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# Best proximity points of contractive mappings on a metric space with a graph and applications

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Abstract

*We establish an existence and uniqueness theorem on best proximity point for contractive mappings on a metric space endowed with a graph. As an application of this theorem, we obtain a result on the existence of unique best proximity point for uniformly locally contractive mappings. Moreover, our theorem subsumes and generalizes many recent fixed point and best proximity point results.*

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## 1. Introduction

Fixed point theory plays an important role for solving equations of the form $Tx = x$ where $T$ is defined on a subset of a metric space, partially ordered metric space, topological vector space or some suitable space. Given two non-empty subsets $A$ and $B$ of a metric space $(X, d)$, consider a non-self mapping $T : A \to B$. If $T(A) \cap A = \varnothing$, there does not exist a solution of the equation $Tx = x$. Then it is interesting to find a point $x \in A$ that is closest to $Tx$ in some sense. Best approximation and best proximity point results have been established in this direction. The well-known best approximation theorem due to Ky Fan [3] states that for a given non-empty compact convex subset $C$ of

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a normed linear space $E$ and a continuous mapping $F : C \to E$, there exists $x^* \in C$ such that $\|x^* - Fx^*\| = d(Fx^*, C) = \inf\{\|Fx^* - x\| : x \in C\}$. Though this result gives the existence of an approximate solution of $Fx = x$, such solution need not be optimal in the sense that $\|x - Fx\|$ is minimum.

Naturally, for the map $T$, one can think of finding an element $x^* \in A$ such that $d(x^*, Tx^*) = \min\{d(x, Tx) : x \in A\}$. Since for all $x \in A$, $d(x, Tx) \geq d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$. An optimal solution of $\min\{d(x, Tx) : x \in A\}$ is one for which the value $d(A, B)$ is attained. An element $x^* \in A$ is called a *best proximity point* for the mapping $T$ if $d(x^*, Tx^*) = d(A, B)$. Hence a best proximity point of the map $T$ is not only an approximate solution of $Tx = x$, but also optimal in the sense that $d(x, Tx)$ is minimum. Clearly, a best proximity point theorem is a natural generalization of a fixed point theorem. Some interesting best proximity point results can be found in [7, 11, 14] and for applications, one can refer to [5, 6].

Recently, Jachymski [4] established the existence of fixed points for contractive mappings on a metric space endowed with a graph. This result unified various fixed point theorems for contractive mappings on metric spaces and partially ordered metric spaces. For some more fixed point results on a metric space with a graph, one can refer to [1, 13].

1.1. **Our contribution.** Following Jachymski [4], in this article we prove an existence and uniqueness theorem on best proximity point for non-self contractive mappings on a metric space endowed with a graph. As an application of this result, we obtain a generalization of the fixed point theorem for uniformly locally contractive mappings due to Edelstein [2, Theorem 5.2]. Also, our result enables us to obtain a best proximity point result for non-self mappings on partially ordered metric spaces. Further, our result subsumes a very recent result on existence of a unique best proximity point on a metric space due to V. Sankar Raj [11, Theorem 3.1].

## 2. Preliminaries

In this section, let us recall some definitions and notations which are needed for our results.

Let $(X, d)$ be a metric space. For given non-empty subsets $A$ and $B$ of $(X, d)$, we denote by $A_0$ and $B_0$ the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

For sufficient conditions which ensure the non-emptiness of $A_0$ and $B_0$, one can refer to [7].

Let $(A, B)$ be a pair of non-empty subsets of $(X, d)$ such that $A_0 \neq \varnothing$. Then the pair $(A, B)$ is said to have the $P$-property [11] if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \;\Rightarrow\; d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

It is easy to verify that for a non-empty subset $A$ of $(X, d)$, the pair $(A, A)$ has the $P$-property. Every pair of non-empty closed convex subsets of a real Hilbert space $H$ has the $P$-property (see [11]).

Consider a directed graph $G$ where the set $V(G)$ of its vertices coincides with $X$, the set $E(G)$ of its edges is such that $E(G) \supseteq \Delta$ (where $\Delta = \{(x, x) : x \in X\}$) and $E(G)$ has no parallel edges. We denote by $\tilde{G}$ the undirected graph obtained from $G$ by ignoring the direction of edges. For given two vertices $x$ and $y$, we say that there is a *path* in $G$ of length $N$ (where $N \in \mathbb{N} \cup \{0\}$) between them if there exists a sequence $(x^i)_{i=0}^{N}$ such that $x^0 = x$, $x^N = y$ and $(x^{i-1}, x^i) \in E(G)\ \forall\ i = 1, 2, \ldots, N$. The graph $G$ is called connected if there is a path between any two vertices and weakly connected if $\tilde{G}$ is connected. For $x \in V(G) = X$, we denote

$$[x]_G^N = \{y \in X : \text{there is a path in } G \text{ of length } N \text{ from } x \text{ to } y\}.$$

## 3. Main results

Throughout this section we assume that $(X, d)$ is a metric space endowed with a directed graph $G$ where $V(G) = X$, $E(G) \supseteq \Delta$ and $G$ has no parallel edges. We now introduce a notion of Banach contraction (for non-self map) with respect to the graph $G$ for which we prove our main results.

**Definition 3.1.** Let $A$ and $B$ be two non-empty subsets of $(X, d)$. A mapping $T : A \to B$ is said to be a *Banach G-contraction* or simply *G-contraction* if for all $x, y \in A$, $x \neq y$ with $(x, y) \in E(G)$:

(a) $d(Tx, Ty) \leq \alpha d(x, y)$ for some $\alpha \in [0, 1)$;

(b) $$\left.\begin{array}{l} d(x_1, Tx) = d(A, B) \\ d(y_1, Ty) = d(A, B) \end{array}\right\} \;\Rightarrow\; (x_1, y_1) \in E(G), \quad \text{for all } x_1, y_1 \in A.$$

**Theorem 3.2.** *Let $(X, d)$ be complete metric space, $A$ and $B$ be two non-empty closed subsets of $(X, d)$ such that $(A, B)$ has the $P$-property. Let $T : A \to B$ be a $G$-contraction such that $T(A_0) \subseteq B_0$. Assume that for some $N \in \mathbb{N}$,*

*(i) there exist $x_0$ and $x_1$ in $A_0$ such that there is a $N$-length path $(y_0^i)_{i=0}^{N} \subseteq A_0$ in $G$ between them and $d(x_1, Tx_0) = d(A, B)$;*

*(ii) for any sequence $\{s_n\}_{n\in\mathbb{N}}$ in $A$ with $s_n \to s$ and $s_{n+1} \in [s_n]_G^N$, there is a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ such that $(s_{n_k}, s) \in E(G)\ \forall k \in \mathbb{N}$.*

*Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $d(x_{n+1}, Tx_n) = d(A, B)$ for $n \in \mathbb{N}$, converging to a best proximity point of $T$. Furthermore, $T$ has a unique best proximity point if for any two elements $x$ and $y$ in $A_0$, there exists a path $(y^i)_{i=0}^{l} \subseteq A_0$ in $\tilde{G}$ between them.*

*Proof.* By (i), there exist two points $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and a sequence $(y_0^i)_{i=0}^N$ containing points of $A_0$ such that $y_0^0 = x_0$, $y_0^N = x_1$ and $(y_0^{i-1}, y_0^i) \in E(G)$ $\forall 1 \leq i \leq N$. As $y_0^1 \in A_0$ and $T(A_0) \subseteq B_0$, there exists $y_1^1 \in A_0$ such that $d(y_1^1, Ty_0^1) = d(A, B)$. Similarly, for $i = 2, \cdots, N$, there exists $y_1^i \in A_0$ such that $d(y_1^i, Ty_0^i) = d(A, B)$.

As $(y_0^0 = x_0, y_0^1) \in E(G)$ and $T$ is a $G$-contraction, it follows from the above that $(x_1, y_1^1) \in E(G)$. In a similar way, it follows that $(y_1^{i-1}, y_1^i) \in E(G)$ for $i = 2, \cdots, N$. Let $x_2 = y_1^N$. Thus $(y_1^i)_{i=0}^N$ is a path from $x_1 (= y_1^0)$ to $x_2 (= y_1^N)$.

Again, for each $i = 1, 2, \cdots, N$, since $y_1^i \in A_0$ and $Ty_1^i \in T(A_0) \subseteq B_0$, there exists $y_2^i \in A_0$ such that $d(y_2^i, Ty_1^i) = d(A, B)$. Also, we have $d(x_2, Tx_1) = d(A, B)$. As shown in the previous paragraph, it follows that $(x_2, y_2^1) \in E(G)$ and $(y_2^{i-1}, y_2^i) \in E(G)$ $\forall i = 2, \cdots, N$. Set $x_3 = y_2^N$. Thus $(y_2^i)_{i=0}^N$ is a path from $x_2 (= y_2^0)$ to $x_3 (= y_2^N)$.

Continuing in this manner for all $n \in \mathbb{N}$, we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ where $x_{n+1} \in [x_n]_G^N$ and $d(x_{n+1}, Tx_n) = d(A, B)$ by producing a path $(y_n^i)_{i=0}^N$ from $x_n (= y_n^0)$ to $x_{n+1} (= y_n^N)$ in such way that

$$d(y_{n+1}^i, Ty_n^i) = d(A, B) \quad \forall\, i = 0, \cdots, N. \tag{3.1}$$

Using the $P$-property of $(A, B)$, it follows from equation (3.1) that for each $n \in \mathbb{N}$,

$$d(y_n^{i-1}, y_n^i) = d(Ty_{n-1}^{i-1}, Ty_{n-1}^i) \quad \forall\, 1 \leq i \leq N. \tag{3.2}$$

Now for any positive integer $n$,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(y_n^0, y_n^N) \\ &\leq d(y_n^0, y_n^1) + d(y_n^1, y_n^2) + \cdots + d(y_n^{N-1}, y_n^N) \\ &= \sum_{i=1}^N d(y_n^{i-1}, y_n^i) = \sum_{i=1}^N d(Ty_{n-1}^{i-1}, Ty_{n-1}^i). \end{aligned}$$

Since for all $n \in \mathbb{N}$ and $1 \leq i \leq N$, $(y_{n-1}^{i-1}, y_{n-1}^i) \in E(G)$ and $T$ is a $G$-contraction, it follows from the above inequalities that for $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq \alpha \sum_{i=1}^N d(y_{n-1}^{i-1}, y_{n-1}^i) \quad \text{for some } \alpha \in [0, 1).$$

Repeating the process, it follows that for all $n \in \mathbb{N}$,

$$d(x_n, x_{n+1}) \leq \alpha^n \sum_{i=1}^N d(y_0^{i-1}, y_0^i) = M\alpha^n \quad \text{where } M = \sum_{i=1}^N d(y_0^{i-1}, y_0^i).$$

Now for $m \geq n$, $n \in \mathbb{N}$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq M\alpha^n + \cdots + M\alpha^{m-1} \\ &= M\alpha^n [1 + \cdots + \alpha^{m-n-1}] \leq M \frac{\alpha^n}{1 - \alpha}. \end{aligned}$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore $\{x_n\}_{n\in\mathbb{N}}$ converges to some point $x^* \in A$ as $n \to \infty$. By (ii), there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(x_{n_k}, x^*) \in E(G)$ $\forall k \in \mathbb{N}$. Hence,

$$d(Tx_{n_k}, Tx^*) \leq \alpha d(x_{n_k}, x^*) \quad \text{for } k \in \mathbb{N}.$$

Thus taking $k \to \infty$, $Tx_{n_k} \to Tx^*$. Using the continuity of the metric function, we get $d(x_{n_{k+1}}, Tx_{n_k}) \to d(x^*, Tx^*)$ as $k \to \infty$. Now $\{d(x_{n_{k+1}}, Tx_{n_k})\}$ is nothing but a constant sequence with value $d(A, B)$. Therefore $d(x^*, Tx^*) = d(A, B)$.

Suppose that $p$ and $q$ are two best proximity points of $T$. Consider two sequences $\{p_n\}_{n\in\mathbb{N}}$ and $\{q_n\}_{n\in\mathbb{N}}$ where $p_n = p$ and $q_n = q$ for all $n \geq 1$. Clearly, $d(p_{n+1}, Tp_n) = d(A, B)$ and $d(q_{n+1}, Tq_n) = d(A, B)$ for all $n \geq 1$. As $p, q \in A_0$, it follows from the hypothesis that there is a path $(y_1^i)_{i=0}^M \subseteq A_0$ in $\tilde{G}$ between $p_1 = p$ and $q_1 = q$. For each $i = 1, 2, \cdots, M-1$, since $y_1^i \in A_0$ and $T(y_1^i) \in T(A_0) \subseteq B_0$, we can obtain $\{y_n^i\}_{n\in\mathbb{N}}$ such that $d(y_{n+1}^i, Ty_n^i) = d(A, B)$ $\forall n \in \mathbb{N}$. It is easy to verify that $T$ is also a $\tilde{G}$-contraction. Also, we have $(y_1^{i-1}, y_1^i) \in E(\tilde{G})$ for $1 \leq i \leq M$. Thus it follows that $(y_2^i)_{i=0}^M$ is a path in $\tilde{G}$ between $p_2 (= y_2^0)$ and $q_2 (= y_2^M)$. Similarly, it follows that $\forall n \in \mathbb{N}$, $(y_n^i)_{i=0}^M$ is a path in $\tilde{G}$ from $p_n (= y_n^0)$ to $q_n (= y_n^M)$. Now for $n \in \mathbb{N}$,

$$\begin{aligned} d(p, q) &= d(p_{n+1}, q_{n+1}) \leq \sum_{i=1}^{M} d(y_{n+1}^{i-1}, y_{n+1}^i) = \sum_{i=1}^{M} d(Ty_n^{i-1}, Ty_n^i) \\ &\leq \alpha \sum_{i=1}^{M} d(y_n^{i-1}, y_n^i) \leq \cdots \leq \alpha^n \sum_{i=1}^{M} d(y_1^{i-1}, y_1^i). \quad [\text{where } \alpha \in [0, 1)] \end{aligned}$$

This implies that $p = q$ and this completes the proof. □

*Remark* 3.3. Theorem 3.2 still holds true if we replace the condition (ii) by the continuity of the function $T$ on the set $A$.

The above Theorem 3.2 yields the following result due to Jachymski [4].

**Theorem 3.4** (see [4])**.** *Let $(X, d)$ be complete and $f : X \to X$ be a map such that for all $x, y \in X$ with $(x, y) \in E(G)$, $(fx, fy) \in E(G)$ and $d(fx, fy) \leq kd(x, y)$ where $k \in [0, 1)$. Assume that for any $\{y_n\}_{n\in\mathbb{N}}$ in $X$ with $y_n \to y^*$ and $(y_{n+1}, y_n) \in E(G)$ $\forall n \geq 1$, there exists a subsequence $\{y_{n_p}\}_{p\in\mathbb{N}}$ such that $(y_{n_p}, y^*) \in E(G)$ for all $p \in \mathbb{N}$. Then the following statements hold:*

- *(i) $\{f^n(x)\}_{n\in\mathbb{N}}$ converges to a fixed point of $f$ if $(x, fx) \in E(G)$;*
- *(ii) if $G$ is weakly connected and there exists $x_0 \in X$ such that $(x_0, fx_0) \in E(G)$, then $\forall x \in X$, $\{f^n(x)\}_{n\in\mathbb{N}}$ converges to a unique fixed point of $f$.*

Further, we get the following result due to V. Sankar Raj [11] as a corollary to the Theorem 3.2 by taking $E(G) = X \times X$.

**Corollary 3.5** ([11, Theorem 3.1])**.** *Let $(X, d)$ be a complete metric space, $A$ and $B$ be two non-empty closed subsets of $(X, d)$ such that $A_0 \neq \varnothing$ and $(A, B)$*

*satisfies $P$-property. Suppose that $T : A \to B$ is such that $T(A_0) \subseteq B_0$ and*

$$d(Tx, Ty) \leq kd(x, y) \quad \forall x, y \in A \text{ and for some } k \in [0, 1). \tag{3.3}$$

*Then there exists a unique $x^*$ in $A$ such that $d(x^*, Tx^*) = d(A, B)$. Further, for any fixed $x_0 \in A_0$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $d(x_n, Tx_{n-1}) = d(A, B)$ for $n \in \mathbb{N}$, converging to $x^*$.*

The following example shows that our Theorem 3.2 is an extension of the above result due to V. Sankar Raj [11].

**Example 3.6.** Consider $X = \mathbb{R}^2$ with usual metric and suppose that

$$A = \left\{ \left(0, \frac{1}{n}\right) : n \in \mathbb{N} \right\} \cup \{(0, 0)\},$$
$$B = \left\{ \left(1, \frac{1}{n}\right) : n \in \mathbb{N} \right\} \cup \{(1, 0)\}.$$

It is easy to check that the pair $(A, B)$ has the $P$-property. Suppose that a map $T : A \to B$ is defined as follows:

$$T((0, x)) = \left(1, \frac{x}{2}\right), \quad \text{for all } (0, x) \in A \text{ with } x \neq 1,$$
$$T((0, 1)) = (1, 1).$$

Consider a graph $G$ with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : d(x, y) < \frac{1}{2}\}$. Let $x = (0, x')$ and $y = (0, y')$ be two elements in $A$ with $(x, y) \in E(G)$. Then,

$$d(T(x), T(y)) = d\left(\left(1, \frac{x'}{2}\right), \left(1, \frac{y'}{2}\right)\right) \leq \frac{1}{2} d(x, y).$$

If $x_1 = (0, x'_1)$ and $y_1 = (0, y'_1)$ are two elements in $A$ such that

$$d(x_1, T(x)) = d(y_1, T(y)) = dist(A, B).$$

Then by using the $P$-property of $(A, B)$, it follows from the above equation that $d(x_1, y_1) = d(T(x), T(y)) \leq \frac{1}{2} d(x, y) < \frac{1}{2}$. Hence the pair $(x_1, y_1) \in E(G)$. This proves that $T$ is a non-self $G$-contraction with $\alpha = \frac{1}{2}$. Clearly, $(X, d)$ is complete and $A$ and $B$ are closed subsets of $X$. Also, note that in this case $A_0 = A, B_0 = B$ and $T(A_0) = T(A) \subseteq B = B_0$. Let $x_0 = (0, \frac{1}{2})$, $x_1 = (0, \frac{1}{4})$ and $N = 1$. Then $d(x_1, T(x_0)) = dist(A, B) = 1$ and the pair $(x_1, x_0) \in E(G)$. Hence, the condition (i) of Theorem 3.2 holds. Also, let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence in $A$ such that $s_n \to s$ as $n \to \infty$. Then there exists a positive integer $M$ such that $d(s_n, s) < \frac{1}{2}$ $\forall n \geq M$. Let $n_k = M + k$ for $k \geq 1$. Consequently, $\{s_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of the sequence $\{s_n\}_{n\in\mathbb{N}}$ such that $(s_{n_k}, s) \in E(G)$ $\forall k \in \mathbb{N}$. This implies that the condition (ii) of Theorem 3.2 is also satisfied. Therefore Theorem 3.2 guarantees the existence of a best proximity point of $T$. Note that $(0, 0)$ and $(0, 1)$ are two best proximity points. However,

$$d(T(0, 0), T(0, 1)) = d((1, 0), (1, 1)) = 1 > kd((0, 0), (0, 1)),$$

for any $k \in [0, 1)$. This proves that $T$ does not satisfy the contractive condition (3.3).

## 4. Applications

Let $A$ and $B$ be two non-empty subsets of a metric space $(X,d)$. A mapping $f: A \to B$ is called $(\epsilon,k)$-*uniformly locally contractive* [2] (where $k \in [0,1)$ and $\epsilon > 0$) if $d(fx,fy) \leq kd(x,y)$ for all $x,y \in A$ with $d(x,y) < \epsilon$. An $(\epsilon,k)$-uniformly locally contractive mapping need not be a contraction, for example one can refer to [2, 8]. As an application of Theorem 3.2, we now establish the following result for uniformly locally contractive mappings.

**Theorem 4.1.** *Let $(X,d)$ be complete metric space, $A$ and $B$ be closed subsets of $(X,d)$ such that $A_0 \neq \varnothing$ and $(A,B)$ satisfies $P$-property. Suppose that $T: A \to B$ is an $(\epsilon,k)$-uniformly locally contractive mapping satisfying $T(A_0) \subseteq B_0$. Then $T$ has a unique best proximity point if the space $(A_0,d)$ is $\epsilon$-chainable, that is, given $a,b \in A_0$, there exist $N \in \mathbb{N}$ and a sequence $(y^i)_{i=0}^N$ in $A_0$ such that $y^0 = a,\ y^N = b$ and $d(y^{i-1},y^i) < \epsilon$ for each $i = 1,2,\cdots,N$.*

*Proof.* Consider the graph $G$ where $V(G) = X$ and $E(G)$ as follows:

$$E(G) = \{(x,y) \in X \times X : d(x,y) < \epsilon\}.$$

It is clear that $E(G) \supseteq \Delta$ and $G$ has no parallel edges. Also, in this case $G = \tilde{G}$. Let $x,y \in A$ be such that $(x,y) \in E(G)$ and for all $x_1,y_1 \in A$,

$$d(x_1,Tx) = d(A,B) \text{ and } d(y_1,Ty) = d(A,B).$$

Since $(x,y) \in E(G)$, $d(Tx,Ty) \leq kd(x,y)$ where $k \in [0,1)$. Hence and by the $P$-property of $(A,B)$, we have $d(x_1,y_1) < \epsilon$. Therefore $T$ is a $G$-contraction. Since $A_0 \neq \varnothing$ and $T(A_0) \subseteq B_0$, there exist $x_0$ and $x_1$ in $A_0$ such that $d(x_1,Tx_0) = d(A,B)$. The $\epsilon$-chainability of $(A_0,d)$ implies that there exist a natural number $N$ and a sequence $(y^i)_{i=0}^N$ containing points of $A_0$ such that $y^0 = x_0$, $y^N = x_1$ and $d(y^{i-1},y^i) < \epsilon$ for $i = 1,\cdots,N$. Thus $(y^i)_{i=0}^N \subseteq A_0$ is a path in $G$ between $x_0$ and $x_1$. If $\{s_n\}_{n\in\mathbb{N}}$ is a sequence in $A$ such that $s_n \to s$, then there exists $M \in \mathbb{N}$ such that $d(s_n,s) < \epsilon\ \forall n \geq M$. Hence we can obtain a subsequence $\{s_{n_p}\}_{p\in\mathbb{N}}$ such that $(s_{n_p},s) \in E(G)\ \forall p \in \mathbb{N}$. Also, it is clear from the $\epsilon$-chainability of $(A_0,d)$ that for every $x,y \in A_0$, there is a path $(q^i)_{i=0}^l \subseteq A_0$ in $\tilde{G}$ (i.e., $G$) between them. Thus $T$ has a unique best proximity point by Theorem 3.2. □

As a corollary to the above theorem, we get the following theorem due to Edelstein [2] by considering $A = B = X$.

**Theorem 4.2** ([2, Theorem 5.2])**.** *Let $(X,d)$ be a complete metric space. An $(\epsilon,k)$- uniformly locally contractive mapping $f: X \to X$ has a unique fixed point if $(X,d)$ is $\epsilon$-chainable.*

In the last part of this section we establish the following result for non-self contractive mapping on a partially ordered metric space.

Let $(X,d)$ be a metric space endowed with a partial order $\preceq$ and $A$ and $B$ be two non-empty subsets of $(X,d)$. By $X_\preceq$, we denote the following set:

$$X_\preceq = \{(x,y) \in X \times X : x \preceq y \text{ or } x \succeq y\}.$$

 

Following [10], we say that a mapping $T : A \to B$ is a *proximally monotone* mapping if for all $x_1, x_2 \in A$ with $x_1 \preceq x_2$:

$$\left.\begin{aligned} d(y_1, Tx_1) = d(A,B) \\ d(y_2, Tx_2) = d(A,B) \end{aligned}\right\} \quad \Rightarrow \quad (y_1, y_2) \in X_{\preceq}, \quad \text{for all } y_1, y_2 \in A.$$

**Theorem 4.3.** *Let $(X, d)$ be complete metric space, $A$ and $B$ be two closed subsets of $(X, d)$ such that $(A, B)$ has the $P$-property. Let $T : A \to B$ be a proximally monotone map such that $T(A_0) \subseteq B_0$ and*

$$d(Tx, Ty) \leq kd(x, y) \quad \text{for all } x \preceq y \text{ and for some } k \in [0, 1).$$

*Assume that either $T$ is continuous on $A$ or for any $\{y_n\}_{n\in\mathbb{N}}$ in $A$ with $y_n \to y^*$ and $(y_n, y_{n+1}) \in X_{\preceq}$ for $n \in \mathbb{N}$, there exists $(y_{n_p})_{p\in\mathbb{N}}$ such that $(y_{n_p}, y^*) \in X_{\preceq}$ for $p \in \mathbb{N}$. Then $T$ has a best proximity point if there exist $x_0$ and $x_1$ in $A_0$ such that $d(x_1, Tx_0) = d(A, B)$ and $(x_0, x_1) \in X_{\preceq}$. Moreover, the best proximity point of $T$ is unique if for $x, y \in A_0$, there exists $z \in A_0$ such that $(x, z), (y, z) \in X_{\preceq}$.*

*Proof.* By considering the graph $G$ where $V(G) = X$ and

$$E(G) := \{(x, y) \in X \times X : \ x \preceq y \vee y \preceq x\},$$

the proof follows by Theorem 3.2 and Remark 3.3. □

The above result includes the fixed point results for mappings on a partially ordered metric space due to Ran and Reurings [12] and J. J. Nieto and R. R. López [9].

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## References

[1] T. Dinevari and M. Frigon, Fixed point results for multivalued contractions on a metric space with a graph, J. Math. Anal. Appl. 405 (2013), 507–517.

[2] M. Edelstein, An extension of Banach's contraction principle, Proc. Amer. Math. Soc. 12 (1961), 7–10.

[3] K. Fan, Extensions of two fixed point theorems of F. E. Browder, Math. Z. 122 (1969), 234–240.

[4] J. Jachymski, The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc. 136 (2008), 1359-1373.

[5] W. K. Kim and K. H. Lee, Existence of best proximity pairs and equilibrium pairs, J. Math. Anal. Appl. 316 (2006), 433–446.

[6] W. K. Kim, S. Kum and K. H. Lee, On general best proximity pairs and equilibrium pairs in free abstract economies, Nonlinear Anal. TMA 68 (2008), 2216–2227.

 

[7] W. A. Kirk, S. Reich and P. Veeramani, Proximinal retracts and best proximity pair theorems, Numer. Funct. Anal. Optim. 24 (2003), 851–862.
[8] L. Máté, The Hutchinson-Barnsley theory for certain non-contraction mappings, Period. Math. Hungar. 27 (1993), 21–33.
[9] J. J. Nieto and R. Rodríguez-López, Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005), 223–239.
[10] V. Pragadeeswarar and M. Marudai, Best proximity points: approximation and optimization in partially ordered metric spaces, Optim. Lett. 7 (2013), 1883–1892.
[11] V. Sankar Raj, Best proximity point theorems for non-self mappings, Fixed Point Theory 14 (2013), 447–454.
[12] A. C. M. Ran and M. C. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004), 1435–1443.
[13] A. Sultana and V. Vetrivel, Fixed points of Mizoguchi-Takahashi contraction on a metric space with a graph and applications, J. Math. Anal. Appl. 417 (2014), 336–344.
[14] A. Sultana and V. Vetrivel, On the existence of best proximity points for generalized contractions, Appl. Gen. Topol. 15 (2014), 55–63.

 