L_c(X)$. In [20, Proposition 3.1], it is shown that if I is a minimal ideal of $C(X)$, then $I = eC(X)$, where e is an idempotent such that $e(x) = 1$ and $e(X \setminus \{x\}) = 0$, where x is an isolated point of X. Clearly $C(X) = eC(X) \oplus (1-e)C(X)$, $eC(X) = eL_c(X) = eC_c(X)$, and $L_c(X) = eL_c(X) \oplus (1-e)L_c(X)$. We also note that $(1-e)L_c(X) = (1-e)C(X) \cap L_c(X)$ is a maximal ideal in $L_c(X)$, therefore $eL_c(X) = eC(X)$ is a minimal ideal in $L_c(X)$. Hence every minimal ideal of $C(X)$ is a minimal ideal in $L_c(X)$, too. Therefore $C_F(X)$ is an ideal in $L_c(X)$, and $C_F(X) \subseteq \mathrm{Soc}\big(L_c(X)\big)$. We should also emphasize that since $\mathrm{Soc}\big(L_c(X)\big)$ is a semisimple $L_c(X)$-module, hence $C_F(X)$ is a direct summand of $\mathrm{Soc}\big(L_c(X)\big)$ as a $L_c(X)$-module. In the proof of Theorem 5.4, we shall briefly observe that $\mathrm{Soc}\big(L_c(X)\big) \subseteq \mathrm{Soc}\big(C_c(X)\big)$. Let us also recall that $C^F(X)$ is the subring of $C(X)$ whose elements have finite image. Hence, we have $C_F(X) \subseteq C^F(X) \subseteq C_c(X) \subseteq L_c(X) \subseteq C(X)$ and $\mathrm{Soc}\big(C_c(X)\big) = \mathrm{Soc}\big(C^F(X)\big)$, see [10], [11]. The following lemma which is similar to [11, Lemma 5.1], somehow determines the minimal ideals of $L_c(X)$. Let us first remind the reader that if I is a nonzero minimal ideal in a reduced commutative ring R, then $I = eR$, where $e \in R$ is an idempotent (note, $I = (a) = (a^2)$, for every $0 \neq a \in I$, and $a = a^2 r$, for some $r \in R$, now put $e = ar$).

In [11, Lemma 5.1], it is shown that if $0 \neq e$ is an idempotent, then $eC_c(X)$ is a minimal ideal in $C_c(X)$ if and only if $Z(1-e)$ is connected. In the next lemma the minimal ideals in $L_c(X)$ are characterized, too.

Lemma 5.1. *Let I be a nonzero minimal ideal in $L_c(X)$, then $I = eL_c(X)$ where e is an idempotent in $L_c(X)$ such that $Z(1-e)$ is connected. Conversely, if $I = eL_c(X)$ where $e \neq 0$ is an idempotent in $L_c(X)$ such that $Z(1-e)$ is a constant subset of X with respect to $L_c(X)$, then I is a minimal ideal in $L_c(X)$.*

Proof. Let I be a nonzero minimal ideal in $L_c(X)$. Since $L_c(X)$ is reduced, $I = eL_c(X)$, where e is an idempotent in $L_c(X)$. If $Z(1-e)$ is not connected, there exists a nonempty clopen subset $A \subsetneq Z(1-e)$ (note, A is clopen in X, too). Now define the idempotent $e_1 \in L_c(X)$ such that $A = Z(1-e_1)$. Clearly $Z(1-e_1) \subsetneq Z(1-e)$. Consequently, $e_1 = ee_1$ but $e \neq e_1 e$. Hence $e_1 L_c(X) \subsetneq eL_c(X) = I$ and this contradicts the minimality of I. Conversely, let $I = eL_c(X)$, where $e \in L_c(X)$ such that $Y = Z(1-e) \subseteq X$ is a constant subset of X with respect to $L_c(X)$ (i.e., $L_c(X) \subseteq C_1(Y)$, see Definition 3.16). We are to show that I is minimal in $L_c(X)$. It suffices to show that $(1-e)L_c(X)$ is a maximal ideal in $L_c(X)$. Now we define $\varphi : L_c(X) \to \mathbb{R}$ by $\varphi(f) = f(Y)$.

Clearly, $ker\varphi = (1-e)L_c(X)$ and $\frac{L_c(X)}{(1-e)L_c(X)} \cong \mathbb{R}$, hence $(1-e)L_c(X)$ is maximal in $L_c(X)$. ☐

In [11, Proposition 5.3], the socle of $C_c(X)$ is characterized and in [11, Remark 5.2, and the introduction of Section 5], it's observed that $C_F(X) \subseteq \mathrm{Soc}\big(C_c(X)\big) = \mathrm{Soc}\big(C^F(X)\big)$. Next, we topologically characterize the socle of $L_c(X)$. The next proof is similar to the proof of [11, Proposition 5.3], but it's given for the sake of the reader.

Proposition 5.2. *Let $f \in L_c(X)$ be a nonunit element. If $f \in \mathrm{Soc}\big(L_c(X)\big)$, then $X\backslash Z(f) \subseteq \bigcup_{i=1}^n A_i$, where $n \in \mathbb{N}$ and $\{A_1, A_2, \ldots, A_n\}$ is a set of mutually disjoint clopen connected subsets of X. Conversely, if $X\backslash Z(f) \subseteq \bigcup_{i=1}^n A_i$, where $n \in \mathbb{N}$ and $\{A_1, A_2, \ldots A_n\}$ is a set of mutually disjoint clopen constant subsets of X with respect to $L_c(X)$, then $f \in \mathrm{Soc}\big(L_c(X)\big)$. In Particular, $\mathrm{Soc}\big(L_c(X)\big)$ is a z_l-ideal in $L_c(X)$.*

Proof. We put $\mathrm{Soc}\big(L_c(X)\big) = \sum_{i\in I} \oplus e_i L_c(X)$, where each e_i is an idempotent in $L_c(X)$, and $e_i L_c(X)$ is a nonzero minimal ideal in $L_c(X)$. Let $f = e_{i_1} f_1 + e_{i_2} f_2 + \ldots + e_{i_n} f_n$ be an element in $\mathrm{Soc}\big(L_c(X)\big)$, where $f_k \in L_c(X)$ and $i_k \in I$, $k = 1, 2, \ldots, n$. We put $A_{i_k} = Z(1 - e_{i_k})$, for each $i_k \in I$, $k = 1, 2, \ldots, n$. Clearly, A_{i_k}, $k = 1, 2, \ldots, n$, are clopen and connected, by Lemma 5.1. Since the idempotent elements $\{e_i : i \in I\}$ are mutually orthogonal, we infer that $\{A_i : i \in I, A_i = Z(1-e_i)\}$ is a set of mutually disjoint clopen connected subsets of X. If $x \notin \bigcup_{i=1}^n A_i$, then $e_{i_k}(x) = 0$, $k = 1, 2, \ldots, n$, hence $x \in Z(f)$. Therefore $X\backslash Z(f) \subseteq \bigcup_{i=1}^n A_i$. Conversely, let $X\backslash Z(f) \subseteq \bigcup_{i=1}^n A_i$, where $\{A_i : i \in I\}$ is a set of mutually disjoint clopen constant subsets in X with respect to $L_c(X)$, we show that $f \in \mathrm{Soc}\big(L_c(X)\big)$. Since each A_i is a clopen set, there exists an idempotent e_i, such that $A_i = Z(1 - e_i)$, where $i = 1, 2, \ldots, n$. We also note that each A_i is constant with respect to $L_c(X)$, hence there is a set of idempotents in $L_c(X)$, $\{e_1, \ldots, e_n\}$ say, which are mutually orthogonal and each $e_i L_c(X)$ is a minimal ideal in $L_c(X)$, by Lemma 5.1. Clearly, $f = e_1 f + e_2 f + \ldots + e_n f \in L_c(X)$ which belongs to $\mathrm{Soc}\big(L_c(X)\big) = \sum_{i\in I} \oplus e_i L_c(X)$. ☐

Remark 5.3. One can easily observe that if in the previous two results we trade off $L_c(X)$ with any $\mathbb{R}$-subalgebra of $L_c(X)$, A say, which contains $C_c(X)$, then the two results are also valid for A.

The next result determines spaces X such that the socles of $L_c(X)$, $C_c(X)$ and hence of $C^F(X)$ coincide.

Theorem 5.4. $\mathrm{Soc}\big(L_c(X)\big) = \mathrm{Soc}\big(C_c(X)\big)$ *if and only if the clopen connected subsets of X coincide with the clopen constant subsets of X with respect to $L_c(X)$.*

Proof. $\mathrm{Soc}\big(L_c(X)\big) \subseteq \mathrm{Soc}\big(C_c(X)\big)$, for if I is a minimal ideal in $\mathrm{Soc}\big(L_c(X)\big)$, then $I = eL_c(X)$ where e is an idempotent such that $Z(1-e)$ is connected, by Lemma 5.1. Hence I is a minimal ideal in $C_c(X)$, by [11, Lemma 5.1]. Now, let I be a nonzero minimal ideal in $\mathrm{Soc}\big(C_c(X)\big)$, so $I = eC_c(X)$, where $e \neq 0, 1$ is an idempotent and $Z(1-e)$ is a clopen connected subset in X, by [11, Lemma 5.1]. Hence by our hypothesis $Z(1-e)$ is constant with respect to $L_c(X)$. Therefore $I = eL_c(X) = eC_c(X)$ is a minimal ideal of $L_c(X)$, by Lemma 5.1, hence it is in $\mathrm{Soc}\big(L_c(X)\big)$. Conversely, let $\mathrm{Soc}\big(L_c(X)\big) = \mathrm{Soc}\big(C_c(X)\big)$, and $\varnothing \neq Y \subseteq X$ be a clopen constant subspace of X with respect to $L_c(X)$ we are to show that Y is connected. Clearly, there exists $e \in L_c(X)$ such that $e(Y) = 1$, $e(X \backslash Y) = 0$. But $Y = Z(1-e)$, hence by Lemma 5.1, we infer that $e \in \mathrm{Soc}\big(L_c(X)\big) = \mathrm{Soc}\big(C_c(X)\big)$. Consequently, $Y = Z(1-e)$ must be connected, by [11, Proposition 5.3]. □

We remind the reader in [11, Theorem 6.6], it is proved that $\mathrm{Soc}\big(C_c(X)\big) = C_F(X)$ if and only if each clopen connected subsets of X consists of a single isolated point. Motivated by this fact and Theorem 5.4, we present the next result.

Theorem 5.5. *If every proper nonempty clopen connected subset of X is singleton, (e.g., any totally disconnected space), then $\mathrm{Soc}\big(L_c(X)\big) = C_F(X)$. Conversely, if $\mathrm{Soc}\big(L_c(X)\big) = C_F(X)$, then every proper nonempty clopen constant subspace of X with respect to $L_c(X)$ is singleton.*

Proof. Let every proper nonempty clopen connected subset of X be singleton, we are to show that $\mathrm{Soc}\big(L_c(X)\big) = C_F(X)$. It is evident that $C_F(X) \subseteq \mathrm{Soc}\big(L_c(X)\big)$. Let I be a nonzero minimal ideal in $\mathrm{Soc}\big(L_c(X)\big)$, so by Lemma 5.1, $I = eL_c(X)$, where $e \neq 0, 1$ is an idempotent and $Z(1-e)$ is a clopen connected subset in X. Hence by our hypothesis $Z(1-e)$ is singleton. Therefore $I = eL_c(X) = eC(X)$ is a minimal ideal in $C_F(X)$, by [20, Proposition 3.3]. Conversely, let $C_F(X) = \mathrm{Soc}\big(L_c(X)\big)$, and $\varnothing \neq Y \subseteq X$ be a clopen constant subspace of X with respect to $L_c(X)$. There exists $e \in L_c(X)$ such that $e(Y) = 1$, $e(X \backslash Y) = 0$. Clearly, by Lemma 5.1, $e \in \mathrm{Soc}\big(L_c(X)\big) = C_F(X)$, hence $eC(X)$ is a minimal ideal in $C_F(X)$, therefore $Y = Z(1-e)$ is singleton. □

The following remark is now immediate.

Remark 5.6. $C_F(X) = \mathrm{Soc}\big(L_c(X)\big) = \mathrm{Soc}\big(C_c(X)\big)$ if and only if each clopen connected subset of X consists of a single isolated point. Consequently, if X is zero-dimensional or totally disconnected, we have $C_F(X) = \mathrm{Soc}\big(L_c(X)\big) = \mathrm{Soc}\big(C_c(X)\big)$.

Let us recall that an ideal in a commutative ring R is essential if it intersects every nonzero ideal of R nontrivially. It is well known and easy to show that a nonzero ideal I in a reduced ring R (i.e., no nonzero element in R is nilpotent) is essential if and only if $Ann(I) = 0$, see [3, Background and preliminary results]. The proof of the following corollary is similar to [11, Corollary 5.4], but we include the proof for the sake of the reader.

Corollary 5.7. *Let X be a lc-completely regular space, and $\mathrm{Soc}\big(L_c(X)\big) = \sum_{i \in I} \oplus e_i L_c(X)$, where $e_i L_c(X)$ is a nonzero minimal ideal of $L_c(X)$, and e_i*

is an idempotent for each $i \in I$. Put $Y = \bigcup_{i\in I} Z(1-e_i)$, then $\operatorname{Soc}\big(L_c(X)\big)$ is essential in $L_c(X)$ if and only if Y is dense in X.

Proof. Let $Y = \bigcup_{i\in I} Z(1-e_i)$ be dense in X, we are to show that $\operatorname{Soc}\big(L_c(X)\big)$ is essential in $L_c(X)$. Since $L_c(X)$ is reduced, in order to prove that $\operatorname{Soc}\big(L_c(X)\big)$ is essential in $L_c(X)$ it suffices to show that $\operatorname{Ann}(\operatorname{Soc}\big(L_c(X)\big)) = (0)$. We note that $f \in \operatorname{Ann}(\operatorname{Soc}\big(L_c(X)\big))$ if and only if $fe_i = 0$ for each $i \in I$. Now, if $fe_i = 0$, then $f(Z(1-e_i)) = 0$, hence $f(Y) = \{0\}$. Since Y is dense in X we infer that $f = 0$, and we are done. Conversely, let $\operatorname{Soc}\big(L_c(X)\big)$ be essential in $L_c(X)$, hence $\operatorname{Ann}(\operatorname{Soc}\big(L_c(X)\big)) = (0)$ in $L_c(X)$. Let us now take $x \in X\backslash\overline{Y}$ and obtain a contradiction. By lc-complete regularity of X, there exists $0 \neq f \in L_c(X)$ with $f(\overline{Y}) = f(Y) = 0$. Therefore $f(Z(1-e_i)) = 0$, hence $fe_i = 0$ for all $i \in I$. Thus $0 \neq f \in \operatorname{Ann}(\operatorname{Soc}\big(L_c(X)\big)) = (0)$, which is a contradiction. □

We recall that $C_F(X)$ is never a prime ideal of $C(X)$, see [8, Proposition 1.2], or [3, Remark 2.4]. The following result characterizes spaces X such that $C_F(X) \neq 0$ is a prime ideal in $L_c(X)$ (note, $C_F(X) \neq 0$ if and only if X has isolated points).

Proposition 5.8. *Let $|I(X)| < \infty$, where $I(X)$ is the set of isolated points in X. If $0 \neq C_F(X)$ is a prime ideal in $L_c(X)$, then $X\backslash I(X)$ is connected in X. Conversely, if $X\backslash I(X)$ is constant with respect to $L_c(X)$, then $0 \neq C_F(X)$ is prime in $L_c(X)$.*

Proof. Let $Y = X\backslash I(X) = A \cup B$, where A, B are two nonempty infinite disjoint clopen subsets of Y and seek a contradiction. Since Y is clopen in X we infer that A, B are also clopen in X. Clearly, $X = I(X) \cup A \cup B$. Now define $f, g \in L_c(X)$ such that $f(A \cup I(X)) = 1$, $f(B) = 0$ and $g(A \cup I(X)) = 0$, $g(B) = 1$. Clearly $fg = 0 \in C_F(X)$, but by [20, Proposition 3.3], we infer that $f, g \notin C_F(X)$, which is a contradiction. Conversely, let $Y = X\backslash I(X)$ be constant with respect to $L_c(X)$ and take $f, g \in L_c(X)$ such that $fg \in C_F(X)$. Clearly $X = Y \cup I(X)$, so $X\backslash Z(fg) \subseteq I(X)$ and $fg(Y) = 0$. Since f and g are constant on Y, we infer that either $f(Y) = 0$ or $g(Y) = 0$, i.e., $X\backslash Z(f) \subseteq I(X)$ or $X\backslash Z(g) \subseteq I(X)$, therefore $f \in C_F(X)$ or $g \in C_F(X)$, by [20, Proposition 3.3], and we are done. □

In the following corollary, we consider spaces X, such that $C_F(X)$ is not a prime ideal in $L_c(X)$.

Corollary 5.9. *If $I(X)$ is an infinite set or $Y = X\backslash I(X)$ is disconnected, then $C_F(X)$ is never a prime ideal in $L_c(X)$.*

Proof. Let $I(X)$ be an infinite set and take $A = \{x_n : n \in \mathbb{N}\}$, $B = \{y_n : n \in \mathbb{N}\}$ to be two disjoint countably infinite subsets of $I(X)$. We now define $f(x) = \begin{cases} \frac{1}{n} & , \ x = x_n \in A \\ 0 & , \ x \notin A \end{cases}$ and $g(x) = \begin{cases} \frac{1}{n} & , \ x = y_n \in B \\ 0 & , \ x \notin B \end{cases}$. Let $\epsilon > 0$ be given, then there exists $k \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$, for all $n \geq k$. Now, for the clopen subsets $G = X\backslash\{x_1, x_2, \ldots, x_k\}$, $H = X\backslash\{y_1, y_2, \ldots, y_k\}$ and for each $x \in G$, $y \in H$, we have $|f(x)| < \epsilon$, $|g(y)| < \epsilon$, hence $f, g \in C(X)$. Clearly, $f, g \in C_c(X)$. Therefore $f, g \in L_c(X)$ and $0 = fg \in C_F(X)$, but $f, g \notin C_F(X)$, by [20, Proposition 3.3]. Consequently, in this case $C_F(X)$ is not prime in $C_c(X)$, a fortiori, in $L_c(X)$). Finally let $|I(X)| < \infty$ and $X \setminus I(X)$ be disconnected, hence by Proposition 5.8, we are done. □

In the next result, which is our main theorem in this section, we consider the maximality of $C_F(X)$ in $L_c(X)$. First, let us recall that if $\varphi : C(X) \to C(Y)$ is a ring homomorphism with $\varphi(1) = 1$, then $\varphi(C_c(X)) \subseteq C_c(Y)$. This is an easy consequence of the fact that whenever $f \in C_c(X)$, then $\mathrm{Im}(\varphi(f)) \subseteq \mathrm{Im}(f)$ (note, let $r \in \mathrm{Im}(\varphi(f))$, then $\varphi(f) - r$ is non-unit, but $\varphi(f - r) = \varphi(f) - r$, hence $f - r$ is non-unit too, i.e., $Z(f - r) \neq \varnothing$, and we are done), see also[11, the comment following Corollary 3.5].

Theorem 5.10. *Let $C_F(X)$ be a maximal ideal in $L_c(X)$. Then $C^F(X) = C_c(X) = L_c(X)$ and $L_c(X)$ is isomorphic to a finite direct product of fields, each of which, is isomorphic to $\mathbb{R}$ and X has a unique infinite clopen connected subset. Conversely, let X have a unique infinite clopen connected subset, and assume that every element of $L_c(X)$ is constant on it, and $L_c(X) \cong \prod_{i=1}^{n} \mathbb{F}_i$, where each $\mathbb{F}_i$ is a field. Then $C^F(X) = C_c(X) = L_c(X)$, and $C_F(X)$ is maximal in $L_c(X)$.*

Proof. Let $C_F(X)$ be a maximal ideal in $L_c(X)$. Let us first take care of the case, when $C_F(X) = 0$. Clearly in this case $L_c(X) = \mathbb{R}$ (note, in this case X is connected and $C_c(X) = C^F(X) = \mathbb{R}$), and we are done. Hence, we may assume that that $C_F(X) \neq 0$. In view of the previous corollary we infer that $I(X)$, the set of isolated points of X must be finite. Let us assume that $|I(X)| = n$, where n is a positive integer (note, $C_F(X) \neq 0$ if and only if $I(X) \neq \varnothing$, see [20, Proposition 3.3]). Hence $C_F(X)$ is a finitely generated ideal in $C(X)$ (note, by [20, Proposition 3.1], there is a one-one correspondence between $I(X)$ and the set of nonzero minimal ideals in $C(X)$). Consequently, $C_F(X) = \sum_{i=1}^{n} \oplus e_i C(X)$, where each e_i is an idempotent and $e_i C(X) = e_i L_c(X)$ is a minimal ideal in $C(X)$ as well as in $L_c(X)$, see the comment preceding Lemma 5.1. Clearly, $C_F(X) = eC_F(X) = eC(X) = eL_c(X)$, where $e = e_1 + e_2 + \cdots + e_n$ (note, $e_i e_j = 0$ for $i \neq j$). Since $C_F(X)$ is maximal in $L_c(X)$, we infer that $e \neq 1$, which implies that $(1-e)L_c(X)$ is a nonzero minimal ideal in $L_c(X)$. Inasmuch as $C_F(X)$ is maximal in $L_c(X)$ and $C_F(X) \subseteq \mathrm{Soc}\big(L_c(X)\big)$, we infer that either $C_F(X) = \mathrm{Soc}\big(L_c(X)\big)$ or $L_c(X) = \mathrm{Soc}\big(L_c(X)\big)$. We claim that $C_F(X) = \mathrm{Soc}\big(L_c(X)\big)$ leads us to a contradiction. To see this, we note that $(1-e)L_c(X)$ is a nonzero minimal ideal in $L_c(X)$. Hence if the latter equality holds, we infer that $(1-e)L_c(X)$ must be in $C_F(X)$. But $C_F(X) \cap (1-e)L_c(X) = 0$, which is absurd. Consequently, we must have $L_c(X) = \mathrm{Soc}\big(L_c(X)\big) = C_F(X) \oplus (1-e)L_c(X)$. Now, for each e_i we can easily show that $e_i L_c(X) \cong \mathbb{R} \cong (1-e)L_c(X)$. To see this, let $x \in Z(1 - e_i)$ and define $\varphi : L_c(X) \to \mathbb{R}$ by $\varphi(f) = f(x)$ for all $f \in L_c(X)$. Hence, $(1-e_i)L_c(X) \subseteq ker\varphi$. Since $(1-e_i)L_c(X)$ is maximal in $L_c(X)$, we infer that $(1-e_i)L_c(X) = ker\varphi$, hence $e_iL_c(X) \cong \frac{L_c(X)}{(1-e_i)L_c(X)} \cong \mathbb{R}$. Similarly $(1-e)L_c(X) \cong \frac{L_c(X)}{eL_c(X)} \cong \mathbb{R}$. Consequently, we have already shown that $L_c(X) \cong \prod_{i=1}^{n+1} R_i$, where $R_i = \mathbb{R}$. In view of Lemma 5.1 , and by the fact that $(1-e)L_c(X)$ is minimal in $L_c(X)$, we infer that $Z(e)$ is connected. Consequently, by the comment preceding Lemma 5.1, $(1-e)C_c(X)$ is a minimal ideal in $C_c(X)$. Hence $C_c(X) = C_F(X) \oplus (1-e)C_c(X)$, which is equal to $\mathrm{Soc}\big(C_c(X)\big) = \mathrm{Soc}\big(C^F(X)\big) \subseteq C^F(X)$, see the comment preceding Proposition 5.2. Thus $C^F(X) = C_c(X)$. But, $C_c(X) = C_F(X) \oplus (1-e)C_c(X)$ is the direct sum of $n+1$ minimal ideals in $C_c(X)$, hence by the above proof for $L_c(X)$, we can also show that $C_c(X) \cong \prod_{i=1}^{n+1} R_i$, where $R_i = \mathbb{R}$. Let us consider the natural isomorphism $\varphi : C_c(X) \to L_c(X) \subseteq C(X)$. Now in view of the comment preceding the theorem we have $L_c(X) \subseteq C_c(X)$, hence $L_c(X) = C_c(X) = C^F(X)$. Finally, in view of [20, Proposition 3.3], it is clear that the connected clopen set $Z(e)$ is infinite (in fact $Z(e) = X \setminus I(X)$). It is also manifest that every non-singleton connected subset of X must be a subset of $Z(e)$, hence $Z(e)$ is the only clopen connected subset of X which is infinite, and we are done. Conversely, since $L_c(X) \cong \prod_{i=1}^{n} \mathbb{F}_i$, where each $\mathbb{F}_i$ is a field, we infer that $L_c(X) = \sum_{i=1}^{n} \oplus u_iL_c(X) = \mathrm{Soc}\big(L_c(X)\big)$, where each $u_iL_c(X)$ is a nonzero minimal ideal in $L_c(X)$, and each u_i is idempotent with $1 = u_1 + u_2 \cdots u_n$. Now let $1 \neq u \in L_c(X)$ be an idempotent such that $Z(1-u)$ is the unique infinite clopen subset of X, on which, every element of $L_c(X)$ is constant. Consequently, $uL_c(X)$ is a minimal ideal in $L_c(X)$, by Lemma 5.1. Multiplying, $1 = u_1 + u_2 \cdots u_n$ by u, we get $u = uu_1 + uu_2 + \cdots uu_n$. Clearly, $u \neq 0$, hence $uu_i \neq 0$ for some i. We now claim that there is a unique i, with $1 \leq i \leq n$ such that $uu_i \neq 0$. To see this, let $uu_i \neq 0 \neq uu_j$ for some $i \neq j$ and obtain a contradiction. But $uu_i \neq 0$ implies that $uL_c(X)u_iL_c(X) \neq 0$, hence $uL_c(X)u_iL_c(X) = uL_c(X) = u_iL_c(X)$ and similarly $uL_c(X) = u_jL_c(X)$, which is a contradiction. Consequently, we may assume that $uu_i = 0$ for $1 \leq i \leq n-1$ and $uu_n \neq 0$. This means that $uL_c(X) = u_nL_c(X)$. In view of [20, Proposition 3.3], and the fact that $Z(1-u)$ is infinite, we infer that $u \notin C_F(X)$, i.e, $u_n \notin C_F(X)$. By Lemma 5.1, and the fact that each $u_iL_c(X)$ for $1 \leq i \leq n-1$ is minimal, we infer that each $Z(1-u_i)$ is connected, which by our assumption is not an infinite set, hence it must be a singleton. Consequently, in view of [20, Proposition 3.3], $u_i \in C_F(X)$ for $1 \leq i \leq n-1$. Inasmuch as $L_c(X) = \sum_{i=1}^{n} \oplus u_iL_c(X) = \mathrm{Soc}\big(L_c(X)\big)$, we infer that $C_F(X) = \sum_{i=1}^{n-1} \oplus u_iL_c(X) \oplus u_nL_c(X) \bigcap C_F(X)$. Since $u_nL_c(X)$ is minimal, we infer that either $u_nL_c(X) \bigcap C_F(X) = 0$ or $u_nL_c(X) \bigcap C_F(X) = u_nL_c(X)$. The latter equality is impossible, for by what we have already observed above $u_n \notin C_F(X)$. Consequently, $C_F(X) = \sum_{i=1}^{n-1} \oplus u_i L_c(X)$ and $L_c(X) = C_F(X) \oplus u_n L_c(X)$, hence $C_F(X)$ is maximal in $L_c(X)$. Now by the proof of the first part we also have $C^F(X) = C_c(X) = L_c(X)$, hence we are done. $\square$

The following theorem shows that for the spaces X in which there exist certain constant subsets with respect to $L_c(X)$, $L_c(X)$ can not be isomorphic to any $C(Y)$.

Theorem 5.11. *Let $|I(X)| < \infty$ and $X \setminus I(X)$ be constant with respect to $L_c(X)$ (note, in this case $L_c(X) = C^F(X)$). Then there is no space Y with $L_c(X) \cong C(Y)$.*

Proof. Let $|I(X)| < \infty$ and $X \setminus I(X)$ be constant with respect to $L_c(X)$, then $C_F(X)$ is a prime ideal in $L_c(X)$ by Proposition 5.8. If there exists a space Y such that $L_c(X) \cong C(Y)$, then $\mathrm{Soc}\big(L_c(X)\big) \cong C_F(Y)$. Now, since $\mathrm{Soc}\big(L_c(X)\big)$ is a z_l-ideal containing a prime ideal $C_F(X)$, $\mathrm{Soc}\big(L_c(X)\big)$ is a prime ideal in $L_c(X)$, by Theorem 3.14. Hence $C_F(Y)$ is a prime ideal in $C(Y)$, which is a contradiction, see the comment preceding Proposition 5.11. $\square$

Remark 5.12. If we replace $L_c(X)$ by $L_F(X)$ or by $L_1(X)$ in this section, then some of the results of this section remain valid for these two rings, too.

Remark 5.13. Let $X = W \cup \{x_1, x_2, \ldots, x_n\}$, where W is constant with respect to $L_1(X)$ (e.g., if we take W as in Remark 2.10) and $x_1, x_2, \ldots, x_n$ are the only isolated points of X (note, W is connected and has no isolated point) i.e., $|I(X)| < \infty$ and $X \setminus I(X) = W$ is a constant subset of X with respect to $L_1(X)$. Hence, by Theorem 5.11, Remark 5.12, $L_1(X)$ can not be isomorphic to any $C(Y)$, in general. But in some special cases, namely, $L_1(W)$ and $L_1(X)$ we have $L_1(W) = \mathbb{R}$ and $L_1(X) \cong \prod_{i=1}^{n} R_i$, where $R_i = \mathbb{R}$, for $i = 1, 2, \ldots, n$. That is to say $L_1(W) = C(Y)$, where Y is a singleton, and $L_1(X) = C(Z)$, where $|Z| < \infty$. But, we should remind the reader that we are interested in infinite spaces.

Remark 5.14. Let K be a subring of $\mathbb{R}$, then $L_c(X, K)$ is a subring of $L_c(X)$ whose elements take values in K. We denote $L_c(X, \mathbb{Z})$, $L_c(X, \mathbb{Q})$ by $L_i(X)$ and $L_r(X)$, respectively. Clearly, $L_i(X) = C(X, \mathbb{Z}) = C_i(X)$ and $L_r(X) = C(X, \mathbb{Q}) = C_r(X)$, see also [10, the comment following Definition 2.1]. It is manifest that $L_i(X) \subseteq L_r(X) \subseteq C(X, F) \subseteq C_c(X) \subseteq L_c(X)$, where F is a countable subfield of $\mathbb{R}$ and $L_i(X) \subseteq L_r(X) \subseteq L_c(X, K) \subseteq L_c(X)$, where K is a proper subfield of $\mathbb{R}$. But unfortunately, apart from $C_c(X)$ and $L_c(X)$, these are not $\mathbb{R}$-subalgebras of $C(X)$, see [10, Remark 7.5], and are not of our interest, in general.

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